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Amenability, unimodularity, and the spectral radius of random walks on infinite graphs

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1 Introduction

In this paper, we consider a vertex-transitive graph $G=(X, E)$, which is locally finite, infinite and connected. We exclude multiple edges in E, while loops are permitted. In particular, all vertices have the same finite degree (number of neighbours), denoted by D. The vertex set X carries an integer-valued metric: $d(x, y)$ is the shortest length (number of edges) of a path in G which connects x and y. An *automorphism* of G is a self-isometry of X with respect to this metric. By $\mathrm{AUT}(G)$ we denote the full group of automorphisms of G. This is a locally compact Hausdorff group with the topology of pointwise convergence. A neighbourhood base of the identity is given by the family of all pointwise stabilizers in $\mathrm{AUT}(G)$ of some finite set of vertices. These subgroups are open compact, and $\mathrm{AUT}(G)$ is totally disconnected. For more details concerning the topology of $\mathrm{AUT}(G)$, see e.g. Trofimov [Tr]; for integration on locally compact groups, the reader is referred to the treatise by Hewitt and Ross [H-R].

Throughout this paper, Γ will be a closed subgroup of $\mathrm{AUT}(G)$ which acts transitively on X. We want to relate properties of Γ with the norm (spectral radius) $\|\mathscr{P}\|$ on $\ell^2(X)$ of the "simple random walk" operator $\mathscr{P}$ of G. The latter is given by

$$\mathscr{P} f(x)=\frac{1}{D} \sum_{y \sim x} f(y), \tag{1.1}$$

where f is an arbitrary complex-valued function on X and $\sim$ denotes the neighbourhood relation in G. Note that $\|\mathscr{P}\| \leqq 1$ always.

If Γ is a discrete group and G is its Cayley graph with respect to some symmetric set of generators, then it is well known that Γ is amenable if and only if $\|\mathscr{P}\|=1$, see Kesten [K2] and Day [Da]. We prove that in general, $\|\mathscr{P}\|=1$ if and only if Γ is amenable and unimodular (Sect. 2, Theor. 1). Indeed, we exhibit a *right* convolution operator $\mathscr{R}_\Phi$, acting on $L^2(\Gamma)$ with respect to *left* Haar measure, such that $\|\mathscr{R}_\Phi\|=\|\mathscr{P}\|$. Calculating $\|\mathscr{R}_\Phi\|$ in terms of a left convolution operator involves the modular function of Γ. We can then use the caracterization by Berg and Christensen [B-C] of amenable locally compact groups in terms of norms of convolution operators to obtain our result.

Using Trofimov's [Tr] combinatorial formula for the modular function and the result of Woess [W2] on amenable group actions on infinite graphs, we then prove that $\|\mathscr{P}\|<1$ if G is vertex-transitive and has more than two ends (Sect. 3, Theor. 2).

The spectral radius of $\mathscr{P}$ is related with the isoperimetric number of G, see Dodziuk [Do], Biggs et al. [B-M-S] and Gerl [Ge]. Thus, our results bear some analogy with those of Brooks [Br] relating amenability of the fundamental group of a closed manifold with the spectrum of the Laplacian of the universal cover.

Furthermore, we obtain estimates for the norms of more general transition operators linked with the graphs structure (Sect. 4, Theor. 3 and Coroll. 3), and we show that on a graph with $\|\mathscr{P}\|<1$, *every* irreducible, Γ-invariant random walk is transient (Sect. 4, Coroll. 4). Finally, we give a number of examples (Sect. 4), where our results allow explicit calculation of the norms.

2 Amenability and unimodularity

Recall that Γ is called *amenable*, if there is a non-negative set function μ, defined on the family of all Borel sets of Γ, such that (a) $\mu(\Gamma)=1$, (b) μ is finitely additive and (c) μ is invariant under left action of Γ. For all details concerning amenable locally compact groups, cf. Pier [Pi].

The stabilizer in Γ of a vertex x will be denoted by Γ_x. We now fix, once and for all, a "root" vertex $o\in X$. The homogeneous space Γ/Γ_o is discrete and can be identified with X by the mapping $\gamma\Gamma_o\mapsto\gamma o$ $(\gamma\in\Gamma)$. In general, $\mathscr{P}$ cannot be described as a left convolution operator of Γ on Γ/Γ_o. However, Γ_o is compact open, and we can fix a left Haar measure $d\gamma$ on Γ such that $\int_{\Gamma_o} d\gamma=1$. The Hilbert space $L^2(\Gamma)$ is considered with respect to this measure.

Now consider the *modular function* Δ of Γ. Recall that Δ is a continuous homomorphism of Γ into the multiplicative group of positive real numbers, determined by

$$\int_\Gamma F(\alpha)\,d\alpha=\Delta(\gamma)\int_\Gamma F(\alpha\gamma)\,d\alpha$$

where F varies in the space of compactly supported continuous functions on Γ. Implicit in [Tr], there is a combinatorial formula for Δ. For the convenience of the reader, we state it with a proof.

Lemma 1. *If* $\gamma\in\Gamma$ *and* $\gamma o=x$ *then* $\Delta(\gamma)=t(x)$, *where*

$$t(x)=\frac{|\Gamma_o x|}{|\Gamma_x o|}.$$

Proof. First observe that $\Gamma_x=\gamma\Gamma_o\gamma^{-1}$ so that $\Delta(\gamma)\int_{\Gamma_x} d\alpha=\int_{\Gamma_o} d\alpha(=1)$. Now, $(\Gamma_x)_o=\Gamma_x\cap\Gamma_o$, and $|\Gamma_x o|$ is equal to the number of cosets in $\Gamma_x/(\Gamma_x\cap\Gamma_o)$. Each of these cosets is compact open in Γ. In the same way, $|\Gamma_o x|=|\Gamma_o/(\Gamma_o\cap\Gamma_x)|$, and we obtain

$$|\Gamma_o x|\int\limits_{\Gamma_o\cap\Gamma_x} d\alpha=\int\limits_{\Gamma_o} d\alpha=\Delta(\gamma)\int\limits_{\Gamma_x} d\alpha=\Delta(\gamma)\,|\Gamma_x o|\int\limits_{\Gamma_o\cap\Gamma_x} d\alpha. \quad \square$$

By transitivity, for every neighbour u of o in X we can fix a $\gamma_u\in\Gamma$, such that $\gamma_u o=u$. We now define a function Φ on Γ by

$$\Phi=\frac{1}{D}\sum_{u\sim o}\chi_{\Gamma_o\gamma_u^{-1}}, \tag{2.1}$$

where χ_A denotes the indicator function of the set A. Then Φ is continuous with compact support $S=\bigcup\limits_{u\sim o}\Gamma_o\gamma_u^{-1}$. With Φ, we associate the right convolution operator $\mathscr{R}_\Phi$ on $L^2(\Gamma)$ defined by

$$\mathscr{R}_\Phi F(\gamma)=F*\Phi(\gamma)=\int\limits_{\Gamma} F(\gamma\alpha)\,\Phi(\alpha^{-1})\,d\alpha, \qquad F\in L^2(\Gamma). \tag{2.2}$$

Lemma 2. *The support S of Φ is a symmetric set which generates Γ as a semigroup.*

Proof. First of all, note that $S=\{\gamma\in\Gamma\,|\,\gamma o\sim o\}$. In particular, $S=S^{-1}$.

Now, we prove by induction on $n=d(x,o)\geqq 1$, that for every $x\in X\setminus\{o\}$ there is a $\gamma_x\in S^n(=S^{-n})$, such that $\gamma_x o=x$.

1) For $n=1$, we have $\gamma_u\in S$ for every $u\sim o$.

2) $n\to n+1$. If $d(o,x)=n+1$ then $d(o,y)=n$ for some $y\sim x$. By assumption, $\gamma_y o=y$ for some $\gamma_y\in S^n$. But γ_y is a bijection from the neighbours of o onto the neighbours of y. In particular, $\gamma_y u=x$ for some $u\sim o$. We obtain $\gamma_x=\gamma_y\gamma_u\in S^{n+1}$ and $\gamma_x o=x$.

Next, we show that $\Gamma_o\subset S^2$. If $u\sim o$ then, as above, $\gamma_u v=o$ for some $v\sim o$. Thus, $\gamma_u\gamma_v\in\Gamma_o$, and $\Gamma_o=\Gamma_o\gamma_v^{-1}\gamma_u^{-1}\subset S^2$.

Finally, if $\gamma\in\Gamma\setminus\Gamma_o$, then $\gamma^{-1}o=x\neq o$, and $\gamma\in\Gamma_o\gamma_x^{-1}\subset\bigcup\limits_{n\geqq 1}S^n$. $\square$

Next, we define the "averaging" operator $\mathscr{S}\colon L^2(\Gamma)\to\ell^2(X)$ by

$$\mathscr{S}F(x)=\int\limits_{\Gamma_o} F(\gamma_x\alpha)\,d\alpha, \qquad F\in L^2(\Gamma), \tag{2.3}$$

where $\gamma_x\in\Gamma$ is such that $\gamma_x o=x$, for all $x\in X$.

Lemma 3. *$\mathscr{S}F(x)$ does not depend on the particular choice of γ_x, and $\|\mathscr{S}\|\leqq 1$.*

Proof. If $\gamma'_x\in\Gamma$, $\gamma'_x o=x$, then $\gamma_x^{-1}\gamma'_x\in\Gamma_o$, and by left invariance of the Haar measure we have

$$\int\limits_{\Gamma_o} F(\gamma'_x\alpha)\,d\alpha=\int\limits_{\Gamma_o} F(\gamma_x\gamma_x^{-1}\gamma'_x\alpha)\,d\alpha=\int\limits_{\Gamma_o} F(\gamma_x\alpha)\,d\alpha.$$

Furthermore, as $\int\limits_{\Gamma_o} d\alpha=1$,

$$\begin{aligned}\|\mathscr{S}F\|_2^2&\leqq\sum_{x\in X}\Big(\int\limits_{\Gamma_o}|F(\gamma_x\alpha)|\,d\alpha\Big)^2\leqq\sum_{x\in X}\int\limits_{\Gamma_o}|F(\gamma_x\alpha)|^2\,d\alpha\\&=\sum_{x\in X}\int\limits_{\gamma_x\Gamma_o}|F(\alpha)|^2\,d\alpha=\|F\|_2^2,\end{aligned}$$

so that $\|\mathscr{S}\|\leqq 1$. $\square$

The operator $\mathcal{S}$ has a "quasi-inverse":

(2.4) $$\mathcal{T} f(\gamma)=f(\gamma o),\ \gamma\in\Gamma,\quad f\in\ell^2(X).$$

If we denote

(2.5) $$L_o^2(\Gamma)=\{F\in L^2(\Gamma)\,|\,F \text{ is constant on each } \gamma\Gamma_o,\ \gamma\in\Gamma\},$$

then $\mathcal{T}$ maps $\ell^2(X)$ onto $L_o^2(\Gamma)$. The following statements are obvious.

Lemma 4. (a) $\|\mathcal{T}\|=1$.
(b) $\mathcal{S}\mathcal{T}f=f$ *for every* $f\in\ell^2(X)$.
(c) $\mathcal{T}\mathcal{S}F=F$ *for every* $F\in L_o^2(\Gamma)$.

We now can relate the operators $\mathcal{R}_\Phi$ and $\mathcal{P}$.

Proposition 1. (a) *For every* $f\in\ell^2(X)$, $\mathcal{R}_\Phi\mathcal{T}f\in L_o^2(\Gamma)$,
(b) $\mathcal{S}\mathcal{R}_\Phi\mathcal{T}=\mathcal{P}$, *and*
(c) $\mathcal{R}_\Phi\mathcal{T}\mathcal{S}=\mathcal{R}_\Phi$.

Proof. Let $\gamma\in\Gamma$, $f\in\ell^2(X)$. Then

$$\mathcal{R}_\Phi\mathcal{T}f(\gamma)=\int_\Gamma \mathcal{T}f(\gamma\alpha)\,\Phi(\alpha^{-1})\,d\alpha=\frac{1}{D}\sum_{u\sim o}\int_{\gamma_u\Gamma_o}\mathcal{T}f(\gamma\alpha)\,d\alpha$$
$$=\frac{1}{D}\sum_{u\sim o}\int_{\gamma_u\Gamma_o} f(\gamma u)\,d\alpha=\frac{1}{D}\sum_{u\sim o} f(\gamma u).$$

As above, γ maps the neighbours of o bijectively to the neighbours of $x=\gamma o$. Thus, the last term is $\mathcal{P}f(\gamma o)$, and

$$\mathcal{R}_\Phi\mathcal{T}f(\gamma)=\mathcal{P}f(\gamma o).$$

This proves (a). Furthermore, if $\gamma_x o=x$ then

$$\mathcal{S}\mathcal{R}_\Phi\mathcal{T}f(x)=\int_{\Gamma_o}\mathcal{P}f(\gamma_x\alpha o)\,d\alpha=\mathcal{P}f(x),$$

and we have obtained (b). We prove (c). First observe that $\Delta(\alpha)=1$ for $\alpha\in\Gamma_o$. We calculate $\mathcal{T}\mathcal{S}F(\gamma)=\int_{\Gamma_o}F(\gamma\alpha)\,d\alpha$ and

$$\mathcal{R}_\Phi\mathcal{T}\mathcal{S}F(\gamma)=\int_\Gamma\int_{\Gamma_o}F(\gamma\beta\alpha)\,\Phi(\beta^{-1})\,d\alpha\,d\beta=\int_{\Gamma_o}\int_\Gamma F(\gamma\beta)\,\Phi(\alpha\beta^{-1})\,\Delta(\alpha)\,d\beta\,d\alpha$$
$$=\int_{\Gamma_o}\int_\Gamma F(\gamma\beta)\,\Phi(\alpha\beta^{-1})\,d\beta\,d\alpha.$$

By definition, $\Phi(\alpha\beta^{-1})=\Phi(\beta^{-1})$ for every $\beta\in\Gamma$, $\alpha\in\Gamma_o$, and the last term is equal to

$$\int_{\Gamma_o}\int_\Gamma F(\gamma\beta)\,\Phi(\beta^{-1})\,d\beta\,d\alpha=\int_{\Gamma_o}\mathcal{R}_\Phi F(\gamma)\,d\alpha=\mathcal{R}_\Phi F(\gamma).\quad\square$$

We can now prove the main result of this section.

Theorem 1. (a) $\|\mathscr{P}\| = \|\mathscr{R}_{\Phi}\|$,

(b) $\|\mathscr{P}\| \leqq \frac{1}{D} \sum_{u \sim o} \frac{1}{\sqrt{t(u)}}$, *and equality holds if and only if* Γ *is amenable,*

(c) $\|\mathscr{P}\| = 1$ *if and only if* Γ *is both amenable and unimodular.*

Proof. (a) From Prop. 1(b), Lemma 3 and Lemma 4(a) we get

$$\|\mathscr{P}\| \leqq \|\mathscr{S}\| \cdot \|\mathscr{R}_{\Phi}\| \cdot \|\mathscr{T}\| \leqq \|\mathscr{R}_{\Phi}\|.$$

On the other hand, by Lemma 4(c) and Prop. 1(a) we have $\mathscr{T}\mathscr{S}\mathscr{R}_{\Phi}\mathscr{T} = \mathscr{R}_{\Phi}\mathscr{T}$. Hence, using Prop. 1(b), (c) we see that

$$\mathscr{T}\mathscr{P}\mathscr{S} = \mathscr{T}\mathscr{S}\mathscr{R}_{\Phi}\mathscr{T}\mathscr{S} = \mathscr{R}_{\Phi}\mathscr{T}\mathscr{S} = \mathscr{R}_{\Phi}.$$

Hence, $\|\mathscr{R}_{\Phi}\| \leqq \|\mathscr{P}\|$, and we have equality.

(b) Let

$$\tilde{\Phi}(\gamma) = \Phi(\gamma^{-1})\sqrt{\Delta(\gamma^{-1})}. \tag{2.6}$$

Routine calculations (as for instance in [H-R, §20]) show that $\|\mathscr{R}_{\Phi}\| = \|\mathscr{L}_{\tilde{\Phi}}\|$, where $\mathscr{L}_{\tilde{\Phi}}$ is the left convolution operator $F \mapsto \tilde{\Phi} * F$ on $L^2(\Gamma)$. We calculate

$$\int_{\Gamma} \tilde{\Phi}(\gamma)\, d\gamma = \frac{1}{D} \sum_{u \sim o} \int_{\gamma_u \Gamma_o} \frac{1}{\sqrt{\Delta(\gamma)}}\, d\gamma = \frac{1}{D} \sum_{u \sim o} \frac{1}{\sqrt{t(u)}}, \tag{2.7}$$

as $\Delta(\gamma) = t(u)$ on $\gamma_u \Gamma_o$ by Lemma 1.

Let ι denote the unit element of Γ (the identity mapping on X). First suppose that $\iota \in \operatorname{supp}(\tilde{\Phi}) = S = S^{-1}$. This is the case if and only if G has a loop at each vertex. Then, by Lemma 2, the hypotheses of [B-C, Theor. 1] are satisfied, and Γ is amenable if and only if

$$\|\mathscr{L}_{\tilde{\Phi}}\| = \int_{\Gamma} \tilde{\Phi}(\gamma)\, d\gamma.$$

Thus, (a) and (2.7) yield the required formula in the presence of loops. Otherwise, we may add a loop at each vertex to obtain a new graph G' with the same group of automorphisms. Its simple random walk operator is

$$\mathscr{P}' = \frac{1}{D+1}\mathscr{I} + \frac{D}{D+1}\mathscr{P},$$

where $\mathscr{I}$ is the identity operator. As $\mathscr{P}$ is self-adjoint, we obtain

$$\|\mathscr{P}'\| = \frac{1}{D+1} + \frac{D}{D+1}\|\mathscr{P}\|. \tag{2.8}$$

According to (2.1) and (2.6), we get associated convolution operators $\mathscr{R}_{\Phi'}$ and $\mathscr{L}_{\tilde{\Phi}'}$ with the same norm as $\mathscr{P}'$. Now, $t(o) = 1$ by Lemma 1, so that

$$\int_{\Gamma} \tilde{\Phi}'(\gamma)\, d\gamma = \frac{1}{D+1}\left(1 + \sum_{u \sim o} \frac{1}{\sqrt{t(u)}}\right). \tag{2.9}$$

Again by [B-C, Theor. 1], Γ is amenable if and only if $\|\mathscr{L}_{\tilde{\Phi}'}\| = \int_\Gamma \tilde{\Phi}'(\gamma)\,d\gamma$. Equating (2.8) and (2.9), we see that (b) is also true in the absence of loops.

(c) In general, it is known (see [H-R, 20.14]) that $\|\mathscr{L}_{\tilde{\Phi}}\| \leqq \int_\Gamma \tilde{\Phi}(\gamma)\,d\gamma$.

Claim 1. $\int_\Gamma \tilde{\Phi}(\gamma)\,d\gamma \leqq 1$, and equality holds if and only if Γ is unimodular.

Proof of Claim 1. Consider $S = \operatorname{supp}(\Phi)$. By Lemma 1, we have

$$D = \int_{S^{-1}} d\gamma = \int_S d\gamma = \sum_{u\sim o} \int_\Gamma \Delta(\gamma_u^{-1})\,\chi_{\gamma_u\Gamma_o}(\gamma)\,d\gamma = \sum_{u\sim o} \frac{1}{t(u)}. \tag{2.10}$$

On the other hand, from the Cauchy-Schwarz inequality and (2.10) we get

$$\int_\Gamma \tilde{\Phi}(\gamma)\,d\gamma = \frac{1}{D} \sum_{u\sim o} \frac{1}{\sqrt{t(u)}} \leqq \frac{1}{\sqrt{D}} \left(\sum_{u\sim o} \frac{1}{t(u)} \right)^{1/2} = 1, \tag{2.11}$$

and equality holds if and only if $(t(u))_{u\sim o}$ is a constant vector: $t(u) = C > 0$ for every $u \sim o$. From Lemma 2 we know that $\gamma_u \gamma_v \in \Gamma_o$ for some $u, v \sim o$. Hence, $1 = \Delta(\gamma_u \gamma_v) = t(u)\,t(v) = C^2$, and $C = 1$. Again by Lemma 2, Γ must be unimodular ($\Delta \equiv 1$) in this case. Combining these considerations with (2.11), we see that Claim 1 is true.

Now (a) and Claim 1 yield (c). □

3 Amenability and ends

In this Section, we combine Theorem 1 with the results of [W2] concerning the action of an amenable group of automorphisms on the space of *ends* of G. We give a brief description of the latter, see Freudenthal [Fr], Halin [H1] and Jung [J1].

An *infinite path* is a one-sided infinite sequence of successively adjacent vertices without repetitions. Two such paths are *equivalent* if, for every finite $U \subset X$, they can be connected by some finite path in $G \setminus U$. An *end* is an equivalence class of infinite paths; the space of ends is denoted by Ω. It is known that an infinite, connected, vertex-transitive graph has one, two or infinitely many ends [H2], [Fr]. If $U \subset X$ is finite and B is the set of vertices of a component of $G \setminus U$, then we add to B all ends which have a representative infinite path with all vertices in B. Thus, we obtain a set $C \subset (X \setminus U) \cup \Omega$. If $z \in (X \setminus U) \cup \Omega$, then it lies in exactly one such set C, denoted by $C(U, z)$ (the component of z after removing U). If z' is another element of $(X \setminus U) \cup \Omega$ and $C(U, z) \neq C(U, z')$, then we say that U separates z and z'. Varying U (finite) and z, the sets $C(U, z)$ constitute a basis of a topology which makes $\bar{G} = X \cup \Omega$ a compact Hausdorff space.

For every end ω, one can find a sequence $\{U_n\}$ of finite subsets of X, such that

$$C(U_{n-1}, \omega) \supset U_n \cup C(U_n, \omega) \qquad \text{for every } n. \tag{3.1}$$

In this case, $\{C(U_n, \omega)\}$ is a neighbourhood base at ω, and $\{U_n\}$ is called *contracting* towards ω. In particular, with respect to the reference vertex o we can define

(3.2) $$B_n = \{x \in X \mid d(x, o) \leqq n\}, \qquad S_n = B_n \setminus B_{n-1},$$

and

$$D_n(\omega) = C(B_{n-1}, \omega) \cap S_n, \qquad n \geqq 1.$$

Then $\{D_n(\omega)\}$ has property (3.1). In the sequel, regardless of the topology of $\bar{G}$, by a connected subset of $\bar{G}$ we mean a set whose intersection with X induces a connected subgraph of G. The proof of the following Lemma is essentially due to Jung [J2]. Recall that $\Gamma \leqq \mathrm{AUT}(G)$ is always assumed to act transitively on X.

Lemma 5. *If $U \subset X$ is finite and disconnects $\bar{G}$ into at least two infinite connected components C_i, $i = 0, \ldots, k$ $(k \geqq 1)$, then for every i there is a $\sigma = \sigma_i \in \Gamma$ such that $\sigma(U \cup C_i) \subset C_i$.*

Proof. Choose a finite connected set $V \subset U$. By transitivity, we can find $\alpha \in \Gamma$ such that $C_i \supset \alpha V \supset \alpha U$ and $\alpha V \cap V = \varnothing$. As V is connected, there is some i' such that $\alpha C_{i'} \supset V \supset U$.

Case 1. If $i' \neq i$, then we must have $\alpha(U \cup C_i) \subset C_i$, and we may choose $\sigma = \alpha$.

Case 2. If $i' = i$, then $C_j \subset \alpha(U \cup C_i)$ for some $j \neq i$. As above, we can find $\beta \in \Gamma$, such that $\beta V \subset C_j \setminus V$, and $\beta C_{j'} \supset V \cup C_i \supset U \cup C_i$ for some j'. If $j' = i$ then we may choose $\sigma = \beta^{-1}$. If $j' \neq i$ then $\beta(U \cup C_i) \subset C_j \subset \alpha C_i$, and we may choose $\sigma = \alpha^{-1} \beta$. $\square$

An automorphism as given by Lemma 5 is called a *shift* [W2]; $\{\sigma^n U\}$ and $\{\sigma^{-n} U\}$ are contracting towards two ends ω_0, ω_1, respectively, which are the unique ends fixed by σ (see [H2]).

Proposition 2. *Let $\Gamma \leqq \mathrm{AUT}(G)$ be closed and act vertex-transitively. If G has more than two ($\equiv$ infinitely many) ends, then Γ is amenable if and only if it fixes a unique end.*

Proof. If Γ fixes the end ω, then by Lemma 5 there must be a shift $\sigma \in \Gamma$ with respect to some finite $U \subset X$, such that $\{\sigma^n U\}$ is contracting towards ω, compare with [W2]. Thus, in the terminology of [W2], ω has finite diameter bounded by $\mathrm{diam}(U)$. Hence, Γ is amenable by [W2, Theor. 2 and Coroll. 2].

Conversely, if Γ is amenable, then by [W2, Theor. 1 and Coroll. 1] it fixes a finite set of vertices, an end or a pair of ends. As Γ acts transitively on X, the first case is impossible. We show that it is also impossible that Γ fixes a pair of ends. As G has more than two ends, we can find a finite $U \subset X$, such that $\bar{G} \setminus U$ has infinite components C_i, $i = 0, \ldots, k$, with $k \geqq 2$. By Lemma 5, $\sigma(U \cup C_0) \subset C_0$ for some shift $\sigma \in \Gamma$, and $\{\sigma^n U\}$ is contracting towards some end $\omega_0 \in C_0$ which is fixed by σ. On the other hand, we must have $U \cup C_i \subset \sigma C_i$ for some $i \neq 0$. Without loss of generality, we assume $i = 1$, so that σ^{-1} "shifts into" C_1 and fixes an end $\omega_1 \in C_1$. In the same way, there is a shift $\tau \in \Gamma$ which fixes an $\omega_2 \in C_2$ and some other end in $\Omega \setminus C_2$. From [H2, Theor. 9] we know that a shift cannot fix more than two ends. In particular, σ does not fix ω_2 and τ does not fix at least one of ω_0, ω_1. Thus Γ cannot fix a set of two ends. $\square$

Before we prove the main result of this Section, we need two more technical preliminaries. The first one is probably well known, but we could not find it stated explicitly.

Lemma 6. *Let G be vertex-transitive, $r \geqq 0$. If $\bar{G} \setminus B_r$ has more than two infinite components and C_0 is one of them, then*

$$\lim_{n \to \infty} |S_n \cap C_0| = \infty.$$

Proof. Let C_i, $i \in I = \{0, \ldots, k\}$ $(k \geqq 2)$ be the infinite components of $\bar{G} \setminus B_r$. For every $i \in I$, we choose a $\gamma_i \in \mathrm{AUT}(G)$ such that $\gamma_i o \in C_i$, $d(o, \gamma_i o) = 2r+1$. Then, by connectedness, there must be $i' \in I$ such that $\gamma_i^{-1} B_r \subset C_{i'}$. Hence

$$C_i \supset \gamma_i (B_r \cup \bigcup_{j \neq i'} C_j).$$

Thus, for $n \geqq 3r+2$, $C_i \cap S_n$ has at least k elements. Now, associating as above j' with j for every $j \in I$,

$$\gamma_i C_j \supset \gamma_i \gamma_j (B_r \cup \bigcup_{\ell \neq j'} C_\ell),$$

and for $n \geqq 5r+3$, $C_i \cap S_n$ has at least k^2 elements. Repeating this process, we see that for $n \geqq r+1+m(2r+1)$, $C_i \cap S_n$ has at least k^m elements. $\square$

Lemma 7. *Let G have more than one end. If Γ acts transitively on X and fixes an end ω, then there is an increasing sequence (n_m) of positive integers, such that*

$$\operatorname{diam}(D_{n_m}(\omega)) \leqq M < \infty.$$

Proof. Let r be such that B_r disconnects G into at least two infinite components, and let $C_0 = C(B_r, \omega)$. By Lemma 5 we can find a shift $\sigma \in \Gamma$, such that $\{\sigma^m B_r\}_{m \geqq 1}$ is contracting towards ω. In particular, $d(o, \sigma^n o)$ is increasing. We set

$$n_m = d(o, \sigma^m o) + r + 1, \qquad m \geqq 1.$$

Then $B_{n_m - 1} \supset \sigma^m B_r$, so that $C(B_{n_m - 1}, \omega) \subset C(\sigma^m B_r, \omega)$. As σ fixes ω, $C(\sigma^m B_r, \omega) = \sigma^m C_0$. In particular, $\sigma^m B_r$ separates o from $D_{n_m}(\omega)$. In other words, if $x \in D_{n_m}(\omega)$, then there is $y \in \sigma^m B_r$ on some shortest path from o to x, and

$$n_m = d(o, x) = d(o, y) + d(y, x) \geqq d(o, \sigma^m o) - r + d(y, x).$$

Hence,

$$d(x, \sigma^m o) \leqq d(x, y) + r \leqq n_m - d(o, \sigma^m o) + 2r = 3r+1,$$

and $\operatorname{diam}(D_{n_m}(\omega)) \leqq M = 6r+2$. $\square$

Indeed, Lemma 7 applies to every end in an arbitrary (locally finite) graph which is "hyperbolic" in the terminology of [W2].

Theorem 2. *If G is a vertex-transitive graph with more than two ($\equiv$ infinitely many) ends, then $\|\mathscr{P}\| < 1$.*

Proof. Let $\Gamma = \mathrm{AUT}(G)$. We show that if Γ is amenable, then it must be unimodular.

Indeed, suppose that Γ is amenable. Then it fixes a unique end ω by Prop. 2. We fix $r \geqq 1$ as in Lemma 6 and write $C_0 = C(B_r, \omega)$. For every $y \in X$, we fix $\gamma_y \in \Gamma$ with $\gamma_y o = y$.

Claim 2. If $n \geqq 1$ and $y \in S_n \setminus C_0$, then $d(o, \gamma_y v) \leqq 2r$ for some $v \in D_n(\omega)$.

Proof of Claim 2. If $n \leqq r$ then this is obvious. So assume $n > r$.

We first show that $\gamma_y D_n(\omega) \cap (B_r \cup C_0) \neq \varnothing$. Observe that $D_n(\omega)$ separates o from ω, so that $\gamma_y D_n(\omega)$ separates y from $\gamma_y \omega = \omega$. Assume that the above intersection is void. Then y and o can be connected in $\bar{G} \setminus \gamma_y D_n(\omega)$ by a path of length n. On the other hand, $o \in B_r \cup C_0$ and $\omega \in C_0$, so that y and ω are in the same component of $\bar{G} \setminus \gamma_y D_n(\omega)$, a contradiction.

Now, if $\gamma_y D_n(\omega) \cap B_r \neq \varnothing$, then Claim 2 is true. Otherwise, $\gamma_y v \in C_0$ for some $v \in D_n(\omega)$. However, B_r separates y from $\gamma_y v$, so that there is some $w \in B_r$ on a shortest path between the two, and

$$n = d(o, v) = d(y, \gamma_y v) = d(y, w) + d(w, \gamma_y v).$$

But $d(y, w) \geqq n - r$, and thus $d(w, \gamma_y v) \leqq r$. This proves Claim 2.

Now set $R = 2r + M$, M as in Lemma 7, and $K = |B_M|$. Choose $n = n_m > r$ according to Lemma 7, large enough such that

$$|S_n \setminus C_0| > K^2,$$

which is possible by Lemma 6.

If $y \in S_n \setminus C_0$, then by Claim 2,

$$\gamma_y^{-1} o \in A = \{x \in S_n \mid d(x, D_n(\omega)) \leqq 2r\}.$$

As $\operatorname{diam}(D_n(\omega)) \leqq M$, there are at most K elements in A, and there must be an $x \in A$ such that

$$B = \{y \in S_n \setminus C_0 \mid \gamma_y^{-1} o = x\}$$

has at least $K+1$ elements. If $y, w \in B$, then $\gamma_w \gamma_y^{-1} y = w$ and $\gamma_w \gamma_y^{-1} o = o$. Thus

$$|\Gamma_o y| \geqq K + 1.$$

On the other hand, $\Gamma_y = \gamma_y \Gamma_o \gamma_y^{-1}$, and $|\Gamma_y o| = |\Gamma_o x|$. Now, every $\alpha \in \Gamma_o$ permutes the infinite components of $\bar{G} \setminus B_{n-1}$ and fixes ω, so that $\alpha D_n(\omega) = D_n(\omega)$. Thus $\alpha x \in A$ if $x \in A$ and $\alpha \in \Gamma_o$. We get that

$$|\Gamma_y o| = |\Gamma_o x| \leqq K.$$

By Lemma 1, Γ is nonunimodular. □

4 Applications, extensions and examples

As in the preceding sections, Γ is a closed group of automorphisms of G which acts transitively on the vertex set X of G. We recall the definition of the *isoperi-*

metric number $i(G)$ of G, see [Do, B-M-S and Ge]. If $U \subset X$, then the *boundary* ∂U is the set of all vertices in U which have a neighbour in $X \setminus U$, and

$$(4.1) \qquad i(G) = \inf\left\{\frac{|\partial U|}{|U|} \,\middle|\, U \subset X, \text{ finite}\right\}.$$

In [B-M-S] and [Ge] it is shown that $i(G)=0$ if and only if $\|\mathscr{P}\|=1$.

Corollary 1. $i(G)=0$ *if and only if* Γ *is both amenable and unimodular.*

Another consequence of Theorem 1 is the following.

Corollary 2. *If some vertex-transitive closed group of automorphisms of G is amenable and unimodular, then this is true for every such group.*

The examples below show that Corollary 2 does not remain true if of the assumptions we drop either amenability or unimodularity. Note that $\mathscr{I}-\mathscr{P}$ may be considered as the discrete Laplacian of G. In view of this observation, compare Corollaries 1 and 2 with the results of [Br] concerning Riemannian manifolds.

Instead of the "simple random walk" operator $\mathscr{P}$ defined in (1.1), we may consider more generally a transition operator $\mathscr{Q}$, given by a matrix $(q(x,y))_{x,y\in X}$ with the following properties:

(i) *stochasticity:* $q(x,y) \geqq 0$ and $\sum_y q(x,y)=1$ for every $x \in X$,

(ii) *Γ-invariance:* $q(\gamma x, \gamma y)=q(x,y)$ for every $x, y \in X$, $\gamma \in \Gamma$,

(iii) *symmetry:* $q(x,y)=q(y,x)$ for every $x, y \in X$, and

(iv) *irreducibility:* If $x \sim y$ then $q^{(n)}(x,y)>0$ for some $n>0$.

Here, $q^{(n)}(x,y)$ is the (x,y)-entry of the nth matrix power of $\mathscr{Q}$. The action of $\mathscr{Q}$ is given by

$$(4.2) \qquad \mathscr{Q}f(x)=\sum_y q(x,y)f(y), \qquad f \in \ell^2(G).$$

The following result is proved along the same lines as Theorem 1.

Theorem 3. *Let* $\Gamma \leqq \mathrm{AUT}(G)$ *be closed and vertex-transitive, and let* $\mathscr{Q}$ *satisfy properties* (i) *through* (iv). *Then*

$$\|\mathscr{Q}\| \leqq \sum_u \frac{q(o,u)}{\sqrt{t(u)}} \leqq 1.$$

The first inequality is an equality if and only if Γ *is amenable, and the second one is an equality if and only if* Γ *is unimodular.*

Proof. For every u with $q(o,u)>0$, fix $\gamma_u \in \Gamma$ such that $\gamma_u o = u$. As in (2.1), associate with $\mathscr{Q}$ a function Φ on Γ by

$$(4.3) \qquad \Phi = \sum_u q(o,u)\, \chi_{\Gamma_o \gamma_u^{-1}}.$$

Then, using properties (i)–(iv), one verifies that $\Phi(\gamma^{-1})=\Phi(\gamma)$ for every $\gamma \in \Gamma$, that $\mathrm{supp}(\Phi)$ generates Γ as a semigroup, and that Lemma 4 and Prop. 1 remain valid with $\mathscr{Q}$ in the place of $\mathscr{P}$. As in (2.7) we associate $\tilde{\Phi}$ with Φ. In this setting, [B-C, Theor. 1] still applies; in the case when $q(x,x)=0$ for some ($\equiv$all) x, we may consider $\mathcal{Q}'=a\mathcal{I}+(1-a)\mathcal{Q}$, $0<a<1$, and use the same argument as before. Identity (2.10) has to be replaced by

$$1=\int_{\Gamma}\Phi(\gamma^{-1})\,d\gamma=\int_{\Gamma}\Phi(\gamma)\,d\gamma=\sum_{u}\frac{q(o,u)}{t(u)}. \tag{4.4}$$

Finally,

$$\int_{\Gamma}\tilde{\Phi}(\gamma)\,d\gamma=\sum_{u}\frac{q(o,u)}{\sqrt{t(u)}}\leqq\Big(\sum_{u}q(o,u)\Big)^{1/2}\left(\sum_{u}\frac{q(o,u)}{t(u)}\right)^{1/2}=1$$

by (4.4), and equality holds if and only if Γ is unimodular. □

If $\mathcal{Q}$ has only properties (i) and (ii), then still something can be said.

Corollary 3. *If Γ is amenable and $\mathcal{Q}$ is stochastic and Γ-invariant, then*

$$\|\mathcal{Q}\|=\sum_{u}\frac{q(o,u)}{\sqrt{t(u)}}.$$

Proof. If Φ is associated with $\mathcal{Q}$ as in (4.3), then we still have $\|\mathcal{Q}\|=\|\mathcal{R}_{\Phi}\|$. If Γ is amenable then we can use a result of Leptin [Le] (see also [Pi, Coroll. 9.7]) to infer that

$$\|\mathcal{R}_{\Phi}\|=\int_{\Gamma}\tilde{\Phi}(\gamma)\,d\gamma=\sum_{u}\frac{q(o,u)}{\sqrt{t(u)}}.\quad\square$$

Indeed, for Corollary 3 it suffices in (i) to have $q(x,y)\geqq 0$ and $\sum_{u}q(x,y)=c>0$ for every $x\in X$. Furthermore, with obvious modifications the results also apply to the norm of $\mathcal{Q}$ as an operator on $\ell^{p}(X)$, $1<p<\infty$.

Corollary 4. *If $i(G)>0$, then every transition operator $\mathcal{Q}$ with properties* (i), (ii) *and* (iv) *gives rise to a transient random walk.*

Proof. We have to show that $\sum_{n}q^{(n)}(o,o)$ is finite. If Φ is defined as in (4.3) then Prop. 1 may be applied with $\mathcal{Q}$ in the place of $\mathcal{P}$. Now, if $F,G\in L_0^2(\Gamma)$, then it is easy to see that $\langle\mathcal{S}F,\mathcal{S}G\rangle=\langle F,G\rangle$, where the inner product is taken in $\ell^2(X)$ and $L^2(\Gamma)$, respectively. Hence, if $\Phi^{(n)}$ denotes the nth convolution power of Φ, then one calculates

$$\begin{aligned}q^{(n)}(x,y)&=\langle\mathcal{Q}^{n}\delta_y,\delta_x\rangle=\langle\mathcal{R}_{\Phi}^{n}\mathcal{T}\delta_y,\mathcal{T}\delta_x\rangle\\&=\int_{\Gamma_o}\int_{\Gamma_o}\Phi^{(n)}(\alpha^{-1}\gamma_y^{-1}\gamma_x\beta)\,d\alpha\,d\beta.\end{aligned}\tag{4.5}$$

If we define $\mu(d\gamma)=\Phi(\gamma^{-1})\,d\gamma$, then μ is a probability measure on Γ. By (iv), the support of μ generates Γ as a semigroup, and if $\mu^{(n)}$ denotes its nth convolution power, then (4.5) yields $q^{(n)}(o,o)=\mu^{(n)}(\Gamma_o)$.

By Corollary 1, Γ is nonamenable or nonunimodular. Thus, Γ is a transient group, see Guivarc'h, Keane and Roynette [G-K-R, Theor. 47 and Theor. 51]. In particular, as Γ_o is compact, $\sum_{n}\mu^{(n)}(\Gamma_o)$ is finite. □

We now turn our attention to graphs with infinitely many ends. Note that Corollary 2 applies in this case, by Theorem 2. If G is the Cayley graph of a finitely generated group Γ, then by definition the ends of Γ are the ends of G. (This is independent of the particular choice of the – locally finite – Cayley graph, see [Fr] and Stallings [St].) A discrete group is of course unimodular, and it is amusing to point out that Theorems 1 and 2 may be used to obtain the following corollary.

Corollary 5. *Every finitely generated group with infinitely many ends is nonamenable.*

This is of course well known. Indeed, it is enough to detect a free subgroup on two generators in Γ; compare with [St] or with the graph-theoretical approach to Stallings' structure theorem by Dunwoody [Du].

It will be of interest to describe in detail those graphs G with infinitely many ends which admit an amenable vertex-transitive group $\Gamma \leqq \mathrm{AUT}(G)$. Note that such a group cannot be discrete and must have very big (uncountable) vertex stabilizers, compare with [H2] and [T1]. This could also be interpreted as having a rich "radial" structure. Our feeling is that such a graph should be a "layer" of finitely many *distance-regular* graphs (see Ivanov [Iv]), compare with the examples given below. In particular, G should be tree-like in the sense of Woess [W1].

Example 1. The homogeneous tree.

We consider $G = T = T_r$, the homogeneous tree of degree $D = r+1$, $r \geqq 2$. For many details concerning the geometry of T, see Cartier [C1]. On one hand, T admits the vertex-transitive action of a discrete group (the free product of $r+1$ copies of the two-element group), which is unimodular and nonamenable.

On the other hand, fix some end ω of T, and consider the group $\Gamma = \Gamma(\omega)$ of all automorphisms of T which fix ω. This group has been studied in detail bey Nebbia [Ne]; it is amenable and nonunimodular. The end ω induces a partial ordering in the vertex set X of T: if $x, y \in X$ then we write $x \geqq y$ if y lies on the unique infinite path which represents ω and starts at x. Observe that $\gamma \in \mathrm{AUT}(T)$ is in Γ if and only if it preserves this ordering. Furthermore, for every $x \in X$ and $n \in \mathbb{N}_0$, there is a unique $w \in X$ such that $x > w$ and $d(x, w) = n$; we write $w = x - n$. Finally, the ordering is directed: every pair $x, y \in X$ has an infimum $c(x, y)$ (the *confluent* of x and y with respect to ω). The *horocycle* of x with respect to ω is the set

$$H(x, \omega) = \{y \in X \mid d(x, c) = d(y, c),\ c = c(x, y)\}.$$

With respect to the "root" vertex o, the horocycles are given by

$$H_m(\omega) = \{x \in X \mid d(x, c) - d(o, c) = m,\ c = c(o, x)\}, \quad m \in \mathbb{Z}. \tag{4.6}$$

The geometry of the horocycles is of interest in the study of radial harmonic analysis on T, see e.g. Cartier [C2] and Betori, Faraut and Pagliacci [B-F-P].

Lemma 8. *If Δ is the modular function of $\Gamma = \Gamma(\omega)$, $\gamma \in \Gamma$ and $\gamma o \in H_m(\omega)$, then $\Delta(\gamma) = r^m$.*

Proof. If x, $y \in X$ and $y = x - n$, $n \in \mathbb{N}_0$, then $\Gamma_y x = \{x' \mid y = x' - n\}$, which has r^n elements, while $\Gamma_x y = \{y\}$. This shows that $t(x) = r^m$ if $x \in H_m(\omega)$ and $x > o$ or $o > x$. Now let $x \in H_m(\omega)$ be incomparable with o. Set $c = c(x, o)$, $n' = d(x, c)$ and $n'' = d(o, c)$. Then n', $n'' > 0$, $c = x - n' = o - n''$ and $\Gamma_o c = \Gamma_x c = \{c\}$. Hence $\Gamma_o x = \{x' \in X \mid c = x' - n' = c(o, x')\}$, which has $(r-1) r^{n'-1}$ elements. In the same way, $|\Gamma_x o| = (r-1) r^{n''-1}$. Thus, $t(x) = r^{n'-n''} = r^m$. □

As an application, we can easily calculate the norm of the simple random walk operator: there are r neighbours of o in $H_1(\omega)$, one is in $H_{-1}(\omega)$, and by Theorem 1,

$$\|\mathscr{P}\| = \frac{1}{r+1}\left(r \cdot \frac{1}{\sqrt{r}} + 1 \cdot \sqrt{r}\right) = \frac{2\sqrt{r}}{r+1}.$$

This is of course well known and can be obtained in a much more elementary way, see e.g. Kesten [K1].

Observe that the only transition operators $\mathscr{Q}$ which satisfy (i), (ii) and (iii) with respect to $\Gamma = \Gamma(\omega)$ are the *radial* ones: $q(x, y)$ depends only on $d(x, y)$. The norms of these operators have been computed by Sawyer [Sa] and Cohen [Co]; for their harmonic analysis, see Figà-Talamanca and Picardello [F-P]. It is enough to know the norms of the operators $\mathscr{Q}_n$, $n \geq 1$, defined by

(4.7) $$q_n(x, y) = \begin{cases} 1/|S_n|, & d(x, y) = n, \\ 0, & \text{otherwise}, \end{cases}$$

where S_n is as in (3.2). From Corollary 3 and Lemma 8 we obtain

(4.8) $$\|\mathscr{Q}_n\| = \frac{1}{|S_n|} \sum_{m=-n}^{n} |H_m(\omega) \cap S_n| \, r^{-m/2}.$$

The simple combinatorial task to count the points in $H_m(\omega) \cap S_n$ is left to the reader. Other operators whose norms can be calculated in application of Corollary 3 are for example $\mathscr{H} f(x) = f(x-1)$ and its left inverse $\mathscr{H}' f(x) = \frac{1}{r} \sum_{y-1=x} f(y)$: they have norms $\sqrt{r}$ and $1/\sqrt{r}$, respectively. □

Let G be a graph with infinitely many ends, which admits a vertex-transitive, closed group Γ that fixes an end ω. Let t be the function on X which corresponds to the modular function of Δ according to Lemma 1. Then Lemma 8 suggests to define the horocycles with respect to ω as those sets of vertices x, where $t(x)$ is constant.

In Example 1, it is not the *whole* automorphism group of the tree which is amenable. Based on a suggestion of W. Imrich and the example in [Tr], we now exhibit an example of an infinite, vertex-transitive graph with infinitely many ends, such that $\mathrm{AUT}(G)$ is amenable.

Example 2. Let $T = (X, E)$ be the homogeneous tree as above, ω an end of T and $\Gamma = \Gamma(\omega)$ its stabilizer in $\mathrm{AUT}(T)$. Consider the partial ordering induced

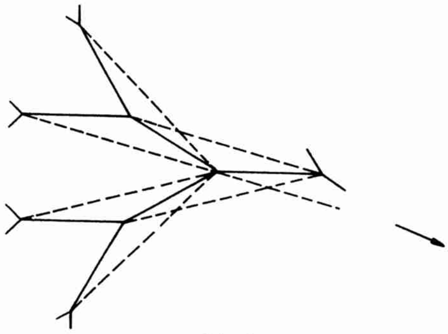

Fig. 1

by ω. We define a new graph $G=(X, \bar{E})$ with the same vertex set and edge set $\bar{E}=E \cup E'$, where

$$E'=\{[x-2, x] \mid x \in X\}. \tag{4.9}$$

Clearly, G has the same space of ends as T. Call *black* edges the edges in E and *red* edges those in E', see Fig. 1.

Lemma 9. $\mathrm{AUT}(G)=\Gamma$.

Proof. Clearly, $\Gamma \leqq \mathrm{AUT}(G)$ (as it preserves the ordering).

Let $\gamma \in \mathrm{AUT}(G)$. A red edge is member of exactly one triangle in G, while a black edge is a side of $r+1$ triangles. Hence, γ must send red edges to red edges and black edges to black edges. In particular, $\gamma \in \mathrm{AUT}(T)$. Consider an edge $[x-1, x]$ in E. Suppose that γ reverses its order with respect to ω: $\gamma(x-1)=y, \gamma x=y-1$. Let $A=\{v \in X \mid v-1=x\}$. Then $\gamma A=\{w \sim y-1 \mid w \neq y\}$, and there must be $v \in A$ such that $\gamma v=w \neq y-2$, i.e. $w-1=y-1$. But then $x-1=v-2$, and the image of the red edge $[x-1, v]$ is $[y, w]$. As neither $y=w-2$ nor $w=y-2$, we have a contradiction. Thus, γ preserves the ordering of X with respect to ω, and is contained in Γ. $\square$

The graph G has several interesting features. First of all, the red edges span two disjoint homogeneous trees with degree $1+r^2$, so that G is a layer of three homogeneous trees. Secondly, $\mathrm{AUT}(G)$ is nonunimodular, and no discrete group can act vertex-transitively on G. In particular, G is not a Cayley graph. $\square$

In general, it is an interesting question, how far from a Cayley graph an arbitrary vertex-transitive graph can be. Trofimov [T1] has shown that such a graph is very close to the Cayley graph of a nilpotent group, if it has polynomial growth: in this case, G admits an imprimitivity system with finite blocks with respect to $\mathrm{AUT}(G)$, such that the image of $\mathrm{AUT}(G)$ on the factor graph is nilpotent-by-finite and has finite vertex stabilizers on the latter. The graph of Example 2 shows that in general, one cannot find a vertex-transitive group $\Gamma \leqq \mathrm{AUT}(G)$ and a corresponding imprimitivity system with finite blocks, such that the image of Γ on the factor graph is finitely generated and has finite vertex-stabilizers. Hence, we conclude with a problem.

Question. Is there an infinite, connected, locally finite graph, which is vertex-transitive but not quasi-isometric (in the sense of Gromov [Gr]) with a Cayley graph of some finitely generated group?

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