Proof. Again we will let α and γ have cycle structure (13^3), β have cycle structure $(1^2 2^4)$, and prove that $\alpha\beta=\gamma$ implies the above for α, β, and γ^{-1}. We may assume as usual that $\alpha=(0)(123)(456)(789)$.

Assume that 0 is fixed by β, so that 0 is the fixed point of γ. Another must also be fixed by β, and we can assume that it is 1; by (2.7), (23) cannot be part of β, so we may assume that (24) is. Then γ sends 1 to 2 to 5, and so sends 5 to 1; hence β must send 5 to 3, so that (35) is part of β. Then γ sends 4 to 3 to 6, and so 6 to 4, forcing β to fix 6, a contradiction. Hence 0 is not fixed by β, and we may assume 1 is.

If the other fixed point of β is 2, then (123) is a part of γ, so 3 must be fixed by β; if the other fixed point of β is 3, then (312) is part of γ, and 2 must be fixed by β. Either way we have a contradiction, so we may assume that the other fixed point of β is 4.

Where is 0 sent to by β? If (02) is in β, then (120) is in γ, so (03) is in γ; if (03) is in β, then (3012 ...) is part of γ. Similar contradictions occur if (05) or (06) are in β. Hence we may assume (07) is in β.

Where is 2 sent by β? If (23) is in β, then (12) is in γ. If (26) is in β, then γ contains (1245 ...). If (28) is in β, then (129) is in γ, so (39) is in β, forcing (0837) in γ. If (29) is in β, then (12708 ...) is in γ. This leaves only (25) as a possibility.

So far $\beta=(1)(4)(07)(25)(\ldots)(\ldots)$, and so γ is now determined: $\gamma=(126)(453)(708)(9)$. This forces the rest of β to be (36) (89), giving the result above. □

This finally serves to rule out case (2.8.4), and #93. The four permutations in this case must be [(0) (123) (456) (789)], [(14) (25) (36) (07) (89)], [(14)], and [(162) (354) (078) (9)], up to conjugacy, by Lemma (2.11). In this case $\{0, 7, 8, 9\}$ is left stable by each of these, so the subgroup generated by them is not transitive.

3. The sufficiency of the existence of the J-map

The last several cases considered in the previous section were ruled out essentially because the J-map from the base curve $\mathbb{P}^1$ to the moduli space $\mathbb{P}^1$ could not exist. In this section I will indicate that a converse to the arguments used above exists: if one can construct an appropriate J-map, then a rational elliptic surface with the prescribed singular fibers exists.

What does an "appropriate" J-map mean? One can take a hint from the 'm' column of Table (1.1). Suppose a list of singular fibers is given, and the task is to construct a rational elliptic surface with exactly those singular fibers. Assume that the list of fibers satisfies the various numerical criteria of Sect. 1. Let $d=\text{degree}(J)$, which is computed using (1.7). Let us say that a map $J:\mathbb{P}^1\to\mathbb{P}^1$ *belongs to* the list of singular fibers if the multiplicities over 0, 1, and ∞ are as follows:

over 0: $(ii+iv^*)$ points of multiplicity 1,
$(iv+ii^*)$ points of multiplicity 2, and
$(d-ii-iv^*-2iv-2ii^*)/3$ points of multiplicity 3;
over 1: $(iii+iii^*)$ points of multiplicity 1, and
$(d-iii-iii^*)/2$ points of multiplicity 2;
over ∞: $(i_n+i_n^*)$ points of multiplicity $n (n\geqq 1)$.

The construction of the surface proceeds in two steps. First, we pull back via an appropriate J-map one of the surfaces with J = the identity. Second, we "twist" extraneous fibers away. Let us describe these steps in turn.

The rational elliptic surface with Weierstrass equation

(3.1) $$y^2 = x^3 - 3t(t-1)^3 x + 2t(t-1)^5$$

has $J=t$, and has exactly three singular fibers: a fiber of type II over $t=0$, a fiber of type III^* over $t=1$, and a fiber of type I_1 over $t=\infty$. Upon a base change from this surface, one will have singular fibers over the points going to 0, 1, and ∞, and the types of the singular fibers are determined merely by the multiplicity of the base change map at these points. In particular, we have the following (see [MP, Table (7.1)):

(3.2) over a point above 0:

if $m=1$: II; if $m=2$: IV; if $m=3$: I_0^*

over a point above 1:

if $m=1$: III^*; if $m=2$: I_0^*

over a point above ∞:

I_m if multiplicity m.

After the pull-back, one makes a "twist" of the resulting surface to adjust the fibers of type IV^*, III^*, II^*, and I_n^*. There are really two processes going on here. One was described earlier in the discussion of # 56; it will be referred to as "transfer of *". Assume that the surface is given in Weierstrass form as $y^2=x^3+Ax+B$, where A and B are forms in s and t. Suppose that s^2 divides A and s^3 divides B, but either t^2 does not divide A or t^3 does not divide B. Then over $s=0$ there is a singular fiber of the surface, of type I_n^*, II^*, III^*, or IV^*, and over $t=0$ there is a fiber of type I_m, II, III, or IV. The "transfer of *" is effected by replacing A by t^2A/s^2 and B by t^3B/s^3. After making this replacement, over $s=0$ there is a singular fiber of type I_n, II, III, or IV, and over $t=0$ there is a fiber of type I_m^*, II^*, III^*, or IV^*; all other singular fibers remain the same, and the J-map of the surface is unaffected. One can be more precise: the fibers are switched according to the following schedule:

(3.3)
$$I_n \leftrightarrow I_n^* \quad (n \geqq 0)$$
$$II \leftrightarrow IV^*$$
$$III \leftrightarrow III^*$$
$$IV \leftrightarrow II^*$$

Note that "transfer of *" preserves the number of "*" fibers, and keeps the p_g of the surface invariant. The second process, which will be called "deflation of *'s", simultaneously "deflates" two "*" fibers as in (3.3), and so the number of these drops by two; the p_g drops by 1. Suppose that over $t=0$ and over $s=0$ we have "*" fibers; then in the Weierstrass equation, s^2t^2 divides A and s^3t^3 divides B. Replace A by A/s^2t^2 and B by B/s^3t^3; this deflates the fibers over $s=0$ and $t=0$ as in (3.3) (the fiber on the right is replaced by that on the left). All other singular fibers are unaffected, and the J-map remains the same.

The main result for constructing surfaces with prescribed singular fibers can now be stated.

(3.4) Proposition. *Suppose that a list of singular fibers for the rational elliptic surface is given, satisfying the numerical criteria of Sect.* 1, *with* $d = \text{degree}(J) \neq 0$. *Suppose further that a map* $J: \mathbb{P}^1 \to \mathbb{P}^1$ *exists, which belong to the list of singular fibers. Then a rational elliptic surface with section can be constructed with exactly the singular fibers of the list, by first pulling back the surface given by* (3.1) *via* J, *then by applying a suitable number of "deflation of *'s", and finally by applying at most one "transfer of *".*

Proof. Write
$$\begin{aligned} d &= \sum n(i_n + i_n^*) \\ &= ii + 2iv + iv^* + 2ii^* + 3a \\ &= iii + iii^* + 2b. \end{aligned}$$

Let Y be the pullback of the surface (3.1) via the J-map which belongs to the given list of singular fibers. By (3.2), we have the following singular fibers on Y:

over the points over $J=0$:
- $(ii + iv^*)$ fibers of type II
- $(iv + ii^*)$ fibers of type IV
- a fibers of type I_0^*;

over points over $J=1$:
- $(iii + iii^*)$ fibers of type III^*
- b fibers of type I_0^*;

over points over $J=\infty$:
- $(i_n + i_n^*)$ fibers of type $I_n (n \geqq 1)$.

The total number of "*" fibers here is $(a + b + iii + iii^*)$. Let e be the number of "*" fibers in the given list; $e = 0$ or 1. Then by (1.3) we have $2(ii + iv^*) + 3(iii + iii^*) + 4(iv + ii^*) + 6e + d = 12$. Therefore

$$\begin{aligned} &(a + b + iii + iii^*) \\ &\quad = ((12 - 2(ii + iv^*) - 3(iii + iii^*) - 4(iv + ii^*) - 6e) - ii - iv^* - 2iv - 2ii^*)/3 \\ &\qquad + ((12 - 2(ii + iv^*) - 3(iii + iii^*) - 4(iv + ii^*) - 6e) - iii - iii^*)/2 + iii + iii^* \\ &\quad = 10 - 5e - 2(ii + iv^*) - 2(iii + iii^*) - 4(iv + ii^*) \end{aligned}$$

which is even if $e = 0$ and odd if $e = 1$. Therefore, after a suitable number of "deflation of *'s", we can arrange exactly e "*" fibers. If $e = 0$ we are done. If $e = 1$, then after one "transfer of *" operation, we arrive at a surface with the prescribed singular fibers.

This surface has the correct fiber types to be our desired rational elliptic surface, and the only point left to check is that it is indeed rational, and not Enriques (which is the only other serious possibility. However our surface has a section, which an Enriques surface does not; this section is pulled back from the section of the surface given by (3.1), and both the "transfer of *" and "deflation of *" operations preserve the existence of this section. □

The above Proposition reduces the construction of an elliptic surface with prescribed singular fibers to the construction of an appropriate J-map. Suppose first that we have a list of singular fibers with $x=0$, so that if the J-map exists, it is ramified only over 0, 1, and ∞, and the ramification is determined. Then the existence of the J-map is equivalent to the existence of the three permutations σ_0, σ_1, and σ_∞ in S_d, with the appropriate cycle structure, generating a transitive subgroup of S_d, whose product is the identity. Therefore:

(3.5) Corollary. *Suppose a list of singular fibers is given with $d \geqq 1$ and $x=0$, satisfying the numerical criteria of Sect. 1. Then a rational elliptic surface with those singular fibers exist if and only if there are three permutations σ_0, σ_1, and σ_∞ in S_d satisfying the following conditions:*

(3.5.1) *the cycle structure of σ_0 is:*
$(ii+iv^*)$ 1-cycles
$(iv+ii^*)$ 2-cycles
$(d-ii-iv^*-2iv-2ii^*)/3$ 3-cycles;

(3.5.2) *the cycle structure of σ_1 is:*
$(iii+iii^*)$ 1-cycles
$(d-iii-iii^*)/2$ 2-cycles;

(3.5.3) *the cycle structure of σ_∞ is:*
$(i_n+i_n^*)$ n-cycles, for each $n \geqq 1$;

(3.5.4) *$\sigma_0\,\sigma_1\,\sigma_\infty =$ the identity in S_d;*

(3.5.5) *the subgroup of S_d generated by $\{\sigma_0\,\sigma_1, \sigma_\infty\}$ is transitive.*

It is by exhibiting the three permutations that the existence of the rational elliptic surfaces with $x=0$ will be demonstrated.

I claim that examples of rational elliptic surfaces with configurations of singular fibers with $x \geqq 1$ can be constructed by suitably deforming the J-map of a surface with $x=0$, and in fact all configurations which occur can be found this way. The basic observation is the identity

$$(1, 2, \ldots, n) = (1, 2, \ldots, k)(k+1, k+2, \ldots, n)(k, n).$$

Suppose a J-map exists which belongs to a configuration of singular fibers. This J-map may be ramified over 0, 1, and ∞, and possibly elsewhere. Therefore we have permutations σ_0, σ_1, σ_∞, etc., satisfying the conditions of (3.5) (generalized to more than three permutations). Suppose that an n-cycle occurs in one of the permutations. Then, by using the above identity, this n-cycle can be replaced by a product of a k-cycle and an $(n-k)$-cycle in the given permutation, at the cost of introducing an extra permutation which is a transposition. This new set of permutations can then be used to construct a J-map, which belongs to an altered list of singular fibers. Using this method, one sees immediately that certain singular fibers can be "deformed" into two singular fibers, leaving all others alone. The resulting fibers are given in the following table.

Table (3.6)

Fiber	Deforms into
I_n	$I_k + I_{n-k}$, $1 \leqq k \leqq n-1$
I_0^*	$II + IV$ if over $J=0$
I_0^*	$III + III$ if over $J=1$
IV	$II + II$