Werk

Titel: Hermitian surfaces of constant holomorphic sectional curvature.

Autor: Sekigawa, Kouei; Sato, Takuji

Jahr: 1990

PURL: https://resolver.sub.uni-goettingen.de/purl?266833020_0205|log71

Kontakt/Contact

Digizeitschriften e.V.
SUB Göttingen
Platz der Göttinger Sieben 1
37073 Göttingen

info@digizeitschriften.de

Math. Z. 205, 659–668 (1990)

Mathematische Zeitschrift
© Springer-Verlag 1990

Hermitian surfaces of constant holomorphic sectional curvature

Takuji Sato[1] **and Kouei Sekigawa**[2]

[1] Faculty of Technology, Kanazawa University, Kanazawa, 920 Japan
[2] Department of Mathematics, Faculty of Science, Niigata University, Niigata, 950-21 Japan

Received December 4, 1989

1 Introduction

Let $M = (M, J, g)$ be an almost Hermitian manifold and $U(M)$ the unit tangent bundle of M. Then the holomorphic sectional curvature $H = H(x)$ can be regarded as a differentiable function on $U(M)$. If the function H is constant along each fibre, then M is called a space of pointwise constant holomorphic sectional curvature. Especially, if H is constant on the whole of $U(M)$, then M is called a space of constant holomorphic sectional curvature.

An almost Hermitian manifold M with integrable almost complex structure is called a Hermitian manifold. A real 4-dimensional Hermitian manifold is called a Hermitian surface. It is known that there exists an example of a Hermitian manifold with pointwise constant holomorphic sectional curvature which is not globally constant ([4]).

In [2], A. Balas and P. Gauduchon proved that every Hermitian metric of constant *non-positive* holomorphic sectional curvature on a compact complex surface is Kähler (Theorem 1). In the same paper, they also studied the structure of compact Hermitian surfaces of constant *positive* holomorphic sectional curvature (Theorem 2).

It should be noticed that the holomorphic sectional curvature in the above paper is defined by means of the curvature tensor with respect to the *Hermitian connection* (known also as the *Chern connection*). By the definition, the Hermitian connection coincides with the Riemannian one in a Kähler manifold. It is easy to observe that the holomorphic sectional curvature with respect to the Hermitian connection is greater than or equal to the holomorphic sectional curvature with respect to the Riemannian one, and in general, the constancy of the holomorphic sectional curvature with respect to the Hermitian connection does not imply the constancy of the holomorphic sectional curvature with respect to the Riemannian one (cf. [10]). So, it is worthwhile to study the structure of Hermitian surfaces of constant holomorphic sectional curvature with respect to the Riemannian connection.

In the sequel, we assume that the curvature tensor of a Hermitian surface means always the one with respect to the Riemannian connection. In connection with the above results, we shall prove the following

Theorem A. *Let $M = (M, J, g)$ be a compact Hermitian surface of constant non-positive holomorphic sectional curvature. Then M is a Kähler surface.*

Theorem B. *Let $M = (M, J, g)$ be a compact Hermitian surface of constant positive holomorphic sectional curvature. Then the Euler number $\chi(M)$ and the Chern number $c_1(M)^2$ are positive, and the Pontrjagin number $p_1(M)$ is non-negative (and hence, M is an algebraic surface with positive Euler number and non-negative signature).*

The proofs of Theorems A and B are given by applying the Miyaoka's inequality for two Chern numbers, $c_1(M)^2$ and $c_2(M)$ of a compact complex surface M. The authors would like to express their hearty thanks to Professor L. Vanhecke for his valuable suggestions.

2 Preliminaries

Let $M = (M, J, g)$ be a $2n$-dimensional almost Hermitian manifold with the almost Hermitian structure (J, g) and $\Omega = (\Omega_{ij})$ the Kähler form of M defined by $\Omega_{ij} = g_{ik}J_j^k$. We assume that M is oriented by the volume form $dM = ((-1)^n/n!)\Omega^n$. We denote by ∇, $R = (R_{ijk}{}^l)$, $\rho = (\rho_{ij})$ and τ, the Riemannian connection, the Riemannian curvature tensor, the Ricci tensor and the scalar curvature of M, respectively. The Ricci $*$-tensor $\rho^* = (\rho^*_{ij})$ and the $*$-scalar curvature τ^* are defined respectively by

$$\rho^*_{ij} = \tfrac{1}{2} J_j^s R_{isa}{}^b J_b^a \,, \tag{2.1}$$

$$\tau^* = g^{ij}\rho^*_{ij} \,. \tag{2.2}$$

The generalized Chern form $\gamma = (\gamma_{ij})$ is given by

$$8\pi\gamma_{ij} = -4J_j^k\rho^*_{ik} - J^{kl}(\nabla_j J_k^h)\nabla_i J_{lh} \,. \tag{2.3}$$

It is well known the 2-form γ represents the first Chern class of M in the de Rham cohomology group.

For any $p \in M$ and any unit vector $x = (x^i) \in T_p(M)$, the holomorphic sectional curvature is defined by

$$H(x) = -R_{iakb}J_j^a J_l^b x^i x^j x^k x^l \,.$$

Now, we assume that M is a Hermitian surface. Then we have

$$d\Omega = \omega \wedge \Omega \,, \tag{2.4}$$

where $\omega = \delta\Omega \circ J$. The 1-form $\omega = (\omega_i)$ is called the Lee form of M. This form ω satisfies the following ([8, 9]):

$$J^{ij}\nabla_i\omega_j = 0 \,, \tag{2.5}$$

$$2\nabla_i J_{jk} = \omega_a J_j^a g_{ik} - \omega_a J_k^a g_{ij} + \omega_j J_{ki} - \omega_k J_{ji} \,, \tag{2.6}$$

$$\tau - \tau^* = 2\delta\omega + \|\omega\|^2 \,. \tag{2.7}$$

We denote by $\chi(M)$, $c_1(M)$, $c_2(M)$ and $p_1(M)$ the Euler class, the first Chern class, the second Chern class and the first Pontrjagin class of M, respectively. We note that $c_2(M)$ is equal to $\chi(M)$. The following theorems play essential roles in the proofs of our theorems.

Theorem 2.1 ([11]). *Let $M = (M, J)$ be a compact connected almost complex surface. Then we have*

$$p_1(M) + 2\chi(M) = c_1(M)^2 .$$

Theorem 2.2 ([6]). *Let $M = (M, J)$ be a compact connected complex surface. Then we have*

$$c_1(M)^2 \leqq \mathrm{Max}\{2c_2(M), 3c_2(M)\} .$$

3 Some formulas

In this Section, we shall prepare some fundamental formulas for a Hermitian surface of pointwise constant holomorphic sectional curvature (with respect to the Riemannian connection) for later use.

Let $M = (M, J, g)$ be a Hermitian surface of pointwise constant holomorphic sectional curvature $c = c(p)$ $(p \in M)$. Then, taking account of [7] and [8], we have

$$(3.1) \qquad R_{ijkl} = \frac{1}{4}\|\omega\|^2 C_{ijkl} + \left(\frac{c}{4} - \frac{1}{16}\|\omega\|^2\right) H_{ijkl} + \frac{1}{96}\{g_{ik}A_{jl} - g_{il}A_{jk} + g_{jl}A_{ik} - g_{jk}A_{il} + J_{ik}B_{jl} - J_{il}B_{jk} + J_{jl}B_{ik} - J_{jk}B_{il} + 2J_{ij}B_{kl} + 2J_{kl}B_{ij}\} ,$$

where

$$C_{ijkl} = g_{il}g_{jk} - g_{ik}g_{jl} ,$$

$$H_{ijkl} = g_{il}g_{jk} - g_{ik}g_{jl} + J_{il}J_{jk} - J_{ik}J_{jl} - 2J_{ij}J_{kl} ,$$

$$A_{ij} = 21(\nabla_i\omega_j + \nabla_j\omega_i + \omega_i\omega_j) - 3J_i^aJ_j^b(\nabla_a\omega_b + \nabla_b\omega_a + \omega_a\omega_b) ,$$

$$B_{ij} = 7(J_j^a\nabla_i\omega_a - J_i^a\nabla_j\omega_a) - (J_j^a\nabla_a\omega_i - J_i^a\nabla_a\omega_j) + 3(J_j^a\omega_i\omega_a - J_i^a\omega_j\omega_a) .$$

By (3.1) and (2.7), we have

$$(3.2) \qquad \rho_{ij} = \left\{\frac{3}{2}c + \frac{3}{16}(\tau - \tau^*)\right\}g_{ij} - \frac{1}{4}T_{ij} ,$$

$$(3.3) \qquad \rho^*_{ij} = \left\{\frac{3}{2}c - \frac{1}{16}(\tau - \tau^*)\right\}g_{ij} + \frac{1}{4}T^*_{ij} ,$$

where

$$(3.4) \qquad T_{ij} = \nabla_i\omega_j + \nabla_j\omega_i + \omega_i\omega_j - J_i^aJ_j^b(\nabla_a\omega_b + \nabla_b\omega_a + \omega_a\omega_b) ,$$

and

$$(3.5) \qquad T^*_{ij} = \nabla_i\omega_j - \nabla_j\omega_i - J_i^aJ_j^b(\nabla_a\omega_b - \nabla_b\omega_a) .$$

By (3.2), we get

$$\tau + 3\tau^* = 24c\,. \tag{3.6}$$

By (3.6), the formulas (3.2) and (3.3) may be rewritten as

$$\rho_{ij} = \frac{\tau}{4} g_{ij} - \frac{1}{4} T_{ij}\,, \tag{3.2$'$}$$

$$\rho^*_{ij} = \frac{\tau^*}{4} g_{ij} + \frac{1}{4} T^*_{ij}\,. \tag{3.3$'$}$$

From now on we establish some integral formulas which will be needed in the next section. Assume that the manifold M is compact and connected. By (3.4),

$$T_{ij}\omega^i\omega^j = 2\omega^i\omega^j \nabla_i\omega_j + \|\omega\|^4 - 2J_i^a J_j^b(\nabla_a\omega_b)\omega^i\omega^j\,,$$

and it follows that

$$\int_M T_{ij}\omega^i\omega^j\, dM = \int_M \{\|\omega\|^2\delta\omega + \|\omega\|^4 - 2F\}\, dM\,, \tag{3.7}$$

where we define the function F on M by

$$F = J_i^a J_j^b(\nabla_a\omega_b)\omega^i\omega^j\,. \tag{3.8}$$

By (3.2)$'$, (3.3)$'$, (3.5) and (3.7), we get

$$\int_M \{\rho_{ij}\omega^i\omega^j + \rho^*_{ij}\omega^i\omega^j - \tfrac{1}{2}F\}\, dM = \tfrac{1}{4}\int_M \{(\tau + \tau^*)\|\omega\|^2 - \|\omega\|^2\delta\omega - \|\omega\|^4\}\, dM\,. \tag{3.9}$$

By making use of Ricci's identity and Green's theorem, we get

$$\int_M (\nabla_i\omega_j)\nabla^j\omega^i\, dM = \int_M \{(\delta\omega)^2 - \rho_{ij}\omega^i\omega^j\}\, dM\,. \tag{3.10}$$

Since

$$(\nabla_i\omega_j)\nabla^i\omega^j = \tfrac{1}{2}(\nabla_i\omega_j - \nabla_j\omega_i)(\nabla^i\omega^j - \nabla^j\omega^i) + (\nabla_i\omega_j)\nabla^j\omega^i\,,$$

we get

$$\int_M (\nabla_i\omega_j)\nabla^i\omega^j\, dM = \int_M \{\|d\omega\|^2 + (\delta\omega)^2 - \rho_{ij}\omega^i\omega^j\}\, dM\,. \tag{3.11}$$

Taking account of the relation $\nabla_j\tau = 2\nabla^i\rho_{ij}$, from (2.6), (3.10) and (3.11), we have

$$\begin{aligned}\int_M \tau\,\delta\omega\, dM &= \int_M \omega^j\nabla_j\tau\, dM \\ &= \int_M \{\|d\omega\|^2 + 2(\delta\omega)^2 + 2\|\omega\|^2\delta\omega - 2\rho_{ij}\omega^i\omega^j - 2\rho^*_{ij}\omega^i\omega^j + 2F\}\, dM\,.\end{aligned}$$

Combining with (3.9), it follows that

$$\int_M F\, dM = \int_M \{\tau\,\delta\omega + \tfrac{1}{4}\|\omega\|^4 - \tfrac{1}{2}(\tau - \tau^*)^2 + 6c\|\omega\|^2 - \|d\omega\|^2\}\, dM\,. \tag{3.12}$$

Next, we define the functions G_1, G_2 by

$$G_1 = J_i^a J_j^b(\nabla_a\omega_b)\nabla^i\omega^j\,,$$
$$G_2 = J_i^a J_j^b(\nabla_a\omega_b)\nabla^j\omega^i\,.$$

Then, by similar computations to those of (3.12), we get

$$\int_M G_1\,dM = \int_M G_2\,dM = \int_M \{\rho^*_{ij}\omega^i\omega^j - \tfrac{3}{2}F - \tfrac{3}{4}\|\omega\|^2\delta\omega\}\,dM\,. \tag{3.13}$$

By (2.7), (3.9), (3.10), (3.11) and (3.13), we get

$$\int_M \|T\|^2\,dM = \int_M \{4\|d\omega\|^2 + 2(\tau-\tau^*)^2 - 4\tau^*\|\omega\|^2\}\,dM\,, \tag{3.14}$$

and

$$\int_M \|T^*\|^2\,dM = \int_M 4\|d\omega\|^2\,dM\,. \tag{3.15}$$

By (3.14), we have immediately the following

Lemma 3.1. *Let M be a compact Hermitian surface of pointwise constant holomorphic sectional curvature. Then*

$$\int_M \tau^*\|\omega\|^2\,dM \leqq \int_M \{\|d\omega\|^2 + \tfrac{1}{2}(\tau-\tau^*)^2\}\,dM\,. \tag{3.16}$$

Equality holds if and only if $T = 0$ (i.e., M is Einstein).

Lemma 3.2. *Let M be a compact Hermitian surface of pointwise constant holomorphic sectional curvature c. Then the Euler class of M is given by*

$$\chi(M) = \frac{1}{32\pi^2}\int_M \left\{12c^2 - \frac{1}{16}(\tau-\tau^*)^2 + \frac{1}{2}\tau^*\|\omega\|^2\right\}dM\,. \tag{3.17}$$

Proof. We begin with a calculation of the norm of the curvature tensor. Since

$$C_{ijkl}R^{ijkl} = 2\tau\,,$$

$$H_{ijkl}R^{ijkl} = 2(\tau + 3\tau^*)\,,$$

$$\begin{aligned}\{g_{ik}A_{jl} - g_{il}A_{jk} + g_{jl}A_{ik} - g_{jk}A_{il} + J_{ik}B_{jl} - J_{il}B_{jk}\\ + J_{jl}B_{ik} - J_{jk}B_{il} + 2J_{ij}B_{kl} + 2J_{kl}B_{ij}\}\,R^{ijkl}\\ = -4A_{ij}\rho^{ij} - 12B_{ik}J^k_j\rho^{*ij}\,,\end{aligned}$$

and by (3.1) and (3.6), we get

$$\|R\|^2 = 12c^2 + \frac{3}{8}\|\omega\|^2(\tau-\tau^*) - \frac{1}{24}A_{ij}\rho^{ij} - \frac{1}{8}B_{ik}J^k_j\rho^{*ij}\,. \tag{3.18}$$

By direct computations, we get

$$\begin{aligned}A_{ij}\rho^{ij} = {}& -\tfrac{9}{2}\tau(2\delta\omega - \|\omega\|^2) - 6\|\omega\|^4\\ & - 12\{(\nabla_i\omega_j)\nabla^i\omega^j + (\nabla_i\omega_j)\nabla^j\omega^i + \omega^i\nabla_i\|\omega\|^2\} + 12(2F + G_1 + G_2)\,,\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}B_{ik}J^k_j\rho^{*ij} = {}& \tfrac{3}{2}\tau^*(2\delta\omega - \|\omega\|^2)\\ & - 4\{(\nabla_i\omega_j)\nabla^i\omega^j - (\nabla_i\omega_j)\nabla^j\omega^i - G_1 + G_2\}\,.\end{aligned} \tag{3.20}$$

By substituting (3.19), (3.20) into (3.18), we get

$$\|R\|^2 = 12c^2 + \frac{3}{16}(\tau - \tau^*)^2 + (\nabla_i\omega_j)\nabla^i\omega^j + \frac{1}{2}\omega^i\nabla_i\|\omega\|^2 + \tfrac{1}{4}\|\omega\|^4 - F - G_1 . \tag{3.21}$$

By (2.7), (3.9), (3.11), (3.13) and (3.21), we have

$$\int_M \|R\|^2 dM = \int_M \left\{12c^2 + \frac{7}{16}(\tau - \tau^*)^2 + \|d\omega\|^2 - \frac{1}{2}\tau^*\|\omega\|^2\right\} dM . \tag{3.22}$$

By (3.2)′ and (3.14), we get

$$\int_M \|\rho\|^2 dM = \tfrac{1}{4}\int_M \{\tau^2 + \|d\omega\|^2 + \tfrac{1}{2}(\tau - \tau^*)^2 - \tau^*\|\omega\|^2\} dM . \tag{3.23}$$

Now, (3.17) is an immediate consequence of (3.22), (3.23) and the well known Gauss-Bonnet formula

$$\chi(M) = \frac{1}{32\pi^2}\int_M \{\|R\|^2 - 4\|\rho\|^2 + \tau^2\} dM .$$

Lemma 3.3. *Let M be a compact Hermitian surface of pointwise constant holomorphic sectional curvature. Then the square of the first Chern class of M is given by*

$$c_1(M)^2 = \frac{1}{32\pi^2}\int_M \{(\tau^*)^2 + \tau^*\|\omega\|^2 + \|d\omega\|^2\} dM . \tag{3.24}$$

Proof. It is easy to see

$$\gamma \wedge \gamma = \tfrac{1}{4}\{\gamma_{ij}\gamma_{kl}J^{ij}J^{kl} - 2\gamma_{ij}\gamma_{kl}J^{ik}J^{jl}\} dM . \tag{3.25}$$

By (2.3), (2.6) and (3.3)′, we get

$$8\pi\gamma_{ij} = (\tau^* + \|\omega\|^2)J_{ij} - J_i^k(\omega_j\omega_k + J_j^a J_k^b \omega_a\omega_b) + J_i^k T^*_{jk} . \tag{3.26}$$

By (3.26) and (2.5), we get

$$8\pi\gamma_{ij}J^{ij} = 4\tau^* + 2\|\omega\|^2 , \tag{3.27}$$

$$64\pi^2\gamma_{ij}\gamma_{kl}J^{ik}J^{jl} = 4(\tau^* + \|\omega\|^2)^2 - 4(\tau^* + \|\omega\|^2)\|\omega\|^2 + 2\|\omega\|^4 - \|T^*\|^2 . \tag{3.28}$$

Substituting (3.27), (3.28) into (3.25), we have

$$\gamma \wedge \gamma = \frac{1}{32\pi^2}\{(\tau^*)^2 + \tau^*\|\omega\|^2 + \tfrac{1}{4}\|T^*\|^2\} dM .$$

Thus we obtain (3.24).

By virtue of (3.6), Theorem 2.1, Lemma 3.2 and Lemma 3.3, the first Pontrjagin class $p_1(M)$ is given by

$$\begin{aligned} p_1(M) &= \frac{1}{32\pi^2}\int_M \{(\tau^*)^2 + \tfrac{1}{8}(\tau - \tau^*)^2 - 24c^2 + \|d\omega\|^2\} dM \\ &= \frac{1}{32\pi^2}\int_M \left\{\frac{1}{12}(\tau - 3\tau^*)^2 + \|d\omega\|^2\right\} dM . \end{aligned} \tag{3.29}$$

Thus we have the following

Theorem 3.4. *Let M be a compact Hermitian surface of pointwise constant holomorphic sectional curvature. Then the first Pontrjagin class $p_1(M)$ is non-negative. If $p_1(M)$ is equal to zero, then $\tau = 3\tau^*$ and $d\omega = 0$ (and hence, M is a locally conformal Kähler surface).*

4 Proofs of Theorems A and B

First, we shall prove Theorem A. Let $M = (M, J, g)$ be a compact Hermitian surface of constant holomorphic sectional curvature $c \leqq 0$.

First of all, we show that $\chi(M) \geqq 0$. To show this, suppose that $\chi(M) < 0$. Then by virtue of Theorem 2.1, Theorem 2.2 and Theorem 3.4, we have

$$0 \leqq 2\chi(M) - c_1(M)^2 = -p_1(M) \leqq 0 .$$

Thus we have

$$\tau = 3\tau^* , \qquad d\omega = 0 , \tag{4.1}$$

and

$$c_1(M)^2 = 2\chi(M) < 0 . \tag{4.2}$$

By (4.1) and (3.6), we get

$$\tau^* = 4c \leqq 0 , \qquad \tau = 12c \leqq 0 .$$

Hence, we have

$$\int_M \tau^* \|\omega\|^2 dM = 4c \int_M \|\omega\|^2 dM \leqq 0 .$$

On the other hand, taking account of (2.7), we have

$$\begin{aligned} \int_M \tau^* \|\omega\|^2 dM &= 4c \int_M \|\omega\|^2 dM \\ &= 4c \int_M (\tau - \tau^*) dM = 32c^2 \operatorname{Vol}(M) \geqq 0 . \end{aligned}$$

Thus, we have

$$\int_M \tau^* \|\omega\|^2 dM = 0 . \tag{4.3}$$

By (4.3) and Lemma 3.3, we have

$$c_1(M)^2 = \frac{1}{32\pi^2} \int_M (\tau^*)^2 dM \geqq 0 .$$

This is a contradiction to (4.2), which shows $\chi(M) \geqq 0$.

Since $\chi(M) \geqq 0$, by virtue of Theorem 2.2, Lemma 3.1, Lemma 3.2 and Lemma 3.3,

$$
\begin{aligned}
(4.4) \qquad 0 &\leqq 32\pi^2\{3\chi(M) - c_1(M)^2\} \\
&= \int_M \left\{36c^2 - \frac{3}{16}(\tau - \tau^*)^2 + \frac{1}{2}\tau^* \|\omega\|^2 - (\tau^*)^2 - \|d\omega\|^2\right\} dM \\
&\leqq \int_M \left\{\frac{1}{16}(\tau + 3\tau^*)^2 + \frac{1}{16}(\tau - \tau^*)^2 - (\tau^*)^2 - \frac{1}{2}\|d\omega\|^2\right\} dM \\
&= \int_M \left\{3c(\tau - \tau^*) - \frac{1}{2}\|d\omega\|^2\right\} dM \\
&= \int_M \left\{3c\|\omega\|^2 - \frac{1}{2}\|d\omega\|^2\right\} dM \leqq 0 .
\end{aligned}
$$

This shows that

(i) $\qquad c \neq 0 \qquad \text{and} \qquad \omega = 0 ,$

or

(ii) $\qquad c = 0 \qquad \text{and} \qquad d\omega = 0 .$

Since $\omega = 0$ implies that M is Kählerian, it is sufficient to consider the cace (ii). In this case, taking account of $\tau + 3\tau^* = 0$, the second line of (4.4) reduces to

$$
\begin{aligned}
(4.5) \qquad 0 &= \int_M \left\{\frac{1}{4}(\tau - \tau^*)^2 - \frac{1}{2}\tau^* \|\omega\|^2\right\} dM \\
&= \frac{1}{8}\int_M \|T\|^2 dM .
\end{aligned}
$$

Hence, we get $T = 0$. Then, by (3.7), (3.12) and (4.5), we have

$$
\begin{aligned}
0 &= \int_M \{\|\omega\|^2 \delta\omega + \|\omega\|^4 - 2F\} dM \\
&= \int_M \{-2\tau^* \|\omega\|^2 - 2\tau\delta\omega + (\tau - \tau^*)^2\} dM \\
&= -2\int_M \tau\delta\omega \, dM .
\end{aligned}
$$

Since $\tau + 3\tau^* = 0$, we get

$$\int_M \tau\delta\omega \, dM = 0 \qquad \text{and} \qquad \int_M \tau^* \delta\omega \, dM = 0 .$$

Then

$$
\begin{aligned}
\int_M (\tau - \tau^*)^2 dM &= \int_M (\tau - \tau^*)(2\delta\omega + \|\omega\|^2) dM \\
&= \int_M (\tau - \tau^*)\|\omega\|^2 dM \\
&= -4\int_M \tau^* \|\omega\|^2 dM \\
&= -2\int_M (\tau - \tau^*)^2 dM ,
\end{aligned}
$$

from which we have

$$\tau = \tau^* = 0 \qquad \text{and} \qquad 2\delta\omega + \|\omega\|^2 = 0 .$$

Consequently, we have

$$\int_M \|\omega\|^2 \, dM = 0 ,$$

which implies $\omega = 0$ on M. This completes the proof of Theorem A.

Next, we shall prove Theorem B. First of all, we show that $\chi(M) \geqq 0$. Suppose that $\chi(M) < 0$. Then, reviewing the proof of Theorem A, we see that $\tau = 3\tau^*$, $d\omega = 0$ and $c_1(M)^2 = 2\chi(M) < 0$. Thus, we have $\tau^* = 4c > 0$ by virtue of (3.6), and hence

$$\int_M \tau^* \|\omega\|^2 \, dM = 4c \int_M \|\omega\|^2 \, dM \geqq 0 .$$

From this and (3.24), we have $c_1(M)^2 \geqq 0$. But this is a contradiction.

Since $\chi(M) \geqq 0$, by (3.17), we get

$$\int_M \tau^* \|\omega\|^2 \, dM \geqq \int_M \{\tfrac{1}{8}(\tau - \tau^*)^2 - 24c^2\} \, dM . \tag{4.6}$$

From (3.6), (3.24) and (4.6), we get

$$\begin{aligned} 32\pi^2 c_1(M)^2 &\geqq \int_M \{(\tau^*)^2 + \tfrac{1}{8}(\tau - \tau^*)^2 - 24c^2 + \|d\omega\|^2\} \, dM \\ &= \int_M \left\{\frac{1}{12}(\tau - 3\tau^*)^2 + \|d\omega\|^2\right\} dM \geqq 0 . \end{aligned} \tag{4.7}$$

Thus, we have

$$c_1(M)^2 \geqq 0 .$$

Suppose that $c_1(M)^2 = 0$. Then, by (4.7), we have $\tau = 3\tau^*$, $d\omega = 0$ and hence $\tau^* = 4c > 0$ by (3.6). Thus, by (3.24), we have

$$\begin{aligned} 0 = c_1(M)^2 &= \frac{1}{32\pi^2} \int_M \{(\tau^*)^2 + \tau^* \|\omega\|^2\} \, dM \\ &= \frac{1}{32\pi^2} \int_M \{16c^2 + 4c\|\omega\|^2\} \, dM > 0 . \end{aligned}$$

But, this is a contradiction. Thus, we have finally

$$c_1(M)^2 > 0 .$$

Therefore, again, by Theorem 2.2, we have

$$3\chi(M) \geqq c_1(M)^2 > 0 .$$

The rest of Theorem B follows immediately from Theorem 3.4 and the well-known classification of compact complex surfaces ([5, 3]).

References

1. Balas, A.: Compact Hermitian manifolds of constant holomorphic sectional curvature. Math. Z. **189**, 193–210 (1985)
2. Balas, A., Gauduchon, P.: Any Hermitian metric of constant non-positive (Hermitian) holomorphic sectional curvature on a compact complex surface is Kähler. Math. Z. **190**, 39–43 (1985)
3. Bombieri, E., Husemoller, D.: Classification and embeddings of surfaces. Proc. Symp. Pure Math. **29**, 329–420 (1975)
4. Gray, A., Vanhecke, L.: Almost Hermitian manifolds with constant holomorphic sectional curvature. Cas. Pestování Mat. **104**, 170–179 (1979)
5. Kodaira, K.: On the structure of compact analytic surfaces I. Am. J. Math. **86**, 751–798 (1964)
6. Miyaoka, Y.: On the Chern numbers of surfaces of general type. Invent. Math. **42**, 225–237 (1977)
7. Sato, T.: On some almost Hermitian manifolds with constant holomorphic sectional curvature. Kyungpook Math. J. **29**, 11–25 (1989)
8. Sekigawa, K.: On some 4-dimensional compact almost Hermitian manifolds. J. Ramanujan Math. Soc. **2**, 101–116 (1987)
9. Tricerri, F., Vaisman, I.: On some 2-dimensional Hermitian manifolds. Math. Z. **192**, 205–216 (1986)
10. Vaisman, I.: Some curvature properties of complex surfaces. Ann. Mat. Pura Appl. **32**, 1–18 (1982)
11. Wu, W.T.: Sur les classes caractéristiques des structures fibrées sphériques. Actual. Sci. Ind. **1183**, 1–89 (1952)