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Euclidean Submanifolds with Nonparallel Normal Space

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1. Introduction

Let $f\colon M^n \to \mathbb{R}^N$ be an isometric immersion of an n-dimensional connected Riemannian manifold into N-dimensional Euclidean space. At each point $x \in M$, we define the *first normal space* $N_1(x)$ to be the subspace of the normal space $T_x M^\perp$ spanned by the vector valued second fundamental form $\alpha\colon T_x M \times T_x M \to T_x M^\perp$. It is a standard fact that if the subspaces N_1 form a k-dimensional subbundle on an open and connected subset $U \subset M$ which is parallel in the connection induced on the normal bundle, then $f(U)$ has substantial codimension k i.e., $f(U)$ is contained in some affine $(n+k)$-dimensional subspace but not in one of lower dimension.

Isometric immersions satisfying $\dim N_1 \equiv 1$ have been considered by several authors. From a local point of view by Griffone-Morvan (see Theor. 2 and Prop. 2 of [G-M]) and from a global point of view by Rodríguez-Tribuzy [R-T] and Dajczer-Rodríguez [D-R]. Locally, it holds that N_1 is parallel unless there exists an open subset where the index of relative nullity satisfies $\nu \equiv n-1$. Recall that $\nu(x) = \dim \Delta(x)$, where

$$\Delta(x) = \{X \in T_x M : \alpha(X, Y) = 0 \ \forall Y \in T_x M\},$$

and that on any open subset where ν is constant, the leaves of the integrable distribution $x \mapsto \Delta(x)$ are part of affine totally geodesic ν-dimensional subspaces. In particular, $\nu \equiv n-1$ implies that the manifold is flat and ruled by totally geodesic hyperplanes.

In this paper we deal with the local theory of isometric immersions with nonparallel first normal spaces of higher dimension. As an application of our general results we obtain the following improvement of Theorem 4 of [G-M].

Theorem 1. *Let $f\colon M^n \to \mathbb{R}^N$ be an isometric immersion with* $\dim N_1 = 2$ *everywhere. Then M contains an open and dense subset M' such that*

$$M' = U_1 \cup U_2 \cup U_3 \quad \textit{with} \quad U_i \cap U_j = \phi \quad \text{if } i \neq j,$$

where U_1, U_2, U_3 are three open subsets satisfying:

1) *The connected components of* U_1 *have substantial codimension equal to* 2.

2) $f(U_2)$ *is foliated by totally geodesic* $(n-2)$*-dimensional affine subspaces.*

3) *For each connected component* U_λ *of* U_3 *there exists an open subset* V_λ *of* $\mathbb{R}^{n+1}$ *and isometric immersions* $g_\lambda\colon U_\lambda \to \mathbb{R}^{n+1}$, $h_\lambda\colon V_\lambda \to \mathbb{R}^N$, *both with* 1*-dimensional first normal spaces such that* $g_\lambda(U_\lambda) \subset V_\lambda$ *and* $f|_{U_\lambda} = h_\lambda \circ g_\lambda$.

Examples of submanifolds of type 2, i.e., submanifolds satisfying $\nu \equiv n-2$ and nonparallel N_1 can be obtained as follows.

Example 1. Let $f_\mu\colon N^n \to \mathbb{R}^{n+1}$ be a 1-parameter family of non-congruent isometric hypersurfaces with $\nu \equiv n-2$. Such deformable submanifolds have been classified by Sbrana [Sb] and Cartan [Ca]. For $\mu_1 \neq \mu_2$ the isometric immersion

$$F_{\mu_1,\mu_2} = \frac{1}{\sqrt{2}} f_{\mu_1} \oplus \frac{1}{\sqrt{2}} f_{\mu_2}\colon N \to \mathbb{R}^{n+1} \oplus \mathbb{R}^{n+1} \equiv \mathbb{R}^{2n+2}$$

has nonparallel 2-dimensional first normal space and $\nu \equiv n-2$.

Example 2. There are many ruled Euclidean submanifolds with $\nu = n-2$ and $\dim N_1 = 2$, whose substantial codimension can be arbitrary (see Sect. 2 of [D-G]). Moreover, it is shown in [D-G] that if M is complete and f is real analytic then either $M^n = M^3 \times \mathbb{R}^{n-3}$ and $f = f_1 \times \mathrm{id}$ splits with $f_1(M^3)$ unbounded or f is ruled by complete affine subspaces.

2. The Results

We say that an isometric immersion $f\colon M^n \to \mathbb{R}^N$ is *regular* if the first normal spaces form a normal subbundle which splits as the orthogonal sum of two subbundles

$$N_1 = T \oplus S,$$

where we define

$$S(x) = \mathrm{Span}\{\mathrm{Im}\, \phi_\eta\colon \forall \eta \in N_1^\perp\}.$$

Here $\phi_\eta\colon T_xM \to N_1(x)$ is the linear map defined by $\phi_\eta Z = (\nabla_Z^\perp \eta)_{N_1}$, where $\nabla^\perp$ denotes the normal connection and $(\ )_{N_1}$ indicates taking the N_1-component. Observe that $S = 0$ iff N_1 is parallel.

Let α_T and α_S denote the T and S components of the second fundamental form α, respectively. At each point $x \in M$, let $\Delta^s(x)$ be the kernel of α_S i.e.,

$$\Delta^s(x) = \{Y \in T_xM : \alpha_S(Y, Z) = 0\ \forall Z \in T_xM\},$$

and let $\nu^s(x) = \dim \Delta^s(x)$. Observe that $\Delta^s = \Delta$, and thus $\nu^s = \nu$, whenever $T = \{0\}$.

Let us denote by A_δ the tangent valued second fundamental form in the normal direction δ.

Proposition 2. *Let $f\colon M^n \to \mathbb{R}^N$ be a regular isometric immersion. For any $\eta \in N_1^\perp$ we have*

i) $\bigcap_{\delta \in \operatorname{Im}\phi_\eta} \operatorname{Ker} A_\delta = \operatorname{Ker}\phi_\eta$,

ii) $\langle \nabla^\perp_{\operatorname{Ker}\phi_\eta} \operatorname{Im}\phi_\eta, T\rangle = 0$.

Proof. i) Since $A_\eta = 0$, we get from the Codazzi equation that

$$A_{\phi_\eta X} Y = A_{\phi_\eta Y} X,$$

and the result follows easily.

ii) From the Ricci equation and $A_\eta = 0$, we have

$$R^\perp(Z, Y)\eta = \nabla_Z^\perp \nabla_Y^\perp \eta - \nabla_Y^\perp \nabla_Z^\perp \eta - \nabla^\perp_{[Z,Y]}\eta = 0 \tag{1}$$

for any $Y, Z \in TM$. Now let $Y \in \operatorname{Ker}\phi_\eta$, $\xi \in T$. Then

$$\langle \nabla^\perp_{[Z,Y]}\eta, \xi\rangle = \langle \phi_\eta [Z, Y], \xi\rangle = 0, \tag{2}$$

and

$$\langle \nabla_Z^\perp \nabla_Y^\perp \eta, \xi\rangle = Z\langle \phi_\eta Y, \xi\rangle - \langle \phi_\eta Y, \nabla_Z^\perp \xi\rangle = 0, \tag{3}$$

since $\nabla_Z^\perp \xi \in N_1$. Using (1), (2) and (3), we obtain that

$$\langle \nabla_Y^\perp \phi_\eta Z, \xi\rangle = \langle \nabla_Y^\perp \nabla_Z^\perp \eta, \xi\rangle = 0,$$

for all $Z \in TM$. This concludes the proof.

Remark. The local conclusion for the case dim $N_1 \equiv 1$ presented in the Introduction follows easily from part i) of the above Proposition.

Corollary 3. *For a regular isometric immersion $f\colon M^n \to \mathbb{R}^N$ it holds that*

i) $\Delta^s(x) = \bigcap_{\eta \in N_1^\perp} \operatorname{Ker}\phi_\eta$,

ii) *S is parallel in $\mathbb{R}^N$ along Δ^s,*

iii) *T is parallel in $TM^\perp$ along Δ^s.*

Proof. i) Trivial from part i) of the Proposition.

ii) First observe that from part i) we have that $N_1^\perp$, and thus N_1, is parallel in $TM^\perp$ along Δ^s. Therefore, for any $Y \in \Delta^s$, $\delta \in S$, we obtain from the Gauss formula and part ii) of the Proposition that

$$\tilde\nabla_Y \delta = -A_\delta Y + \nabla_Y^\perp \delta = \nabla_Y^\perp \delta \in N_1 \cap T^\perp = S,$$

where $\tilde\nabla$ denotes the standard connection in $\mathbb{R}^N$. This proves ii).

iii) By ii) this is equivalent to N_1 being parallel in $TM^\perp$ along Δ^s.

Theorem 4. *Let $f\colon M^n \to \mathbb{R}^N$ be a regular isometric immersion with* dim $T = t$ *and $v^s = k$, everywhere. Then Δ^s is integrable and each leaf of Δ^s is locally substantial in codimension less or equal than t.*

Proof. Let $Y, \bar Y \in \Delta^s$. For any $\eta \in N_1^\perp$ by the Ricci equation

$$\nabla_Y^\perp \nabla_{\bar Y}^\perp \eta - \nabla_{\bar Y}^\perp \nabla_Y^\perp \eta - \nabla^\perp_{[Y,\bar Y]}\eta = 0.$$

Since $N_1^\perp$ is parallel in the normal connection along Δ^s, it follows that $\nabla^\perp_{[Y,\bar{Y}]}\eta \in N_1^\perp$. Therefore $\phi_\eta[Y,\bar{Y}]=0$ for all $\eta\in N_1^\perp$ and thus $[Y,\bar{Y}]\in\Delta^s$ by part i) of Corollary 3. So Δ^s is integrable.

Along a fixed leaf L of Δ^s consider the orthogonal splitting $TM=TL\oplus TL^\perp$ and the subbundle $\tau=TL^\perp\oplus T$. Clearly τ contains the first normal space of L as a submanifold of $\mathbb{R}^N$. Furthermore, τ is parallel in the normal connection of L. In fact, for any $Y\in TL$, $X\in TL^\perp$ and $\xi\in T$, we have

$$\tilde{\nabla}_Y(X+\xi)=\nabla_Y X+\alpha(Y,X)-A_\xi Y+\nabla_Y^\perp\xi\in TM\oplus T,$$

since $\alpha_S(Y,X)=0$, and $\nabla_Y^\perp\xi\in T$ by part iii) of Corollary 3.

Now let $U\subset M$ be an open subset where the subspaces

$$R(x)=\operatorname{Span}\{(\tilde{\nabla}_Z\delta)_\tau : \forall\delta\in S, Z\in TM\}$$

verify that $\dim R(x)=$constant for all $x\in U$.

Claim. The distribution $x\in L\cap U\mapsto R(x)$ is normal to L and parallel in $\mathbb{R}^N$.

To prove the claim first observe that for any $Y\in TL$, $Z\in TM$ and $\delta\in S$, we have

$$\langle\tilde{\nabla}_Z\delta, Y\rangle=\langle -A_\delta Z+\nabla_Z^\perp\delta, Y\rangle=-\langle Z, A_\delta Y\rangle=0,$$

which implies that R is normal to L. It remains to show that $\tilde{\nabla}_Y(\tilde{\nabla}_Z\delta)_\tau\in R(x)$. We compute

$$\begin{aligned}\tilde{\nabla}_Y\tilde{\nabla}_Z\delta&=\tilde{\nabla}_Y(-A_\delta Z+\nabla_Z^\perp\delta)\\&=-\nabla_Y A_\delta Z-\alpha(Y,A_\delta Z)-A_{\nabla_Z^\perp\delta}Y+\nabla_Y^\perp\nabla_Z^\perp\delta.\end{aligned}\tag{4}$$

On the other hand, by Codazzi's equation we have

$$\nabla_Y A_\delta Z=A_\delta\nabla_Y Z+A_{\nabla_Y^\perp\delta}Z-A_\delta\nabla_Z Y-A_{\nabla_Z^\perp\delta}Y,\tag{5}$$

and from Ricci's equation, we get

$$\alpha(Y,A_\delta Z)=R^\perp(Y,Z)\delta=\nabla_Y^\perp\nabla_Z^\perp\delta-\nabla_Z^\perp\nabla_Y^\perp\delta-\nabla^\perp_{[Y,Z]}\delta.\tag{6}$$

Putting together (4), (5) and (6), we obtain

$$\begin{aligned}\tilde{\nabla}_Y\tilde{\nabla}_Z\delta&=-A_\delta[Y,Z]+\nabla^\perp_{[Y,Z]}\delta-A_{\nabla_Y^\perp\delta}Z+\nabla_Z^\perp\nabla_Y^\perp\delta\\&=\tilde{\nabla}_{[Y,Z]}\delta+\tilde{\nabla}_Z\nabla_Y^\perp\delta.\end{aligned}\tag{7}$$

But

$$\tilde{\nabla}_Y(\tilde{\nabla}_Z\delta)_\tau=\tilde{\nabla}_Y\tilde{\nabla}_Z\delta-\tilde{\nabla}_Y(\tilde{\nabla}_Z\delta-(\tilde{\nabla}_Z\delta)_\tau).$$

Since $\tilde{\nabla}_Z\delta-(\tilde{\nabla}_Z\delta)_\tau\in S\oplus N_1^\perp$, and $S\oplus N_1^\perp$, is parallel in $\mathbb{R}^N$ along TL, we have that $\tilde{\nabla}_Y(\tilde{\nabla}_Z\delta-(\tilde{\nabla}_Z\delta)_\tau)\in S\oplus N_1^\perp$. Using (7), and that $\tilde{\nabla}_Y(\tilde{\nabla}_z\delta)_\tau\in TM\oplus T$, we get

$$\begin{aligned}\tilde{\nabla}_Y(\tilde{\nabla}_Z\delta)_\tau&=(\tilde{\nabla}_Y\tilde{\nabla}_Z\delta)_{TM\oplus T}=(\tilde{\nabla}_{[Y,Z]}\delta+\tilde{\nabla}_z\nabla_Y^\perp\delta)_{TM\oplus T}\\&=(\tilde{\nabla}_{[Y,Z]}\delta)_\tau+(\tilde{\nabla}_z\nabla_Y^\perp\delta)_\tau,\end{aligned}$$

since $\nabla_Y^\perp\delta\in S$. This proves the claim.

Next we argue that $R(x)$ is orthogonal to the first normal space of L in $\mathbb{R}^N$. In fact, for any $Y, \bar{Y} \in TL$ we have using (7) that

$$\langle \tilde{\nabla}_{\bar{Y}} Y, \tilde{\nabla}_Z \delta \rangle = \bar{Y} \langle Y, \tilde{\nabla}_Z \delta \rangle - \langle Y, \tilde{\nabla}_{\bar{Y}} \tilde{\nabla}_Z \delta \rangle = 0.$$

Finally we show that $\dim R \geqq \dim TL^{\perp}$. To see this just observe that for $\delta \in S$, we have

$$(\tilde{\nabla}_Z \delta)_\tau = -A_\delta Z + (\nabla_Z^{\perp} \delta)_T,$$

and that

$$TL^{\perp} = \operatorname{Span}\{A_\delta Z : \forall \delta \in S, Z \in TM\}.$$

Given a point $x_0 \in L$ there exists a neighboorhod V of x_0 in L where $\dim R \geqq \dim R(x_0)$ and thus V reduces codimension by at least $\dim R(x_0) \geqq \dim TL^{\perp}$. So $V \subset L$ has substantial codimension $\leqq \dim \tau - \dim TL^{\perp}$, which concludes the proof.

Remark. For $T = \{0\}$ the above Theorem follows from the result on the relative nullity distribution described in the Introduction.

Corollary 5. *Let f: $M^n \to \mathbb{R}^N$ be a regular isometric immersion with* $\dim S = 1$ *and* $\dim T = t$. *Then there exists a foliation of M by hypersurfaces each of which has substantial codimension at most t.*

Proof. Using Proposition 2 it is easy to see that $\nu^s = n-1$ and $\dim R = 1$ everywhere.

Proposition 6. *Let f: $M^n \to \mathbb{R}^N$ be a regular isometric immersion with* $\dim S = 2$. *Then $\nu^s = n-2$.*

Proof. It is easy to see that there exists $\eta \in N_1^{\perp}$ such that $\operatorname{Im} \phi_\eta = S$, and the result follows from Part i) of Proposition 2.

Remark. It can be shown by similar arguments that if $\dim S = 3$, then $\nu^s \geqq n-3$.

Theorem 7. *Let f: $M^n \to \mathbb{R}^N$ be a regular isometric immersion of a simply connected Riemannian manifold with* $\dim S = 1$, *and* $\dim T = t$. *Then there exists an isometric immersion g: $M^n \to \mathbb{R}^{n+t}$ such that locally $f = h \circ g$, where h: $U \subset \mathbb{R}^{n+t} \to \mathbb{R}^N$ is an isometric immersion with 1-dimensional first normal space everywhere.*

Proof. We define a 1-parameter family of bilinear forms α^λ: $TM \times TM \to T_f^{\perp} M$, and a 1-parameter family of bundle connections $\nabla^{\perp\lambda}$ on $T_f^{\perp} M$ compatible with the metric for each $\lambda \in [0, 1]$ as follows

1) $\alpha^\lambda = \alpha_T \otimes \lambda \alpha_S$, (8)

2) i) $\nabla^{\perp\lambda}|_T = \nabla^{\perp}|_T$,

ii) $\nabla^{\perp\lambda} \delta = \lambda \nabla^{\perp} \delta$ for $\delta \in S$, (9)

iii) $\nabla^{\perp\lambda}|_{N_1^{\perp}} = \nabla^{\perp}|_{N_1^{\perp}}$.

It is a long but straightforward computation to verify that for each $\lambda \in [0, 1]$, the above Riemannian data on $T_f M^{\perp}$ verifies the Gauss, Codazzi and Ricci equations. This verification depends strongly on the fact that $\nu^s = n-1$. Since M is simply connected there exists a 1-parameter family of isometric immersions g_λ: $M^n \to \mathbb{R}^N$ varying smoothly with λ (see [W], p. 298) such that $g_1 = f$ and a 1-parameter family of bundle isomorphisms $\phi_\lambda: T_f M^\perp \to T_{g\lambda} M^\perp$ along g_λ preserving the metric, the second fundamental form α_λ and the bundle connection $\nabla^{\perp\lambda}$. Now we take $g=g_0$. From (8) we have that T is the first normal space of g, and from (9) that T is parallel in the normal connection. This implies that g has substantial codimension t.

Let c: $[a, b]\to M$ be a unit speed curve orthogonal to Δ^s. Notice that the distribution Δ^s is independent of $\lambda\in(0, 1]$. For each $\lambda\in(0, 1]$ we consider the $(n+1)$-dimensional ruled submanifold F_λ of $\mathbb{R}^N$ along $g_\lambda(c([a, b]))$ (maybe with singularities) where the ruling through the point $g_\lambda(c(s))$ is the $(n+t-1)$-dimensional affine subspace $(g_\lambda)_*(\Delta^s(c(s)))\oplus\phi_\lambda T(c(s))$, which by the proofs of Theorem 4 and Corollary 5, contains the leaf of Δ^s through $g_\lambda(c(s))$.

Let $X(s)=c'(s)$ and $S=\operatorname{Span}\{\delta\}$. Then

$$R^\lambda(c(s))=\operatorname{Span}\{(\tilde{\nabla}_X \phi_\lambda \delta)_{\tau^\lambda}\}.$$

But

$$\begin{aligned}(\tilde{\nabla}_X \phi_\lambda \delta)_{\tau^\lambda} &= -\langle A^\lambda_{\phi_\lambda\delta} X, X\rangle (g_\lambda)_* X + (\nabla^{\perp\lambda}_X \phi_\lambda \delta)\phi_\lambda T\\ &=\lambda(-\langle A_\delta X, X\rangle (g_\lambda)_* X+\phi_\lambda(\nabla^\perp_X\delta)_T).\end{aligned}$$

Therefore, for each $\lambda\in(0, 1]$, the one dimensional subbundle R^λ is independent of λ up to ϕ_λ. So the same holds for his orthogonal complement Z^λ in τ^λ. Let $Y_1(s), \ldots, Y_k(s)$ and $\gamma_1(s), \ldots, \gamma_{n+t-k-1}(s)$ be orthonormal bases for $T_{c(s)}L$ and $Z(c(s))=Z^1(c(s))$, respectively. We also consider the map ψ^λ: $\tau\to\tau^\lambda$ defined by

$$\psi^\lambda(u+\theta)=(g_\lambda)_* u+\phi_\lambda\theta,$$

for $u\in TL$, $\theta\in T$. Now we parametrize F_λ, $\lambda\in(0, 1]$ by the map X_λ: $[a, b]\times\mathbb{R}^k\times\mathbb{R}^{n+t-k-1}\to\mathbb{R}^N$ defined by

$$\begin{aligned}&X_\lambda(s, t_1, \ldots, t_k, \mu_1, \ldots, \mu_{n+t-k-1})\\ &\quad=g_\lambda(c(s))+(g_\lambda)_* \sum_{j=1}^{k} t_j Y_j(s)+\psi^\lambda \sum_{i=1}^{n+k+t-1} \mu_i\gamma_i(s).\end{aligned}$$

Claim. All maps X_λ induce the same metric.

It is easy to see that to prove the claim it is sufficient to show that $\dfrac{\partial X_\lambda}{\partial s}$ is independent of λ up to ϕ_λ. Let $Y=\sum t_j Y_j$, $\gamma=\sum\mu_i\gamma_i$. Then

$$\frac{\partial X_\lambda}{\partial s}=(g_\lambda)_* X+(g_\lambda)_* \nabla_X Y+\phi_\lambda\alpha^\lambda(X, Y)+\tilde{\nabla}_X\psi^\lambda\gamma.$$

But $\alpha^\lambda(X, Y)=\alpha^\lambda_T(X, Y)$ since $Y\in TL$. Also $\tilde{\nabla}_X\psi^\lambda\gamma\in TM\oplus\phi_\lambda T$ because for any $\xi\in S\oplus N_1^\perp$

$$\langle\tilde{\nabla}_X\psi^\lambda\gamma, \psi^\lambda\xi\rangle=\langle\psi^\lambda\gamma, \tilde{\nabla}_X\psi^\lambda\xi\rangle=0.$$

Let $\gamma=u+\theta$ with $u\in TL^\perp$ and $\theta\in T$. Then

$$\tilde{\nabla}_X\psi^\lambda\gamma=(g_\lambda)_*(\nabla_X u)+\phi_\lambda\alpha^\lambda(X, u)-(g_\lambda)_* A^\lambda_{\phi_\lambda\theta}X+\phi^\lambda\nabla^{\perp\lambda}_X\theta$$

Now $\cancel{\nabla}_x\psi^\lambda\gamma\in TM\oplus T_\lambda$, implies that

$$\tilde{\nabla}_X\psi^\lambda\gamma=(g_\lambda)_*\nabla_X u+\phi^\lambda\alpha_T(X, u)-(g_\lambda)_* A_\theta X+\phi^\lambda(\nabla^\perp_X\theta)_T,$$

which proves the claim.

It follows from the claim that the induced metric is flat because for the limit $\lambda=0$ what we get with X_0 is just a local parametrization of the $(n+t)$-dimensional affine subspace which contains substantially the immersion g. To conclude the proof of the theorem we define $h=X_1$ and observe that any flat ruled Euclidean submanifold verifies $\dim N_1 \leqq 1$. Since $\dim N_1=1$ along $f(c([a, b]))$, the result follows.

Proof of Theorem 1. Let $U \subset M$ be the open subset of points where N_1 is not parallel. Define $U_1=M-\bar{U}$. Now let $V \subset U$ be the open subset where $\nu<n-2$ and set $U_2=U-\bar{V}$. Since $\dim S=1$ on V by Proposition 6, it follows from Theorem 7 that there exists an open and dense subset $U_3 \subset V$ whose components we may assume to be simply connected, where the claim of the theorem holds.

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