Werk

Titel: I. Discriminant considerations.

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Fibration	Realization	rk MW	Torsion	Reference
$I_2\ 2II\ 6I_1$	A(0;2,0,0)	7		
$I_2\ II\ 8I_1$	A(0;1,0,0)	7		
$I_2\ 10I_1$	A(0;0,0,0)	7		
$6II$		8		
$4II\ 4I_1$		8		
$3II\ 6I_1$		8		
$2II\ 8I_1$		8		
$II\ 10I_1$		8		
$12I_1$		8		

We note that the following list of fibrations are excluded, although they satisfy the criteria 2′) and 3)

$I_7\ III\ II$	$I_6\ 2II$	$I_5\ IV\ III$	$I_5\ I_3\ 2II$	$2IV\ I_3\ I_1$
$2IV\ III\ I_1$	$2IV\ I_2\ II$	$2IV\ II\ 2I_1$	$IV\ 2III\ II$	$IV\ III\ 2II\ I_1$
$IV\ I_2\ 3II$	$IV\ 3II\ 2I_1$	$3I_3\ 3I_1$	$I_3\ III\ I_2\ 2II$	$3III\ II\ I_1$
$2III\ 3II$	$I_2\ 5II$	$5II\ 2I_1$		

The set up

I. Discriminant considerations

Any Jacobian rational elliptic surface can be represented as the double covering of a quadric cone, branched along the vertex and a disjoint tri-section. Thus geometrically we have the data of a cubicsurface C in $\mathbb{P}_3$, a point V outside and a conic K. The point V and the conic K define a quadric cone, and its intersection with C defines the branchcurve (apart from the vertex). Note that C will not be uniquely defined.

By choosing coordinates we can assume $V = (0,0,0,1)$ and C given in Weierstraß form

$$Q(x, y, z) + wP(x, y, z) + w^3 = 0 \tag{1}$$

The discriminant D of the projection will hence be given by

$$27Q^2 + 4P^3 = 0 \tag{2}$$

This is a sextic and it will be singular, the singularities will be of two kinds.

a) (unavoidable) the locus of triple ramification given by $P = Q = 0$
b) (avoidable) the image of the singular points of C.

Proof. a) is clear, choose another point, then necessarily both P, $Q \neq 0$. Choose the point of ramification above, its w-coordinate will satisy

$$P + 3w^2 = 0\,, \tag{3}$$

as it also satisfies (1) we get

$$Q - 2w^3 = 0 \tag{4}$$

forming the gradient of (2) we obtain

$$54Q\nabla Q + 12P^2\nabla P = 0 \tag{5}$$

Using (3) and (4) this can be rewritten as

$$\nabla Q + w\nabla P = 0 \tag{6}$$

(but (3) and (6) just express that the point is singular on the hypersurface (1)).

In general the singularities will be of the unavoidable type and consist of six cusps lying at the complete (transversal) intersection of the conic $P = 0$ and the cubic $Q = 0$, with cuspidal tangents given by the corresponding tangents of the cubic.

Given a conic K in the plane ($w = 0$) a point of the intersection of C with the quadric cone KV can lie above a singular point of the discriminant D contained in K.

We could then in principle describe all rational elliptic surfaces by the intersections of a conic with a discriminant sextic, or as we do not lose any information by restricting to the conic K, by the discriminant forms of a binary quartic and a binary sextic. (Recall that the degrees will double when restricted to a conic). In practice it is not feasible as the curve D encodes more information than is given geometrically by the conic P and the cubic Q.

However over the total ramification it is posible to give an exclusively geometric dictionary. In fact;

Lemma 1.1. *Let p be a point on K above which the projection totally ramifies, (i.e. a point such that the line pV meets C at one point triply). We have*

Intersection of P, Q and K at p	*Kodaira fiber*
K and Q transversal	*II*
K tangent to Q of order two, P K transversal	*III*
K tangent to Q and P of order two	*IV*

Proof. This is straightforward to prove in view of Table 2. In fact by choosing local coordinates we can parametrise the conic K by (t, t^2), the cone over the conic by (t, t^2, s) and thus the intersection of the quadric cone by the cubic will be given by

$$s^3 + P(t, t^2)s + Q(t, t^2) = 0 \tag{7}$$

where both P and Q vanish at $x = y = 0$. Now the lowest degree of terms in (7) is two exactly when K is tangent to Q at the origin of order 2 which exactly expresses the fact that K is tangent to Q. Now the quadratic term of (7), will be a square. iff P can be written under the form

$$P(x, y) = y(a + bx + cy) + dx^2 \tag{8}$$

which says that P is tangent to K as well.

It is now easy to use Bezout's theorem to conclude the following lemma (bearing in mind that P cannot be a factor of Q).