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An Elementary Proof of Harnack's Inequality for Schrödinger Operators and Related Topics

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Dedicated to Professor Dr. Erhard Heinz on the occasion of his 65th birthday

Introduction

For $G \subset \mathbb{R}^N$ an open set we want to study distributional solutions of

$$-\Delta u + Vu = f. \tag{0.1}$$

We make use of the following.

Definition. Let $G \subset \mathbb{R}^N$ ($N \geqq 3$) be an open set and let $V: G \to \mathbb{R}$ be measurable. We say that $V \in K^N(G)$, if for each $t > 0$

$$\varphi_V(t, G) := \sup_{x \in \mathbb{R}^N} \int_{\{|y-x| \leqq t\}} \frac{|V(y)| \chi_G(y)}{|x-y|^{N-2}}\, dy \tag{0.2}$$

is finite and

$$\lim_{t \to 0} \varphi_V(t, G) = 0\,. \tag{0.3}$$

Here, χ_G denotes the characteristic function of G. We say that $V \in K^N_{\mathrm{loc}}(G)$, if (0.2) and (0.3) hold true for each $G_1 \Subset G$. If $N = 2$, $|x-y|^{2-N}$ has to be replaced by $||\ln|x-y||$.

In connection with essential selfadjointness problems for Schrödinger operators of type (0.1) this class was first introduced by Kato [8]. But in fact, it was studied before by Schechter [10]. Throughout this paper we restrict ourselves to the case $N \geqq 3$. The changes necessary for $N = 2$ are obvious. All functions occurring are assumed to be real valued. More precisely we assume

$$\text{(H.1)} \qquad \begin{cases} G \subset \mathbb{R}^N \text{ denotes an open set, } V, f \in K^N_{\mathrm{loc}}(G) \\ u \in L^1_{\mathrm{loc}}(G),\ Vu \in L^1_{\mathrm{loc}}(G) \text{ and } -\Delta u + Vu = f \\ \text{in the sense of distributions}\,. \end{cases}$$

In their meanwhile famous paper Aizenman and Simon [1] proved under assumption (H.1) for $f = 0$ that a solution u of (0.1) is continuous. Moreover they

showed Harnack's inequality as well as a stronger version of it. It is very interesting that necessary and sufficient conditions for the potential V are given in [1] for the strong Harnack inequality to hold (compare the Definition in [1], p. 210 or our Definition 5.3). Further they prove subsolution estimates. These results are excellent, but the mixed analytic-probabilistic proofs are long and complicated and an enormous probabilistic machinery is required. Following the lines of [1], simplifications were made by Zhao [14], [15], [16]. The first purely analytic proof of continuity of solutions and Harnack's inequality is due to Chiarenza, Fabes and Garofalo [2]. Instead of the Laplacian they allow a general uniformly elliptic operator of second order in divergence form with bounded measurable coefficients. Their proof is based on L^p-estimates for Green's functions by Fabes and Stroock [3], requiring hard analysis. Another proof of continuity is due to Wiegner [11]. Very recently, Hinz and Kalf [6] gave nice, rather elementary proofs completely in the classical framework of subsolution estimates as well as Harnack's inequality.

Since the results from [1] are an ideal tool in many applications, it is desirable to have simple proofs for them and to have explicit control on how the constants depend on the local quantities of the potential V. This is the main goal of the present paper. Since the definition of the class K^N_{loc} is formulated using the fundamental solution of the Laplacian, we are rather astonished that until now no direct proof solely based on local representations of the solution u of $-\Delta u + Vu = 0$ by means of the fundamental solution or Green's function was given. In fact, in [1] these tools were used as well as the probabilistic theory. About thirty years ago those direct methods were successfully applied in similar situations in famous papers by Wienholtz [12] (for another use of this method compare [5], p. 195–204) and Ikebe and Kato [7].

It is exactly this that we want to do in the present paper which is completely straightforward. Naturally we begin with proving local boundedness. We also consider inhomogeneous equations. On the basis of a nearly trivial observation (Lemma 2.1) local boundedness is immediately shown (Theorem 2.4). The inhomogeneity f is supposed to satisfy $f \in K^N_{\mathrm{loc}}(G)$. This condition turns out to be sufficient for the distributional solution to be continuous. Conversely, however, if u is continuous, then $f \in K^N_{\mathrm{loc}}(G)$ (Theorem 6.3). Once u is known to be locally bounded, the continuity of u is seen immediately and it is easy to estimate the modulus of continuity (Theorem 3.3). This estimate is basic for our proof of the strong Harnack inequality. For subclasses of potentials V, e.g. Stummel-potentials or $V \in L^p_{\mathrm{loc}}(G)$ with $p > \frac{N}{2}$ Hölder continuity of u follows immediately (Theorem 3.4). Observe that our result is weaker than that by Witte [13]. As soon as u is known to be continuous, we may locally represent u by means of Green's function for the Laplacian in a ball. With the help of a nice estimate for Green's function (Lemmas 4.1, 4.2; observe that Zhao [14] has given a similar, but weaker estimate) we can derive local L^∞-estimates of u solely by local L^1-norms of u and f (Theorems 4.5, 4.6, 4.7). If $f = 0$ and $u \geqq 0$ one readily proves estimates of the type

$$\int_{K_R(x_0)} u(x)dx \leqq \frac{2\omega_N R^N}{N} u(x_0)$$

for $0 < R \leqq R_0$ sufficiently small. Combining the results of sections 3 and 4 we immediately get an improved estimate of the modulus of continuity (Theorem 5.1) and the strong Harnack inequality (Theorem 5.4). Assuming the validity of the strong Harnack inequality and the existence of strictly positive local solutions of $-\Delta u + Vu = 0$, we give a simple proof of $V \in K^N_{\text{loc}}$ (Theorem 6.1). It turns out that a certain condition used in [1] implies the existence of strictly positive solutions (Theorem 6.2). In the last section we indicate how subsolution estimates can be derived by our method. It seems to be clear that other results from [1] like those concerning the Dirichlet problem can also be derived by our method. It strikes us as a rather remarkable fact that from the property $V \in K^N$ we solely use the above definition but no derived properties such as L^1- or L^∞-Δ-smallness or L^2-Δ-formsmallness. The last property was proven by Schechter ([10], p. 138). A more direct proof consistent with the methods used here may be found in [6]. We think that the proofs given here are most elementary, natural and adequate for the class K^N.

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1. Notations and Preliminaries

We write $A \Subset B$ if A and B are open subsets of $\mathbb{R}^N$, $\bar{A}$ compact and $\bar{A} \subset B$. If $R > 0$ and $x \in \mathbb{R}^N$, let us write $K_R(x) := \{y \in \mathbb{R}^N \mid |y - x| < R\}$ and $K_R := K_R(0)$. By ω_N we denote the *surface* of the unit sphere K_1 in $\mathbb{R}^N$. Observe that [4] (p. 8) denote by ω_N the *volume* of K_1! All our proves are based on integral representations of the weak solutions. Therefore, we first prove validity of those formulas. Let $0 \leqq j \in C_0^\infty(\mathbb{R}^N)$, $j(z) = 0$ for $|z| \geqq 1$ and $\int j(z)dz = 1$. For $\varepsilon > 0$ let $j_\varepsilon(z) := \varepsilon^{-N} j(\varepsilon^{-1} z)$ and for $g \in L^1_{\text{loc}}(\mathbb{R}^N)$ denote by $g_\varepsilon(x) := \int j_\varepsilon(x - y) g(y) dy$ the mollified function. Then $g_\varepsilon \in C^\infty(\mathbb{R}^N)$ and for each $K \Subset \mathbb{R}^N$, $\| g - g_\varepsilon \|_{L^1(K)} \to 0$ $(\varepsilon \to 0)$ (see e.g. [4], p. 141).

Lemma 1.1. *Let $G \subset \mathbb{R}^N$ be a bounded open set, $\varphi \in C_0^0(G)$ $g \in L^1_{\text{loc}}(G)$, and $0 < \alpha < N$. Then $F(x) =: \int g(y)\varphi(y) \; |x - y|^{-\alpha} dy$ exists for almost all $x \in G$, $F \in L^1(G)$. Further if $0 < \varepsilon_n, \varepsilon_n \to 0$ and $F_n(x) := \int g_{\varepsilon_n}(y)\varphi(y) \, |x - y|^{-\alpha} dy$, then there exists a subsequence $(\varepsilon_{n'})$ such that $F_{n'}(x) \to F(x)$ for almost all $x \in \mathbb{R}^N$.*

Proof. Let $M := \sup \int\limits_G |x - y|^{-\alpha} dx$. Then since

$$\int\limits_G |g(y)| \, |\varphi(y)| \left(\int\limits_G |x - y|^{-\alpha} dx \right) dy \leqq M \int\limits_G |g(y)| \, |\varphi(y)| dy$$

the assertion concerning F follows by Fubini's theorem. Since

$$\begin{aligned} \int\limits_G |F_n(x) - F(x)| dx &\leqq \int\limits_G |g_{\varepsilon_n}(y) - g(y)| \, |\varphi(y)| \int\limits_G |x - y|^{-\alpha} dx dy \\ &\leqq M \, \| \varphi \|_\infty \, \| g_{\varepsilon_n} - g \|_{L^1(\operatorname{supp} \varphi)} \to 0 \,, \end{aligned}$$

by the Riesz-Fischer theorem there exists a subsequence (ε_n) such that $F_{n_\nu} \to F$ a.e. in G. □

Lemma 1.2. *Let $G \subseteq \mathbb{R}^N$ be an open set, $v \in L^1_{\mathrm{loc}}(G)$, $g \in L^1_{\mathrm{loc}}(G)$ and $-\Delta v = g$ in the sense of distributions. Let $\rho \in C_0^\infty(G)$. Then with*

$$c_N := \frac{1}{(N-2)\omega_N} \tag{1.1}$$

the representation

$$\begin{cases} v(x)\rho(x) = c_N \int g(y)\rho(y)\ |y-x|^{2-N} dy + \\ + c_N \int v(y)(|y-x|^{2-N} \Delta\rho(y) + 2\nabla\rho(y)\cdot \nabla_y |y-x|^{2-N}) dy \end{cases} \tag{1.2}$$

holds for almost all $x \in G$.

Proof. For $w \in C_0^\infty(\mathbb{R}^N)$ we have the representation ([4], p. 18) (1.3) $w(x) = c_N \int |y-x|^{2-N}(-\Delta w(y))dy$. For $0 < \varepsilon < \operatorname{dist}(\operatorname{supp}\rho, \partial G)$ and $y \in \operatorname{supp}\rho$, we have $\Delta v_\varepsilon(y) = (\Delta v)_\varepsilon(y) = g_\varepsilon(y)$. Put $w_\varepsilon := \rho \cdot v_\varepsilon$ in (1.3). Then after one partial integration we get (1.2) with v replaced by v_ε. Choosing a suitable subsequence (ε_n), by Lemma 1.1 we may pass to the limit and get (1.2). □

2. Local Boundedness

Let $\eta \in C_0^\infty(\mathbb{R})$, $0 \leqq \eta(t) \leqq 1$, $\eta(t) = 1$ for $|t| \leqq \frac{1}{2}$ and $\eta(t) = 0$ for $|t| > 1$. For $\delta > 0$ and $z \in \mathbb{R}^N$ define $\eta_\delta(z) := \eta(\delta^{-1}|z|)$. Our proof of local boundedness rests decisively on the following nearly trivial observation.

Lemma 2.1. *Let $G \subset \mathbb{R}^N$ be open and $V \in K^N_{\mathrm{loc}}(G)$. Let $x \in G$, $0 < \delta < \frac{1}{4}\operatorname{dist}(x, \partial G)$. For $z \in \mathbb{R}^N$, $z \neq x$ define*

$$F_\delta(x, z) := \int \frac{|V(y)|\eta_\delta(x-y)\eta_\delta(z-y)}{|x-y|^{N-2}|y-z|^{N-2}}\, dy\,. \tag{2.1}$$

Then there is a constant $c_1 = c_1(N) > 0$ such that

$$F_\delta(x, z) \leqq c_1\, \varphi_V(\delta;\, K_{3\delta}(x))\, \frac{\eta_{4\delta}(x-z)}{|x-z|^{N-2}}\,. \tag{2.2}$$

Proof. Let $\rho := |x-z| > 0$. i) First consider the case $\rho = |x-z| > 2\delta$. Then for $|x-y| \leqq \delta$, $|z-y| \geqq |z-x| - |x-y| > \delta$ and therefore $\eta_\delta(z-y) = 0$. Analogously, if $|y-z| \leqq \delta$, then $|x-y| > \delta$ and $\eta_\delta(x-y) = 0$. Therefore, $\eta_\delta(x-y)\eta_\delta(z-y) = 0$ for $|x-z| > 2\delta$.

ii) Let now $|x-z| = \rho \leqq 2\delta$. Split $\mathbb{R}^N = \left\{y \,|\, |y-x| \leqq \frac{\rho}{2}\right\} \cup \left\{y \,|\, |y-x| > \frac{\rho}{2}\right\}$ $=: G_1 + G_2$ and denote the corresponding integrals by I_1 and I_2. For $y \in G_1$, $|z-y| \geqq |z-x| - |x-y| \geqq \rho - \frac{\rho}{2} = \frac{\rho}{2}$ and therefore

$$I_1 \leqq \left(\frac{2}{\rho}\right)^{N-2} \int\limits_{G_1} \frac{|V(y)|\eta_\delta(x-y)}{|x-y|^{N-2}}\, dy \leqq \left(\frac{2}{\rho}\right)^{N-2} \varphi_V(\delta,\, K_{3\delta}(x))\,.$$

Analogously

$$I_2 \leqq \left(\frac{2}{\rho}\right)^{N-2} \int\limits_{G_2} \frac{|V(y)|\eta_\delta(y-z)}{|y-z|^{N-2}}\,dy \leqq \left(\frac{2}{\rho}\right)^{N-2} \varphi_V(\delta, K_{3\delta}(x))$$

With $c_1 := 2^{N-1}$ we get estimate (2.2) since $\eta_{4\delta}(x-z) = 1$ for $|x-z| = \rho \leqq 2\delta$ and $F_\delta(x, z) = 0$ for $\rho > 2\delta$. □

Lemma 2.2. *Assume (H.1). Let $x_0 \in G$, $R_0 > 0$ such that $K_{2R_0}(x_0) \Subset G$. Then for $0 < \delta \leqq R_0$ and almost all $x \in K_{R_0}(x_0)$*

$$(2.3)\quad \begin{cases} u(x) = c_N \int \dfrac{-V(y)u(y)\eta_\delta(y-x)}{|y-x|^{N-2}}\,dy + c_N \int \dfrac{f(y)\eta_\delta(y-x)}{|y-x|^{N-2}}\,dy \\ + c_N \int \dfrac{u(y)\Delta_y \eta_\delta(y-x)}{|y-x|^{N-2}}\,dy + 2c_N \int u(y)\,\nabla_y \eta_\delta(y-x)\cdot \nabla_y |y-x|^{2-N} dy\ . \end{cases}$$

Proof. For $x_1 \in K_{R_0}(x_0)$ and $0 < \delta \leqq R_0$ let $\rho(y) := \eta_\delta(y - x_1)$. Then $\rho \in C_0^\infty(G)$ and we may apply Lemma 1.2. Since $x_1 \in K_{R_0}(x_0)$ was arbitrary in the resulting expression (1.2) with this choice of ρ we may put $x_1 = x$ and observe $\eta_\delta(0) = 1$. □

Lemma 2.3. *Assume (H.1). Let $x_0 \in G$, $R_0 > 0$ such that $K_{2R_0}(x_0) \Subset G$. Choose $0 < \delta_0 \leqq R_0/4$ so small that*

$$(2.4)\qquad \varphi_V(\delta) := \varphi_V(\delta; K_{2R_0}(x_0)) \leqq (2c_N c_1)^{-1} \quad \text{for } 0 < \delta \leqq \delta_0$$

(where $c_1 > 0$ by Lemma 2.1). Then for $0 < R \leqq R_0/2$, $0 < \delta \leqq \delta_0$ and almost all $z \in K_R(x_0)$

$$(2.5)\quad \begin{cases} \int \dfrac{|V(x)|\,|u(x)|\,\eta_\delta(x-z)}{|x-z|^{N-2}}\,dx \leqq c_3\,\delta^{2-N}\,\|Vu\|_{L^1(K_{R+4\delta}(x_0))} \\ + c_4\,\delta^{-N}\,\|u\|_{L^1(K_{R+2\delta}(x_0))} + c_5\,\varphi_f(\delta) \end{cases}$$

with $c_i = c_i(N)$, $i = 3, 4, 5$ independent of R and δ.

Proof. Consider (2.3) and abbreviate the integrals at the right hand side by $I_k(x)$ ($k = 1, \ldots, 4$). Denote by χ_I the characteristic function for the interval $\{\frac{1}{2} \leqq t \leqq 1\}$. Then by the properties of η_δ we get with a constant $c_2 = c_2(N, \eta) > 0$ for the last two integrals the estimate

$$(2.6)\qquad |I_3(x) + I_4(x)| \leqq c_2 \cdot \delta^{-N} \int \chi_I(\delta^{-1}|y-x|)\,|u(y)|\,dy$$

and therefore with $\varphi_f(\delta) := \varphi_f(\delta; K_{2R_0}(x_0))$ for $x \in K_R(x_0)$

$$(2.7)\quad \begin{cases} |u(x)| \leqq c_N \int \dfrac{|V(y)|\,|u(y)|\eta_\delta(y-x)}{|y-x|^{N-2}}\,dy + c_N \varphi_f(\delta) \\ + c_2 \cdot \delta^{-N} \int |u(y)|\,\chi_I(\delta^{-1}|y-x|)\,dy \end{cases}$$

This last inequality for u we put into the left hand side of (2.5) and apply Fubini's

theorem. For $z \in K_R(x_0)$ we get

$$\int \frac{|V(x)|\,|u(x)|\eta_\delta(x-z)}{|x-z|^{N-2}}\,dx \leqq c_N \int |V(y)|\,|u(y)| \int \frac{\eta_\delta(y-x)\eta_\delta(x-z)|V(x)|}{|y-x|^{N-2}|x-z|^{N-2}}\,dxdz$$

$$+ c_N \varphi_f(\delta) \int \frac{|V(x)|\eta_\delta(x-z)}{|x-z|^{N-2}}\,dx +$$

$$+ c_2 \cdot \delta^{-N} \int \frac{|V(x)|\,\eta_\delta(x-z)}{|x-z|^{N-2}} \int |u(y)|\,\chi_I(\delta^{-1}|y-x|)dydx$$

By the properties of η_δ and χ_I the last expression may be estimated by $c_2 \cdot \delta^{-N} \varphi_v(\delta)\,\|u\|_{L^1(K_{R+2\delta}(x_0))}$. To the first integral we apply Lemma 2.1 and observe $\varphi_v(\delta, K_{3\delta}(x)) \leqq \varphi_v(\delta)$ for $x \in K_{R+\delta}(x_0)$. Further writing $\eta_{4\delta} = \eta_\delta + (\eta_{4\delta} - \eta_\delta)$ we end with

$$(2.8) \quad \begin{cases} \int \dfrac{|V(x)|\;|u(x)|\;\eta_\delta(x-z)}{|x-z|^{N-2}} \leqq c_N \cdot c_1\,\varphi_v(\delta) \int \dfrac{|V(y)|\,|u(y)|\eta_\delta(y-z)}{|y-z|^{N-2}}\,dy \\ + c_N c_1\,\varphi_V(\delta) \int \dfrac{|V(y)|\,|u(y)|(\eta_{4\delta}(y-z) - \eta_\delta(y-z))}{|y-z|^{N-2}}\,dy \\ + c_N \varphi_f(\delta)\varphi_V(\delta) + c_2 \cdot \delta^{-N}\,\varphi_V(\delta)\;\|u\|_{L^1(K_{R+2\delta}(x_0))} \end{cases}$$

Since $\eta_{4\delta}(y-z) - \eta_\delta(y-z) = 0$ for $|y-z| \leqq \dfrac{\delta}{2}$ and $|y-z| \geqq 4\delta$

$$\int \frac{|V(y)|\,|u(y)|(\eta_{4\delta}(y-z) - \eta_\delta(y-z))}{|y-x|^{N-2}} \leqq \left(\frac{\delta}{2}\right)^{2-N} \|Vu\|_{L^1(K_{R+4\delta}(x_0))}$$

We observe (2.4) and derive (2.5) from (2.8) with $c_3 := 2^{N-2}$,

$$c_4 := \frac{c_2}{c_1 \cdot c_N} \quad \text{and} \quad c_5 := c_1^{-1}. \quad \square$$

Theorem 2.4. *Assume (H.1). Let $x_0 \in G$, $R_0 > 0$ such that $K_{2R_0}(x_0) \Subset G$. Choose $0 < \delta_0 \leqq R_0/4$ so small that (2.4) holds. Then for $0 < R \leqq R_0/2$, $0 < \delta \leqq \delta_0$ and $x \in K_R(x_0)$*

$$(2.9) \quad \begin{cases} \|u\|_{L^\infty(K_R(x_0))} \leqq c_3 \delta^{2-N}\,\|Vu\|_{L^1(K_{R+4\delta}(x_0))} + \\ + c_6\,\delta^{-N}\,\|u\|_{L^1(K_{R+2\delta}(x_0))} + c_7\,\varphi_f(\delta) \end{cases}$$

with $c_i = c_i(N) > 0$, $i = 3, 6, 7$.

Proof. Consider (2.7). For $x \in K_R(x_0)$

$$\int |u(y)|\,\chi_I(\delta^{-1}|y-x|)dy \leqq \|u\|_{L^1(K_{R+\delta}(x_0))}.$$

The first expression at the right hand side of (2.7) is estimated by (2.5). From this we derive (2.9) with $c_6 := c_2 + c_4$, $c_7 := c_N + c_5$. $\square$

3. Continuity

Once local boundedness is proved, clearly, u is continuous. Consider (2.3) for $x \in K_R(x_0)$ with suitable R and δ. In the last two integrals no singularities occur and these constitute continuous functions. If $0 < \varepsilon < \delta$, then

$$h_\varepsilon(x) := \int\limits_{\mathbb{R}^N \setminus K_\varepsilon(x)} \frac{-V(y)u(y)\eta_\delta(y-x)}{|y-x|^{N-2}}\, dy \text{ is continuous}$$

and converges locally uniformly to the first integral in (2.3) (compare [1], p. 241). Analogously the second integral defines a continuous function. Our purpose in this chapter is to give an estimate for the modulus of continuity. This turns out to be basic for our proof of the strong Harnack inequality. The nice splitting of the singular integrals in the proof of Lemma 3.2 we learned from the paper of Chiarenza–Fabes–Garofalo [2], p. 423/424.

Lemma 3.1. *Let* $0 < \beta < 1$ *and* $x_0, x, y \in \mathbb{R}^N$ *such that*

$$\lambda := |x - x_0|^{1-\beta} \leqq 2^{-1} \tag{3.1}$$

Then there is a constant $c_8 = c_8(N) > 0$ *such that*

$$\left| \frac{1}{|x-y|^{N-2}} - \frac{1}{|x_0-y|^{N-2}} \right| \leqq c_8 \frac{|x-x_0|^{1-\beta}}{|x_0-y|^{N-2}} \quad \text{for all } y \in \mathbb{R}^N \tag{3.2}$$

such that $|y - x_0| \geqq |x - x_0|^\beta$.

Proof. Let $y \in \mathbb{R}^N$ such that

$$|y - x_0| \geqq |x - x_0|^\beta . \tag{3.3}$$

Then, by (3.1) $|x - x_0| = |x - x_0|^\beta \, |x - x_0|^{1-\beta} \leqq \lambda \cdot |y - x_0|$ and therefore

$$|y - x| \geqq |y - x_0| - |x - x_0| \geqq (1 - \lambda)\, |y - x_0| . \tag{3.4}$$

Then, by (3.4)

$$\left| \frac{1}{|x-y|^{N-2}} - \frac{1}{|x_0-y|^{N-2}} \right| = \big| |x_0 - y| - |x - y| \big| \sum_{k=0}^{N-3} |x_0 - y|^{k-N+2} |x - y|^{-k-1}$$

$$\leqq \frac{|x_0 - x|}{|y - x_0|^{N-1}} \sum_{k=0}^{N-3} \left(\frac{1}{1-\lambda} \right)^{k+1}$$

Let $c_8 := \sum\limits_{k=0}^{N-3} 2^{k+1}$. If we observe (3.3) we get

$$|x_0 - x| = |x_0 - x|^\beta \, |x_0 - x|^{1-\beta} \leqq |y - x_0| \, |x_0 - x|^{1-\beta}$$

and we derive (3.2) since $\lambda \leqq \frac{1}{2}$. □

Lemma 3.2. *Let* $0 \leqq g \in K^N_{\mathrm{loc}}(G)$, $x_0 \in G$, $R > 0$ *such that* $K_R(x_0) \Subset G$. *Let* $\varphi_g(t) := \varphi_g(t; K_R(x_0))$.

Let $\rho \in C_0^\infty(K_R(x_0))$, $0 \leqq \rho \leqq 1$ *and for* $x \in \mathbb{R}^N$ *put*

$$J(x) := \int \frac{g(y)\rho(y)}{|y-x|^{N-2}}\, dy .$$

Then J is continuous at x_0 and if $0 < \beta < 1$ is chosen, then, for $|x - x_0|^{1-\beta} \leqq 2^{-1}$

$$|J(x) - J(x_0)| \leqq c_8 |x - x_0|^{1-\beta} \varphi_g(R) + 2\varphi_g(\tfrac{3}{2}|x - x_0|^\beta) . \tag{3.5}$$

Proof. Without loss of generality let $x_0 = 0$. Choose $0 < \beta < 1$ and let $0 < |x| < 2^{-\frac{1}{1-\beta}}$. Then with $K_R = K_R(0)$,

$$|J(x) - J(0)| \leqq \int_{K_R} g(y) \big| |y - x|^{2-N} - |y|^{2-N} \big| dy$$

$$\leqq \int_{K_R \cap \{y \,|\, |y| \geqq |x|^\beta\}} \ldots + \int_{K_R \cap \{y \,|\, |y| \leqq |x|^\beta\}} \ldots := J_1(x) + J_2(x)$$

By (3.2)

$$J_1(x) \leqq c_8 |x|^{1-\beta} \int_{K_R} \frac{g(y)}{|y|^{N-2}} dy \leqq C_8 |x|^{1-\beta} \varphi_g(R) \tag{3.6}$$

For $|y| \leqq ||x|^\beta$, because of $|x|^{1-\beta} \leqq 2^{-1}$, $|x - y| \leqq |x| + |y| \leqq |x|^\beta(1 + |x|^{1-\beta}) \leqq \frac{3}{2}|x|^\beta$ and therefore

(3.7)

$$J_2(x) \leqq \int_{K_R \cap \{|y - x| \leqq \frac{3}{2}|x|^\beta\}} \frac{g(y)}{|y - x|^{N-2}} dy + \int_{K_R \cap \{|y| \leqq |x|^\beta\}} \frac{g(y)}{|y|^{N-2}} dy \leqq 2\varphi_g(\tfrac{3}{2}|x|^\beta) .$$

(3.5) follows then from (3.6) and (3.7). □

Theorem 3.3. *Assume (H.1). Let $x_0 \in G$, $R > 0$ such that $K_R(x_0) \Subset G$ and $u \in L^\infty(K_R(x_0))$. Then u is continuous at x_0 and chosen $0 < \beta < 1$, there is a constant $c_9 = c_9(N) > 0$, independent of R, such that for $|x - x_0| \leqq \min(R/4, 2^{-\frac{1}{1-\beta}})$ we have*

$$\begin{cases} |u(x) - u(x_0)| \leqq [(c_8 \varphi_v(R) + c_9 R^{-(1-\beta)}) \|u\|_{L^\infty(K_R(x_0))} + \\ + c_8 \varphi_f(R)]|x - x_0|^{1-\beta} + 2\|u\|_{L^\infty(K_R(x_0))} \cdot \varphi_V(\tfrac{3}{2}|x - x_0|^\beta) \\ + 2\varphi_f(\tfrac{3}{2}|x - x_0|^\beta) \end{cases} \tag{3.8}$$

where $\varphi_V(t) := \varphi_V(t; K_R(x_0))$, $\varphi_f(t) := \varphi_f(t, K_R(x_0))$.

Proof. Without loss of generality, let $x_0 = 0$. Denote by η the function introduced at the beginning of Chapter 2. With $\rho := \eta_R$ apply Lemma 1.2 to u. Then, for $|x| \leqq R/4$ we have

$$\begin{cases} u(x) = c_N \int \frac{-V(y)u(y)\eta_R(y)}{|y - x|^{N-2}} dy + c_N \int \frac{f(y)\eta_R(y)}{|y - x|^{N-2}} dy \\ + c_N \int u(y) \, (|y - x|^{2-N} \Delta\eta_R(y) + \nabla_y |y - x|^{2-N} \cdot \nabla\eta_R(y)) dy \\ =: I_1(x) + I_2(x) + I_3(x) \end{cases} \tag{3.9}$$

Since $u \in L^\infty(K_R)$, $V \in K^N(K_R)$ we have $Vu \in K^N(K_R)$ and $\varphi_{Vu}(t) = \varphi_{Vu}(t, K_R) \leqq \|u\|_{L^\infty(K_R)} \varphi_V(t)$.

Therefore, we may apply Lemma 3.2 to I_1 and I_2. Choose $0 < \beta < 1$ and let $|x| \leqq \min\left[R/4, 2^{-\frac{1}{1-\beta}}\right]$. Then by (3.5)

$$\begin{cases} |I_1(x) - I_1(0)| + |I_2(x) - I_2(0)| \leqq \\ \leqq c_8 |x|^{1-\beta} \left(\|u\|_{L^\infty(K_R)} \varphi_V(R) + \varphi_f(R)\right) + \\ + 2\left(\|u\|_{L^\infty(K_R)} \varphi_V\left(\tfrac{3}{2}|x|^\beta\right) + \varphi_f\left(\tfrac{3}{2}|x|^\beta\right)\right) \end{cases} \tag{3.10}$$

To estimate the corresponding expressions for I_3 we observe that $\nabla\eta_R$ and $\Delta\eta_R$ vanish for $|y| \leqq \frac{R}{2}$. For $R \geqq |y| \geqq R/2$ and $|x| \leqq R/4$ let $\phi(t) := |y - tx|^{2-N}$, $t \in [0, 1]$. Then by the mean value theorem there is $t_0 \in (0,1)$ such that

$$\begin{cases} \left||y - x|^{2-N} - |y|^{2-N}\right| = |\phi'(t_0)| \leqq (N-2) \dfrac{|x|}{|y - t_0 x|^{N-1}} \leqq \\ \leqq (N-2) 4^{N-1} \dfrac{|x|}{R^{N-1}} \end{cases} \tag{3.11}$$

because of $|y - t_0 x| \geqq |y| - |x| \geqq R/4$. Analogously

$$|\nabla_y |y - x|^{2-N} - \nabla_y |y|^{2-N}| \leqq d_1 \frac{|x|}{R^N} \quad \text{for } |x| \leqq \frac{R}{4}, \frac{R}{2} \leqq |y| \leqq R\,. \tag{3.12}$$

where $d_1 = d_1(N)$. Further, there is a constant $d_2 = d_2(\eta, N) > 0$ such that

$$|\nabla\eta_R(y)| \leqq d_2 R^{-1}, \; |\Delta\eta_R(y)| \leqq d_2 R^{-2}\,. \tag{3.13}$$

From (3.11)–(3.13) follows the existence of a constant $c_9 = c_9(d_1, d_2, N) > 0$, independent of R, such that

$$\begin{cases} |I_3(x) - I_3(0)| \leqq c_9 R^{-1} \|u\|_{L^\infty(K_R)} \cdot |x| \leqq \\ \qquad \leqq c_9 R^{-1} |x|^{1-\beta} R^\beta \|u\|_{L^\infty(K_R)} \end{cases} \tag{3.14}$$

Combining (3.10) and (3.14) yields (3.8). □

In general there is no natural choice of $0 < \beta < 1$, we may e.g. choose $\beta = \frac{1}{2}$ (then, (3.8) holds for $|x - x_0| \leqq \frac{1}{4}\min(1, R)$). But if φ_V and φ_f are Hölder-continuous we get from (3.8) a Hölder-condition:

Theorem 3.4. *Let the assumptions of Theorem* 3.3 *hold. In addition assume that there is an* α *with* $0 < \alpha < 1$ *and there are constants* $M_V = M(V, K_R(x_0))$ *and* $M_f = M(f, K_R(x_0))$ *such that*

$$\varphi_V(t) \leqq M_V t^\alpha, \; \varphi_f(t) \leqq M_f t^\alpha. \tag{3.15}$$

Then, for $|x - x_0| \leqq \min\left(R/4, 2^{-\frac{2}{1+2}}\right)$ *we have*

$$\begin{cases} |u(x) - u(x_0)| \leqq \left\{\left[c_8 M_V R^\alpha + c_9 R^{-\frac{2}{1+2}} + 2M_V\left(\tfrac{3}{2}\right)^\alpha\right] \|u\|_{L^2(K_R(x_0))} + \right. \\ \left. + M_f\left[c_8 R^\alpha + 2\left(\tfrac{3}{2}\right)^\alpha\right]\right\} |x - x_0|^{\frac{2}{1+2}} \end{cases} \tag{3.16}$$

Proof. Because of (3.15),

$$\varphi_i(\tfrac{3}{2}\, t^\beta) \leqq M_i(\tfrac{3}{2})^\alpha\, t^{\alpha\beta}, \quad i = V, f. \tag{3.17}$$

Then, $\alpha\beta = 1 - \beta$ if $\beta = \dfrac{1}{1+\alpha}$ and $\alpha\beta = \dfrac{\alpha}{1+\alpha}$. By (3.17), estimate (3.16) is immediately read off from (3.8). □

Examples for Condition (3.15). i) Suppose V satisfies *a local Stummel-condition* that is V is measurable in G and there is a number $\gamma \in (0, 4)$ such that for each $G' \Subset G$ there exists a constant $D_{G'} > 0$ such that

$$\sup_{x \in \mathbb{R}^N} \int_{G' \cap K_1(x)} \frac{|V(y)|^2}{|x-y|^{N-4+\gamma}}\, dy \leqq D_{G'}\,, \tag{3.18}$$

If $K_R(x_0) \Subset G$, $0 < t \leqq 1$ then

$$\int_{K_t(x) \cap K_R(x_0)} \frac{|V(y)|}{|x-y|^{N-2}}\, dy \leqq \left(\int_{K_t(x) \cap K_R(x_0)} \frac{|V(y)|^2}{|x-y|^{N+\gamma-4}} \right)^{\frac{1}{2}} \left(\int_{K_t(x)} \frac{dy}{|x-y|^{N-\gamma}} \right)^{\frac{1}{2}} \leqq$$

$$\leqq D^{\frac{1}{2}}_{K_R(x_0)} \left(\frac{\omega_N}{\gamma} \right)^{\frac{1}{2}} t^{\frac{\gamma}{2}}.$$

Therefore, (3.15) is satisfied with $\alpha = \gamma/2$.

ii) Analogously it is readily seen that $V \in L^p_{\mathrm{loc}}(G)$ with $p > \dfrac{N}{2}$ implies (3.15) with $\alpha = 2 - \dfrac{N}{p}$. Observe that this L^p-condition implies the Stummel-condition for $N \geqq 4$ with $0 < \gamma < 4 - \dfrac{2N}{p}$.

4. Some Properties of Green's Function and Integral Representations

Our proof of Harnack's inequality rests on estimates derived by means of Green's function for the Laplacian in a ball. Without loss of generality we may assume the ball to be centered at the origin. For $R > 0$ let $K_R := K_R(0)$. For $N \geqq 3$, $R > 0$ and $x, y \in K_R$ let

$$G(x, y) := \frac{1}{(N-2)\omega_N} \left\{ |x-y|^{2-N} - \left(\frac{|y|^2\,|x|^2}{R^2} - 2\langle x, y\rangle + R^2 \right)^{\frac{2-N}{2}} \right\} \tag{4.1}$$

be Green's function for the Laplacian (compare e.g. [4], p. 19). If $v \in C^2(K_R) \cap C^0(\bar{K}_R)$, $-\Delta v = g \in C^0(G)$, then

$$v(x) = \frac{1}{R\omega_N} \int_{\partial K_R} \frac{R^2 - |x|^2}{|y-x|^N}\, v(y)\, d\omega_y + \int_{K_R} G(x, y)\, g(y)\, dy \tag{4.2}$$

The following properties are immediately derived from (4.1)

$$G(x, y) = G(y, x) \quad \text{for } x, y \in K_R \tag{4.3}$$

$$0 \leqq G(x, y) \leqq c_N |x - y|^{2-N} \ (c_N \text{ by } (1.1)) \tag{4.4}$$

$$|\nabla_y G(x, y) \leqq \frac{2}{\omega_N} |x - y|^{1-N} \tag{4.5}$$

We decisively use a certain estimate concerning the behaviour of G near the boundary:

Lemma 4.1. *For $x, y \in K_R$ and $0 \leqq |x| < |y| \leqq R$*

$$0 \leqq G(x, y) \leqq \frac{2}{\omega_N} \frac{(R - |y|)}{|x - y|^{N-1}} . \tag{4.6}$$

Proof. Let $z := \frac{R}{|y|} \cdot y \in \partial K_R$ and for $t \in [0, 1]$ let $\phi(t) := G(x, ty + (1 - t)z)$. Then, $\phi(1) = G(x, y)$, $\phi(0) = G(x, z) = 0$, because of $z \in \partial K_R$. Since $\left| ty + (1 - t) \frac{R}{|y|} \cdot y \right|^2 \geqq |y|^2 > |x|^2$, ϕ is differentiable and by the mean value theorem there is $t_0 \in [0, 1]$ such that with $y_0 := t_0 y + (1 - t_0) \frac{R}{|y|} y$

$$G(x, y) = \phi'(t_0) = \sum_{i=1}^{N} \left(\frac{\partial G}{\partial y_i} \right) (x, y_0) \left(y_i - \frac{R}{|y|} y_i \right) \tag{4.7}$$

Further, $$|x - y_0|^2 = \left| (x - y) - (1 - t_0) \cdot \left(\frac{R}{|y|} - 1 \right) y \right|^2 =$$

$$= |x - y|^2 - 2(1 - t_0) \cdot \left(\frac{R}{|y|} - 1 \right) \langle x, y \rangle + 2(1 - t_0) \left(\frac{R}{|y|} - 1 \right) \cdot |y|^2$$

$$+ (1 - t_0)^2 \left(\frac{R}{|y|} - 1 \right)^2 |y|^2$$

Since $R \geqq |y| > |x|$ we conclude $-2\langle x, y \rangle > -2|y|^2$ and therefore $|x - y_0| \geqq |x - y|$. Then by (4.5) we derive (4.6) from (4.7) applying Schwarz's inequality. □

An immediate consequence is

Lemma 4.2. *Let $x, y \in K_R$. Then*

$$\left| |y|^2 - |x|^2 \right| G(x, y) \leqq \frac{4}{\omega_N} \frac{(R^2 - |y|^2)}{|x - y|^{N-2}} . \tag{4.8}$$

Proof. If $|x| \geqq |y|$, then $\left| |y|^2 - |x|^2 \right| = |x|^2 - |y|^2 \leqq R^2 - |y|^2$ and (4.8) follows immediately from (4.4). If $|x| < |y|$, then $0 \leqq |y|^2 - |x|^2 = (|y| - |x|)(|y| + |x|) \leqq |y - x| \cdot 2R$ and

$$R - |y| = \frac{R^2 - |y|^2}{R + |y|} \leqq \frac{R^2 - |y|^2}{R}$$

and (4.8) follows by (4.6). □

For the next considerations we make the following hypothesis

(H.2) $\begin{cases} \text{Let } G \subset \mathbb{R}^N \text{ be open } v,\ g \in L^1_{loc}(G),\ -\Delta v = g \text{ in the sense of} \\ \text{distributions. Further, } v \text{ is assumed to be continuous in } G\ . \end{cases}$

Lemma 4.3. *Let $R > 0$, $K_R \Subset G$ and assume (H.2). Then (4.2) holds for v.*

Proof. As in the proof of Lemma 1.2 consider v_ε. Since v is continuous in a neighborhood of $\bar{K}_R$, $v_\varepsilon \to v$ uniformly on ∂K_R. Therefore in (4.2) for $x \in K_R$ the integrals over ∂K_R for v_ε converge to the corresponding one for v. Observing (4.4) the remaining part of the proof is the same as in Lemma 1.2. □

Corollary.

$$v(0) = \frac{1}{R^{N-1}\omega_N} \int_{\partial K_R} v(y)d\omega_y + c_N \int_{K_R} g(y)\,(|y|^{2-N} - R^{2-N})dy\ . \tag{4.9}$$

Lemma 4.4. *Same assumptions as in Lemma 4.3. Then*

$$\int_{K_R} v(x)dx = \frac{R}{N} \int_{\partial K_R} v(y)d\omega_y + \frac{1}{2N} \int_{K_R} g(y)(R^2 - |y|^2)dy\ . \tag{4.10}$$

Proof. Define

$$h(x) := \frac{1}{\omega_N} \int_{\partial K_R} \frac{R^2 - |x|^2}{|y - x|^N} v(y)d\omega_y\ . \tag{4.11}$$

Then $\Delta h = 0$ in K_R and $h|_{\partial K_R} = v|_{\partial K_R}$. By the second mean value property $h(0) = \frac{N}{R^N \omega_N} \int_{K_R} h(x)dx$. On the other hand, by (4.11),

$$h(0) = \frac{1}{\omega_N R^{N-1}} \int_{\partial K_R} v(y)d\omega_y \text{ and therefore}$$

$$\int_{K_R} h(x)dx = \frac{R}{N} \int_{\partial K_R} v(y)d\omega_y \tag{4.12}$$

Integrating (4.2) leads to

$$\int_{K_R} v(x)dx = \frac{R}{N} \int_{\partial K_R} v(y)d\omega_y + \int_{K_R} g(y) \int_{K_R} G(x, y)dxdy \tag{4.13}$$

Consider $\omega(x) := R^2 - |x|^2$. Then, $\omega|_{\partial K_R} = 0$ and $-\Delta\omega = 2N$.

By (4.2) we get with the help of (4.3)

$$R^2 - |y|^2 = 2N \int_{K_R} G(y, x)dx = 2N \int_{K_R} G(x, y)dx$$

Therefore, (4.10) follows immediately from (4.13). □

Theorem 4.5. *Assume (H.1). Let $x_0 \in G$, $R_0 > 0$ such that $K_{2R_0}(x_0) \Subset G$. Let $0 < R_1 \leqq R_0$ be so small that*

$$\varphi_V(R; K_{2R_0}(x_0)) \leqq [2(c_N + 4\omega_N^{-1})]^{-1} \quad \textit{for } 0 < R \leqq R_1\ .$$

Then for $0 < R \leqq R_1$

$$\int_{K_R} (R^2 - |x|^2)|V(x)|\,|u(x)|dx \leqq \frac{8R}{\omega_N}\varphi_V(R)\int_{\partial K_R}|u(y)|d\omega_y + 2c_N R^2\varphi_V(R)\|f\|_{L^1(K_R)}. \tag{4.14}$$

Proof. Without loss of generality, assume $x_0 = 0$. By Lemma 4.3 the representation (4.2) holds for u with $g = -Vu + f$. We multiply (4.2) by $R^2 \div |x|^2$, which leads to

$$\begin{cases} (R^2 - |x|^2)\,|u(x)| \leqq \dfrac{1}{R\omega_N}\displaystyle\int_{\partial K_R}\frac{(R^2 - |x|^2)^2}{|y-x|^N}|u(y)|d\omega_y + \\ + (R^2 - |x|^2)\displaystyle\int_{K_R} G(x,y)|V(y)|\,|u(y)|dy + (R^2 - |x|^2)\int_{K_R} G(x,y)|f(y)|dy \end{cases} \tag{4.15}$$

For $|y| = R$, $R^2 - |x|^2 = (R - |x|)\,(R + |x|) \leqq 2R|y - x|$. Observe this for the integral over ∂K_R. We multiply (4.15) by $|V(x)|$ and integrate:

$$\begin{cases} \displaystyle\int_{K_R}(R^2 - |x|^2)|V(x)|\,|u(x)|dx \leqq \frac{4R}{\omega_N}\int_{\partial K_R}|u(y)|\int_{K_R}\frac{|V(x)|}{|y-x|^{N-2}}dxd\omega_y + \\ + \displaystyle\int_{K_R}|V(y)|\,|u(y)|(R^2 - |y|^2)\int_{K_R}|V(x)|G(x,y)dxdy \\ + \displaystyle\int_{K_R}|V(y)|\,|u(y)|\int(|y|^2 - |x|^2)|\,V(x)|\,G(x,y)dxdy \\ + \displaystyle\int_{K_R}|f(y)|\int_{K_R}(R^2 - |x|^2)|V(x)|\,G(x,y)dxdy =: \sum_{k=1}^{4} I_k \end{cases} \tag{4.16}$$

Then, for $0 < R \leqq R_1$

$$I_1 \leqq \frac{4R}{\omega_N}\varphi_V(R)\int_{\partial K_R}|u(y)|d\omega_y \tag{4.17}$$

By (4.4)

$$I_2 \leqq c_N\varphi_V(R)\int_{K_R}|V(y)|\,|u(y)|\,(R^2 - |y|^2)dy \tag{4.18}$$

By Lemma 4.2,

$$\begin{cases} I_3 \leqq \dfrac{4}{\omega_N}\displaystyle\int_{K_R}|V(y)|\,|u(y)|(R^2 - |y|^2)\int_{K_R}\frac{|V(x)|}{|x-y|^{N-2}}dxdy \leqq \\ \leqq \dfrac{4}{\omega_N}\varphi_V(R)\displaystyle\int_{K_R}|V(y)|\,|u(y)|(R^2 - |y|^2)dy \end{cases} \tag{4.19}$$

$$I_4 \leqq c_N R^2\,\varphi_V(R)\int_{K_R}|f(x)|dy \tag{4.20}$$

Adding up (4.17)–(4.19) leads to (4.14) for $0 < R \leqq R_1$. □

Theorem 4.6. *Same assumption as in Theorem* 4.5. *Let* $0 < \mu < 1$. *Then, for* $0 < R \leqq R_1$ (R_1 *by Theorem* 4.5)

$$\text{(4.21)} \quad \begin{cases} \displaystyle\int_{K_{\mu^2 R}(x_0)} |V(x)u(x)|dx \leqq \frac{16\varphi_V(R)}{R^2\omega_N(1-\mu)^2} \int_{\{\mu R \leqq |y-x_0| \leqq R\}} |u(y)|dy + \\ \qquad + \dfrac{2c_N\varphi_V(R)}{1-\mu} \|f\|_{L^1(K_R(x_0))} \,. \end{cases}$$

Proof. Again we may assume $x_0 = 0$. Consider (4.14) with R replaced by r, $0 < r \leqq R_1$.

We have $\displaystyle\int_{K_r} (r^2 - |x|^2)|V(x)u(x)|dx \geqq \int_{K_{\mu r}} (r^2 - |x|^2)|V(x)u(x)|dx \geqq$

$$\geqq r^2(1-\mu) \int_{K_{\mu r}} |V(x)u(x)|dx \text{ because of } r^2 - |x|^2 =$$

$$= (r - |x|)(r + |x|) \geqq r^2(1-\mu) \quad \text{for } x \in K_{\mu r}.$$

Then, by (4.14), for $0 < r \leqq R \leqq R_1$.

$$\int_{K_{\mu r}} |V(x)u(x)|dx \leqq \frac{8\varphi_V(r)}{\omega_N r(1-\mu)} \int_{\partial K_r} |u(x)|d\omega_x + \frac{2c_N\varphi_V(r)}{1-\mu}\|f\|_{L^1(K_r)}$$

$$\leqq \frac{8\varphi_V(R)r^{N-2}}{\omega_N(1-\mu)} \int_{\partial K_1} |u(r\zeta)|d\omega_\zeta + \frac{2c_N\varphi_V(R)}{1-\mu}\|f\|_{L^1(K_R)}$$

We multiply by r and integrate leading to

$$\text{(4.22)} \quad \begin{cases} \displaystyle\int_{\mu R}^{R} r \int_{K_{\mu r}} |V(x)u(x)|dxdr \leqq \frac{8\varphi_V(R)}{\omega_N(1-\mu)} \int_{\{\mu R \leqq |y| \leqq R\}} |u(y)|dy + \\ + \dfrac{c_N\varphi_V(R)}{1-\mu}(R^2 - \mu R^2) \|f\|_{L^1(K_R)} \,. \end{cases}$$

We observe

$$\int_{\mu R}^{R} r \int_{K_{\mu r}} |V(x)u(x)|dxdr \geqq \int_{K_{\mu^2 R}} |V(x)u(x)|dx \, \tfrac{1}{2}(R^2 - (R\mu)^2)$$

and derive (4.21) from (4.22). □

Now we are in the position to improve the L^∞-estimate from Theorem 2.5.

Theorem 4.7. *Assume* (*H*.1). *Let* $x_0 \in G$, $R_0 > 0$ *such that* $K_{2R_0}(x_0) \Subset G$. *Choose* $R_2 \leqq R_0/2$ *so small that*

$$\text{(4.23)} \quad \varphi_V(R_2, K_{2R_0}(x_0)) \leqq \min([2(c_N + 4\omega_N^{-1})]^{-1}, [2c_N c_1]^{-1}) \,,$$

where c_N *is defined by* (1.1) *and* c_1 *by Lemma* 2.1. *Then, there are constants*

$c_i = c_i(N) > 0$, $i = 10, 11$, *solely depending on* N, *such that for* $0 < R \leqq R_2$

$$\|u\|_{L^\infty(K_R(x_0))} \leqq c_{10} R^{-N} \|u\|_{L^1(K_{2R}(x_0))} + c_{11}\, \varphi_f(2R) \tag{4.24}$$

where $\varphi_f(2R) = \varphi_f(2R, K_{2R_0}(x_0))$.

Proof. For $0 < R \leqq R_2$ let $\varphi_V(R) := \varphi_V(R, K_{2R_0}(x_0))$ and choose $\delta := R/8$. Then the assumptions of Theorems 2.4 and 4.6 are satisfied and we conclude from (2.9)

$$\begin{cases} \|u\|_{L^\infty(K_R(x_0))} \leqq c_3 \left(\dfrac{R}{8}\right)^{2-N} \|Vu\|_{L^1(K_{3R/2}(x_0))} + \\ + c_6 \left(\dfrac{R}{8}\right)^{-N} \|u\|_{L^1(K_{3R/2}(x_0))} + c_7\, \varphi_f\left(\dfrac{R}{8}\right). \end{cases} \tag{4.25}$$

Apply now Theorem 4.6 for $\tilde{R} = 2R$ and $\mu = \sqrt{3}/2$. Because of (4.23) we get from (4.21) with constants $c_i = c_i(N) > 0$, $i = 12, 13$,

$$\int_{K_{3R/2}(x_0)} |V(x)u(x)|\, dx \leqq R^{-2} c_{12} \int_{\{\sqrt{3}R \leqq |y| \leqq 2R\}} |u(y)|\, dy + c_{13} \|f\|_{L^1(K_{2R}(x_0))} \tag{4.26}$$

Since

$$\|f\|_{L^1(K_{2R})} \leqq \int_{\{y \,|\, |y - x_0| \leqq 2R\}} \frac{|f(y)|\, |x_0 - y|^{N-2}}{|x_0 - y|^{N-2}}\, dy \leqq \varphi_f(2R)(2R)^{N-2} \tag{4.27}$$

we derive (4.24) combining (4.25)–(4.27). □

Remark 4.8. An important consequence of (4.24) is now the possibility to estimate in Theorem 3.3 the modulus of continuity by local L^1-norms.

Next theorem is of the type of a maximum principle.

Theorem 4.9. *Same assumptions as in Theorem* 4.5. *Choose* $0 < R_3 \leqq R_1$ *so small that* $\varphi_V(R_3, K_{2R_0}(x_0)) < c_N^{-1}$. *Then, for* $0 < R \leqq R_3$

$$\sup_{K_R(x_0)} |u(x)| \leqq (1 - c_N \varphi_V(R))^{-1} \left(\sup_{\partial K_R} |u(y)| + c_N \varphi_f(R)\right) \tag{4.28}$$

where $\varphi_V(R) := \varphi_V(R, K_{2R_0(x_0)})$.

Proof. Assume again $x_0 = 0$ and consider the representation (4.2) for u. Because of $\dfrac{1}{R\omega_N} \int \dfrac{R^2 - |x|^2}{|y - x|^N}\, d\omega_y = 1$ (Poisson-formula for $w = 1$) we conclude for $|x| < R$

$$|u(x)| \leqq \sup_{y \in \partial K_R} |u(y)| + c_N \varphi_v(R) \sup_{K_R} |u(y)| + c_N \varphi_f(R)$$

implying (4.28). □

Having in mind Theorems 3.3 and 4.7 the next estimate is most important for our proof of the strong Harnack inequality.

Theorem 4.10. *Assume (H.1) and in addition $u \geqq 0$. Let $x_0 \in G$, $R_0 > 0$ such that $K_{2R_0}(x_0) \Subset G$. Choose $0 < R_3 \leqq R_0/2$ so small that*

$$\varphi_V(R_3, K_{2R_0}(x_0)) \leqq \min[\varphi_V(R_2, K_{2R_0}(x_0)),\ (2^{N+3}c_N)^{-1},\ (2\omega_N c_N c_{10})^{-1}] \tag{4.29}$$

where c_{10} is defined by Theorem 4.7. Then there is a constant $c_{14} = c_{14}(N)$ solely depending on N such that for $0 < R \leqq R_3$

$$\int_{K_R(x_0)} u(x)dx \leqq \frac{2}{N}\omega_N R^N u(x_0) + c_{14} R^N \varphi_f(R) \tag{4.30}$$

where $\varphi_f(R) := \varphi_f(R; K_{2R_0}(x_0))$.

Proof. Assume again $x_0 = 0$. Consider R_2 by Theorem 4.7. For $0 < R \leqq R_2$ apply (4.10) to u and observe (4.14), (4.27), leading to

$$\begin{cases} \displaystyle\int_{K_R} u(x)dx \leqq \frac{R}{N}\int_{\partial K_R} u(y)d\omega_y + \frac{1}{2N}\int_{K_R} |V(y)|\,u(y)(R^2 - |y|^2)dy + \\ \qquad + \dfrac{1}{2N}\displaystyle\int_{K_R} |f(y)|(R^2 - |y|^2)dy \\ \leqq \left(1 + \dfrac{4}{\omega_N}\varphi_V(R)\right)\dfrac{R}{N}\displaystyle\int_{\partial K_R} u(y)d\omega_y + \left(\dfrac{2}{N}c_N\varphi_V(R) + \dfrac{1}{2N}\right)R^2\,\|f\|_{L^1(K_R)} \\ \qquad \leqq \tfrac{1}{2}R\displaystyle\int_{\partial K_R} u(y)d\omega_y + \frac{1}{2}R^N\varphi_f(R) \end{cases} \tag{4.31}$$

where we observed (4.23), (4.29) and $N \geqq 3$. By (4.9), applied to u, we get

$$\begin{cases} R\displaystyle\int_{\partial K_R} u(y)d\omega_y \leqq R^N\omega_N u(0) + \\ + R^N\omega_N c_N \displaystyle\int_{K_R} |V(y)|\ u(y)\ (|y|^{2-N} - R^{2-N})dy \\ + R^N\omega_N c_N \displaystyle\int |f(y)|\ (|y|^{2-N} - R^{2-N})dy \end{cases} \tag{4.32}$$

We split the first integral on the right hand side. By (4.24)

$$\begin{cases} R^N \displaystyle\int_{K_{R/2}} |V(y)|\,u(y)\,|y|^{2-N}dy \leqq R^N\varphi_V(R)\,\|u\|_{L^\infty(K_{R/2})} \leqq \\ \leqq c_{10}\,\varphi_V(R)\,\|u\|_{L^1(K_R)} + c_{11}R^N\varphi_V(R)\,\varphi_f(R)\ . \end{cases} \tag{4.33}$$

For $R/2 \leqq |y| \leqq R$, $|y|^{2-N} - R^{2-N} = \dfrac{R^{N-2} - |y|^{N-2}}{|y|^{N-2}R^{N-2}} =$

$$= (R - |y|)\sum_{K=0}^{N-3} R^{k-N+2}|y|^{-1-k} \leqq 2^{N-1}\,(R - |y|)\,R^{1-N} \leqq 2^{N-1}(R^2 - |y|^2)R^{-N}\ .$$

Therefore, again by (4.14), (4.27)

$$\text{(4.34)}\qquad \begin{cases} R^N \int\limits_{K_R \setminus K_{R/2}} |V(y)|\, u(y)\, (|y|^{2-N} - R^{2-N})dy \leqq \\ \leqq 2^{N-1} \int\limits_{K_R} |V(y)|\, u(y)\, (R^2 - |y|^2)dy \leqq \\ \leqq \frac{2^{N+2}}{\omega_N} \varphi_V(R)\, R \int\limits_{\partial K_R} u(y)d\omega_y + 2^N c_N \varphi_V(R) R^N \varphi_f(R) \end{cases}$$

Choose now $0 < R' \leqq R_2'$ so small that $\varphi_V(R') \leqq c_N^{-1} 2^{-N-3}$. Then we derive from (4.32) with the aid of (4.33), (4.34) observing (4.27)

$$R \int\limits_{\partial K_R} u(y)d\omega_y \leqq 2R^N \omega_N\, u(0) + 2c_{10}\omega_N c_N \varphi_V(R) \int\limits_{K_R} u\,dx + c_{15} R^N \varphi_f(R)$$

where $c_{15} = c_{15}(N) > 0$ solely depends on N. Put this inequality into (4.31), then

$$\int\limits_{K_R} u(x)dx \leqq R^N \omega_N u(0) + c_{10}\omega_N c_N \varphi_V(R) \int\limits_{K_R} u dx + \frac{(c_{16}+1)}{2} R^N \varphi_f(R)\,.$$

We choose now finally $0 < R_3 \leqq R'$ so small that $\varphi_V(R_3) \leqq [2\omega_N c_N c_{10}]^{-1}$ and derive (4.30). □

5. Strong Harnack Inequality

Theorem 5.1. *Let $G \subset \mathbb{R}^N$ be open, f, $V \in K_{\mathrm{loc}}^N(G)$. Then for each pair $G_1 \Subset G$ and $d > 0$, there exists a $R_0 = R_0(d, V) > 0$ and continuous, nonnegative, increasing functions $g_{G_1,d,V}$ and $g_{G_1,d,f,V}$ defined on $[0, R_0]$, vanishing at $t = 0$, with the following properties:*

α) Let $x_1 \in G_1$, $d > 0$ such that $K_{2d}(x_1) \subset G$ and let u, $Vu \in L^1_{\mathrm{loc}}(K_{2d}(x_1))$ and assume that u satisfies there $-\Delta u + Vu = f$ in the sense of distributions.

β) $u \geqq 0$ a.e. in $K_{2d}(x_1)$.

Then u is continuous in $K_d(x_1)$ and for each $x_0 \in K_d(x_1)$ and $0 \leqq |x - x_0| \leqq R_0(d, V)$

$$\text{(5.1)}\qquad |u(x) - u(x_0)| \leqq g_{G_1,d,V}(|x - x_0|)u(x_0) + g_{G_1,d,V,f}(|x - x_0|)\,.$$

Proof. i) If $\partial G \neq \phi$, for $x \in \mathbb{R}^N$ let us denote $d(x) := \mathrm{dist}(x, \partial G) := \inf\{|x - y| \mid y \in \partial G\}$. Then

$$\text{(5.2)}\qquad |d(x) - d(x')| \leqq |x - x'| \quad \text{for } x, x' \in \mathbb{R}^N\,.$$

Abbreviate $K_x(x) := K_{3d(x)/4}(x)$ and for $G_1 \Subset G$ define $G_2 := \bigcup\limits_{x \in \bar{G}_1} K_x(x)$. Clearly, G_2 is open and by compactness of $\bar{G}_1$, G_2 is bounded. By (5.2) the function d attains its minimum $d_m > 0$ on $\bar{G}_1$ and for $z \in \bar{G}_2$ we easily conclude $d(z) \geqq \frac{1}{8} d_m > 0$, that is $G_2 \Subset G$. Analogously we see $\bar{G}_1 \Subset G$. If $x \in G_1$, $d > 0$ such that $K_{2d}(x) \subset G$ then $d \leqq \dfrac{d(x)}{2}$. Therefore $\frac{3}{2} d \leqq \frac{3}{4} d(x)$ and $K_{3d/2}(x) \subset K_x(x) \subset G_2$.

ii) If $\partial G = \phi$ that is $G = \mathbb{R}^N$, then given $d > 0$ define $G_2 := G_{2d} := \{y \in \mathbb{R}^N | \text{dist}(y, G_1) < 2d\}$ for $G_1 \Subset \mathbb{R}^N$. We have to keep in mind this dependence of G_2 on d in case of $G = \mathbb{R}^N$. Again, $K_{3d/2} \subset G_2$.

iii) Choose now $R_1 > 0$ so small that

$$\varphi_V(8R_1, G_2) \leqq \tfrac{1}{2}\min[(c_N + 4\omega_N^{-1})^{-1}, (c_N c_1)^{-1}, (2^{N+2}c_N)^{-1}, (\omega_N c_N c_{10})^{-1}] \tag{5.3}$$

and define

$$\varphi_j(r) := \varphi_j(r; G_2) \quad \text{for } r > 0, j = f \text{ or } V. \tag{5.4}$$

Let $x_1 \in G_1$, $d > 0$ such that $K_{2d}(x_1) \subset G$. Choose

$$0 < R_0 = R_0(d) := \tfrac{1}{2}\min(R_1, d/32, 3^{-2}) \tag{5.5}$$

and assume u to satisfy hypotheses α) and β). For $x_0 \in K_d(x_1)$ by (5.5) we may apply Theorem 3.3 to $K_{4R_0}(x_0)$ and $\beta := \frac{1}{2}$. Let for $t \geqq 0$

$$h_1(t) := (c_8\varphi_V(4R_0) + c_9(4R_0)^{\frac{1}{2}})\, t^{\frac{1}{2}} + 2\varphi_V(\tfrac{3}{2}\, t^{\frac{1}{2}}) \quad \text{and} \tag{5.6}$$

$$h_2(t) := c_8\varphi_f(4R_0)\, t^{\frac{1}{2}} + 2\varphi_f(\tfrac{3}{2}\, t^{\frac{1}{2}}) \tag{5.7}$$

Then, by (3.8), for $x \in K_{R_0}(x_0)$

$$|u(x) - u(x)_0| \leqq h_1(|x - x_0|)\, \|u\|_{L^\infty(K_{4R_0}(x_0))} + h_2(|x - x_0|) \tag{5.8}$$

Because of (5.3), (5.5) the assumptions of Theorem 4.7 (applied to $K_{4R_0}(x_0)$) are satisfied and we derive from (4.24)

$$\|u\|_{L^\infty(K_{4R_0}(x_0))} \leqq c_{10}(4R_0)^{-1} \int\limits_{K_{8R_0}(x_0)} u dx + c_{11}\varphi_f(8R). \tag{5.9}$$

Applying now Theorem 4.10 to $K_{8R_0}(x_0)$ (observe $K_{16R_0} \subset K_{3d/4}(x_1) \subset G_2$) we conclude from (4.30)

$$\int\limits_{K_{8R_0}(x_0)} u dx \leqq 2\omega_N(8R_0)^N\, u(x_0) + c_{14}(8R_0)^N \varphi_f(8R_0) \tag{5.10}$$

Combining (5.9), (5.10) we get with constants c_{16}, c_{17} solely depending on N

$$\|u\|_{L^\infty(K_{4R_0}(x_0))} \leqq c_{16}\, u(x_0) + c_{17}\varphi_f(8R_0) \tag{5.11}$$

and finally, from (5.8)

$$\begin{aligned} |u(x) - u(x_0)| \leqq\ & c_{16}h_1(|x - x_0|)\, u(x_0) + c_{17}\varphi_f(8R_0)h_1(|x - x_0|) \\ & + h_2(|x - x_0|). \end{aligned} \tag{5.12}$$

Define $g_{G_1, d, V}(t) := c_{16}h_1(t)$ and

$$g_{G_1, d, V, f}(t) := c_{17}\varphi_f(8R_0)h_1(t) + h_2(t). \quad \square$$

Remark 5.2. If (3.15) is satisfied with $0 < \alpha < 1$ then we may choose $\beta = \dfrac{1}{1+\alpha}$. Then with respect to Theorem 3.3 in addition to (5.5) we have to demand $0 < R_0 \leqq 2^{-\frac{1}{1+\alpha}}$. We derive from (3.16) that u is Hölder-continuous with exponent

$\frac{\alpha}{1+\alpha}$ and in (5.1) the functions g can be replaced by $c(V,f)\,|x-x_0|^{\frac{\alpha}{1+\alpha}}$

Definition 5.3. Let $G \subset \mathbb{R}^N$ be open, $V \in L^1_{\mathrm{loc}}(G)$. We say the operator $-\Delta + V$ obeys a strong Harnack inequality in G if and only if for each pair $G_1 \Subset G$ and $d > 0$ there is an increasing function $g_{G_1,d}\colon [0,d] \to \mathbb{R}$ continuous at $t = 0$, $g_{G_1,d}(0) = 0$, with the properties

i) If $x_1 \in G_1$, $d > 0$ such that $K_{2d}(x_1) \subset G$ and if u, $Vu \in L^1_{\mathrm{loc}}(K_{2d}(x_1))$ with $-\Delta u + Vu = 0$ in the sense of distributions

and

ii) $u \geqq 0$

then u is continuous in $K_d(x_1)$ and

$$|u(x) - u(x_1)| \leqq g_{G_1,d}(|x - x_1|)u(x_1) \quad \text{for } x \in K_d(x_1) \tag{5.6}$$

Theorem 5.4. *Let $G \subset \mathbb{R}^N$ be open, $V \in K^N_{\mathrm{loc}}(G)$. Then $-\Delta + V$ obeys a strong Harnack inequality in G.*

Proof. From the proof of Theorem 5.1 we derive for $f = 0$ in G the existence of a number $0 < R_0 < d/32$ and a continuous increasing function $g := g_{G_1,d,R_0}\colon [0, R_0] \to 0$ with $g(0) = 0$ and

$$|u(x) - u(x_0)| \leqq g(|x - x_0|)u(x_0) \quad \text{for } x \in K_{R_0}(x_0),\ x_0 \in K_d(x_1)\,. \tag{5.7}$$

If $x \in K_d(x_1)$ then there is a $k \in \mathbb{N}$ such that $(k-1)R_0 \leqq |x - x_1| \leqq R_0 k$. For $j = 0, 1, \ldots, k$ define $y_j := x_1 + \frac{j}{k}(x - x_1)$. Then $y_0 = x_1$, $y_k = x$ and $|y_j - y_{j-1}| = \frac{|x - x_1|}{k} \leqq R_0$, $j = 1, \ldots, k$ and $g(|y_j - y_{j-1}|) \leqq g(R_0)$. By (5.7) for $1 \leqq j \leqq k$

$$|u(y_j) - u(y_{j-1})| \leqq g(R_0)\, u(y_{j-1})\,. \tag{5.8}$$

Therefore, $u(y_j) = u(y_{j-1}) + u(y_j) - u(y_{j-1}) \leqq (1 + g(R_0))\ u(y_{j-1})$ and by induction $u(y_j) \leqq (1 + g(R_0))^j\, u(x_1)$. Then by (5.8)

$$|u(y_j) - u(y_{j-1})| \leqq g(R_0)(1 + g(R_0))^j\, u(x_1) \tag{5.9}$$

and therefore

(5.10)

$$|u(x) - u(x_1)| \leqq \sum_{j=1}^{k} |u(y_j) - u(y_{j-1})| \leqq u(x_1) \cdot (1 + g(R_0))[(1 + g(R_0)^k - 1]$$

Since $k \leqq \frac{|x - x_1|}{R_0} + 1$ we define

$$g_{G_1,d}(t) := \begin{cases} g_{G_1,d,R_0}(t) \text{ for } 0 \leqq t \leqq R_0 \\ (1 + g_{G_1,d,R_0}(R_0))(1 + g_{G_1,d,R_0}(R_0))^{1 + \frac{t}{R_0}} - 1 \quad \text{for } R_0 < t \leqq d \end{cases}$$

Then by (5.10) we conclude (5.6). □

Theorem 5.5. (Harnack's inequality). *Let $G \subset \mathbb{R}^N$ be open and $V \in K^N_{\mathrm{loc}}(G)$. Then for each subdomain $G_1 \Subset G$ there exists a constant $C = C(G_1, N, V)$ depending only on local norms of V, such that for any $0 \leqq u \in L^1_{\mathrm{loc}}(G)$ with $Vu \in L^1_{\mathrm{loc}}(G)$ and $-\Delta u + Vu = 0$ in G the sense of distributions*

$$\sup_{x \in G_1} u(x) \leqq C \inf_{x \in G_1} u(x) . \tag{5.11}$$

Proof. Fix $d := \frac{1}{4} \operatorname{dist}(G_1, \partial G)$ if $\partial G \neq \phi$ and let $d := \operatorname{diam} G_1$ otherwise. With this choice of d consider $g = g_{G_1, d}$ according to Theorem 5.4. Choose $0 < \rho \leqq d$ such that $g(\rho) \leqq \frac{1}{2}$. For $x_0 \in G_1$ and $x \in K_\rho(x_0)$ we have by (5.6) $|u(x) - u(x_0)| \leqq g(|x - x_0|)u(x_0)$. Therefore $u(x) \leqq \frac{3}{2} u(x_0)$. Further $u(x) = u(x_0) + u(x) - u(x_0) \geqq \frac{1}{2} u(x_0)$. Therefore $u(x) \leqq \frac{3}{2} u(x_0) \leqq 3\, u(x)$ for $x \in K_\rho(x_0)$ and

$$\sup_{K_\rho(x_0)} u(x) \leqq 3 \inf_{K_\rho(x_0)} u(x) \tag{5.12}$$

$\bar{G}_1$ can be covered by a finite number of balls $K_\rho(x_i)$, $x_i \in G_1$, $i = 1, \ldots, k$. If x_m, $x_M \in \bar{G}_1$ are such that for the continuous function u we have $u(x_m) \leqq u(x) \leqq u(x_M)$ for $x \in \bar{G}_1$, we may connect x_m and x_M by a closed arc $\Gamma \subset \bar{G}_1$. Applying (5.12) to each ball $K_\rho(x_i) \cap \Gamma \neq \phi$, we get (5.11) with $C = 3^k$. □

6. Necessity of the K^N_{loc}-conditions for V and f

Theorem 6.1. *Let $G \subset \mathbb{R}^N$ be open, $V \in L^1_{\mathrm{loc}}(G)$, $V \leqq 0$ a.e. in G and suppose that the operator $-\Delta + V$ obeys the strong Harnack inequality. If for each $x_1 \in G$ there exists a $d > 0$ with $K_{2d}(x_1) \subset G$ and a function $0 \leqq u \in L^1_{\mathrm{loc}}(K_{2d}(x_1))$ such that $Vu \in L^1_{\mathrm{loc}}(K_{2d}(x_1))$, $-\Delta u + Vu = 0$ in $K_{2d}(x_1)$ in the sense of distributions, $u \in C^0(K_d(x_1))$ (by the strong Harnack inequality), and $u(x_1) > 0$, then $V \in K^N_{\mathrm{loc}}(G)$.*

Proof. Let $d' := d/4$. Then for each $x_0 \in K_{d'}(x_1)$ the strong Harnack inequality applies to $u|_{K_{2d'}(x_1)}$ and with $g := g_{G_1, d'}$ we have by (5.6)

$$|u(x) - u(x_0)| \leqq g(|x - x_0|)u(x_0) \quad \text{for } x \in K_{d'}(x_0) \tag{6.1}$$

and all $x_0 \in K_{d'}(x_1)$. Then $u(x) > 0$ for $x \in K_{d'}(x_1)$, because otherwise, by (6.1) u would vanish identically in $K_{d'}(x_1)$ which contradicts $u(x_1) > 0$. Consider the representation (2.3) for $x \in K_{d'}(x_1)$ and $0 < \delta < d'/2$. Write

$$\begin{cases} c_N \int \dfrac{u(y) \Delta_y \eta_\delta(y - x)}{|y - x|^{N-2}} dy = u(x)\, c_N \int \dfrac{\Delta_y \eta_\delta(y - x)}{|y - x|^{N-2}}\, dy \\ \qquad\qquad + c_N \int \dfrac{(u(y) - u(x)) \Delta_y \eta_\delta(y - x)}{|y - x|^{N-2}} dy \end{cases} \tag{6.2}$$

Observe by (1.3)

$$c_N \int \Delta_y \eta_\delta(y - x) |y - x|^{N-2} dy = -\eta_\delta(0) = -1 . \tag{6.3}$$

By (3.13) and (6.1) we get with a constant $c_{17} = c_{17}(N) > 0$

$$c_N \int \frac{|u(y) - u(x)|\,|\Delta_y \eta_\delta(y-x)|}{|y-x|^{N-2}}\,dy \leqq c_{17} g(\delta) u(x) \tag{6.4}$$

Analogously we treat the last integral in (2.3) observing

$$2c_N \int -\Delta_y \eta_\delta(y-x)|\,|y-x|^{2-N}\,dy = 2 \tag{6.5}$$

and

$$2c_N \int |u(y) - u(x)|\,|\nabla_y \eta_\delta(y-x)|\,|\nabla|y-x|^{2-N}|\,dy \leqq c_{18} g(\delta) u(x)\,. \tag{6.6}$$

So we conclude from (2.3) by (6.3), (6.5)

$$\begin{aligned} c_N u(x) \int \frac{-V(y)\eta_\delta(y-x)}{|y-x|^{N-2}}\,dy &= c_N \int \frac{(u(x)-u(y))V(y)\eta_\delta(y-x)}{|y-x|^{N-2}}\,dy \\ &+ C_N \int \frac{(u(x)-u(y))\Delta_y \eta_\delta(y-x)}{|y-x|^{N-2}}\,dy + \\ &+ 2c_N \int (u(x)-(u(y))\nabla_y \eta_\delta j(y-x)\,\nabla_y|y-x|^{2-N}dy \end{aligned}$$

Let δ_0 be chosen so small that $g(\delta) \leqq \frac{1}{2}$ for $0 < \delta \leqq \delta_0$. Then by (6.4) and (6.6)

$$c_N u(x) \int \frac{-V(y)\eta_\delta(y-x)}{|y-x|^{N-2}}\,dy \leqq 2(c_{17} + c_{18}) g(\delta) u(x) \tag{6.7}$$

Divide by $C_N u(x) > 0$ and observe $\eta_\delta(y-x) = 1$ for $y \in K_{\delta/2}(x)$. Then by (6.7)

$$\int\limits_{K^{\delta/2}(x)} \frac{-V(y)}{|y-x|^{N-2}}\,dy \leqq c_{19} g(\delta) \quad \text{for all } x \in K_{d'}(x_1),\ 0 < \delta \leqq \delta_0\,,$$

which proves $V \in K^N_{\mathrm{loc}}(G)$. □

We show now that the following condition (6.8) of the type introduced by Aizenman and Simon ([1], p. 247; compare also p. 210) guaranties the local existence of a positive distributional solution of $-\Delta u + Vu = 0$. Our assumption on V is slightly weaker than that of [1] according to our more general concept of "K^N_{loc}". The following proof is similar to [1], p. 247.

Theorem 6.2. *Let $G \subset \mathbb{R}^N$ be open and let $V \leqq 0$ be measurable in G. Suppose that for each $G_1 \Subset G$ there exist constants $0 \leqq C'_{G_1} < 1$, $C''_{G_1} \geqq 0$ such that*

$$\int\limits_{G_1} |V|\phi^2 \leqq C'_{G_1} \|\nabla\phi\|^2 + C''_{G_1} \|\phi\|^2 \quad \text{for } \phi \in C_0^\infty(G_1). \tag{6.8}$$

Then $V \in L^1_{\mathrm{loc}}(G)$ and for each $x_1 \in G$ there exists a $d > 0$ with $K_{2d}(x_1) \subset G$ and there exists a $u \in L^1_{\mathrm{loc}}(K_{2d}(x_1))$ such that $Vu \in L^1_{\mathrm{loc}}(K_{2d}(x_1))$, u is a distributional solution of $-\Delta u + Vu = 0$ in $K_{2d}(x_1)$ and $u(x) \geqq \gamma > 0$ a.e. in $K_{2d}(x_1)$.

Proof. Let $x_1 \in G$. For $r > 0$ we write $K_r := K_r(x_1)$. Choose $R > 0$ such that $K_{2R} \Subset G$. Choose $\phi \in C_0^\infty(K_{2R})$, $0 \leqq \phi \leqq 1$ such that $\phi = 1$ in K_R. Then by (6.8)

$$\int\limits_{K_R} |V| \leqq \int\limits_{K_R} |V|\phi^2 \leqq C'_{K_{2R}} \|\nabla\phi\|^2 + C''_{K_{2R}} \|\phi\|^2$$

and therefore $V \in L^1_{\mathrm{loc}}(G)$. Let $C' := C'_{K_R}$, $C'' := C''_{K_R}$.

Then by Fatou's theorem we conclude from (6.8)

$$\int_{K_R} |V|\, u^2 \leqq C' \|\nabla u\|^2 + C'' \|u\|^2 \quad \text{for } u \in H_0^{1,2}(K_R)\,. \tag{6.9}$$

Then $\int_{K_R} |V|\,|u| = \int_{K_R \cap \{|u| \leqq 1\}} |V|\,|u| + \int_{K_R \cap \{|u| > 1\}} |V|\,|u| \leqq \int_{K_R} |V| + \int_{K_R} |V| u^2$

and therefore $Vu \in L^1(K_R)$ for $u \in H_0^{1,2}(K_R)$.

Let for $u, \phi \in H^{1,2}(K_R)$

$$B[u, \phi] := \langle \nabla u, \nabla \phi \rangle - \langle |V|^{\frac{1}{2}} u, |V|^{\frac{1}{2}} \phi \rangle \tag{6.10}$$

Choose now $R > 0$ so small that $\dfrac{4R^2}{N^2} \leqq \dfrac{1 - C'}{2C''}$. Because of

$$\|u\| \leqq \frac{2R}{N} \|\nabla u\| \quad \text{for } u \in H_0^{1,2}(K_R) \tag{6.11}$$

we conclude by (6.9) for $u \in H_0^{1,2}(G)$

$$B[u, u] \geqq (1 - C') \|\nabla u\|^2 - C'' \|u\|^2 \geqq \frac{1 - C'}{2} \|\nabla u\|^2 \tag{6.12}$$

On the other hand again by (6.9), (6.11)

$$|B[u, u]| \leqq M \|\nabla u\|^2 \quad \text{for } u \in H_0^{1,2}(K_R) \tag{6.13}$$

where $M := 1 + C' + C'' \dfrac{4R^2}{N^2}$. Let $f \in C^1(\bar{K}_R)$ such that $0 \leqq f \leqq 1$, $f(x) = 0$ for $x \in K_{R/4}$, $f(x) = 1$ for $x \in K_R \setminus K_{3R/4}$. Because of (6.11)–(6.13) by the Lax-Milgram theorem ([4], p. 78) there exists a $u \in H_0^{1,2}(K_R)$ ($H_0^{1,2}(K_R)$ equipped with norm $\|\nabla \cdot\|$) such that

$$B[u, \phi] = \langle f, \phi \rangle \quad \text{for } \phi \in H_0^{1,2}(K_R)\,. \tag{6.14}$$

As seen above $Vu \in L^1(K_R)$. Let $u_- := \min(u, 0)$. Then by the chain rule (see e.g. [4], S. 146 or [9], p. 16), $\nabla u_- = \nabla u$ for $u \leqq 0$ and $\nabla u_- = 0$ for $u > 0$ and therefore, by (6.12) and (6.14)

$$\frac{1 - C'}{2} \|\nabla u_-\|^2 \leqq B[u_-, u_-] = B[u, u_-] = \langle f, u_- \rangle \leqq 0\,. \tag{6.15}$$

Therefore $u_- = 0$ a.e. that is $u \geqq 0$ a.e. Without loss of generality, we may assume $x_1 = 0$. With Green's function (4.1) for K_R define $\omega(x) := \int_{K_R} G(x, y) f(y) dy$. Then $\omega \in C^2(K_R) \cap C^0(\bar{K}_R)$, $-\Delta \omega = f$ in K_R, $\omega|_{\partial K_R} = 0$. By (4.4), (4.5), $\|\omega\|_\infty + \|\nabla \omega\|_\infty \leqq C(R)$. From this we easily conclude $\omega \in H_0^{1,2}(K_R)$. Further, $\omega(x) > 0$ for $x \in K_R$. Otherwise $x \in K_R$ and $\omega(x) = 0$ would imply $G(x, y) f(y) = 0$ for all $y \in K_R$. But this contradicts $f(y) = 1$ for $y \in K_R \setminus K_{3R/4}$ and $G(x, y) > 0$ for

$x, y \in K_R$. Since $\langle \nabla\omega, \nabla\phi \rangle = \langle f, \phi \rangle$ for all $\phi \in H_0^{1,2}(K_R)$ by (6.14) for $\phi \in H_0^{1,2}(K_R)$, $\phi \leqq 0$

$$\langle \nabla(u - \omega), \nabla\phi \rangle = \langle |V|^{\frac{1}{2}} u, |V|^{\frac{1}{2}}\phi \rangle \leqq 0 . \tag{6.16}$$

Choose $\phi := (u - \omega)_- = \min(u - \omega, 0)$ in (6.16) and conclude $\nabla(u - \omega)_- = 0$, therefore $(u - \omega)_- = 0$ and $u(x) \geqq \omega(x) > 0$ a.e. for $x \in K_R$. Choose now $d := \frac{R}{8}$ and

$\gamma := \min_{|x| \leqq R/8} \omega(x) > 0$. Then, by (6.14) u has the required properties in K_{2d}. □

Theorem 6.3. *Let $G \subset \mathbb{R}^N$ be open, $V \in K_{\mathrm{loc}}^N(G)$. Let $0 \leqq f \in L_{\mathrm{loc}}^1(G)$. If $x_1 \in G$, $d > 0$ such that $K_{3d}(x_1) \Subset G$ and u, $Vu \in L^1(K_{3d}(x_1))$ such that u is a distributional solution of $-\Delta u + Vu = f$ and $u|_{K_{2d}(x_1)}$ is continuous. Then $f \in K^N(K_{d/2}(x_1))$.*

Proof. We proceed as in the proof of Theorem 6.1 and consider the representation (2.3) for $x \in K_d(x_1)$ and $0 < \delta < \frac{d}{4}$. As in the proof of Theorem 6.1 we get

$$\begin{aligned} c_N \int \frac{f(y)\eta_\delta(y - x)}{|y - x|^{N-2}}\, dy = {} & c_N \int \frac{V(y)u(y)\eta_\delta(|y - x|)}{|y - x|^{N-2}}\, dy \\ & + c_N \int \frac{(u(x) - u(y))\Delta_y \eta_\delta(y - x)}{|y - x|^{N-2}}\, dy \\ & + 2c_N \int (u(x) - u(y))\nabla_y \eta_\delta((y - x)\nabla_y |y - x|^{2-N} dy \end{aligned} \tag{6.17}$$

Since $u \in C^0(K_{2d}(x_1))$, u is bounded on $K_{3/2\,d}(x_1)$.

For $x \in K_d(x_1)$, $0 < \delta < \frac{d}{4}$ we get from (6.17) by (6.4), (6.3)

$$\begin{aligned} c_N \int \frac{f(y)\eta_\delta(y - x)}{|y - x|^{N-2}}\, dy \leqq {} & c_N \|u\|_{L^\infty(K_{3/2\,d}(x_1))}\, \varphi_V(\delta, K_{3/2\,d}(x_1)) + \\ & + (c_{17} + c_{18}) \cdot \max_{|y-x| \leqq \delta} |u(y) - u(x)| . \end{aligned} \tag{6.18}$$

Let $h_u(\delta) := \max\{|u(y) - u(x)| \,|x, y \in K_{3/2d}(x_1), |x - y| \leqq \delta\}$. Then $h_u(\delta) \to 0$ $(\delta \to 0)$. If $x \notin K_d(x_1)$ then $K_{\delta/2}(x) \cap K_{d/2}(x_1) = \phi$.

If $x \in K_d(x_1)$ then

$$\int_{K_{\delta/2}(x) \cap K_{d/2}(x_1)} \frac{f(y)}{|x - y|^{N-2}}\, dy \leqq \int \frac{f(y)\eta_\delta(y - x)}{|x - y|^{N-2}}\, dy$$

Therefore, abbreviating the right hand side of (6.18) by $\omega(u; \delta)$ we get for $0 < \delta \leqq \frac{\delta}{4}$

$$\int_{K_{\delta/2}(x) \cap K_{d/2}(x_1)} \frac{f(y)}{|x - y|^{N-2}}\, dy \leqq \omega(u; \delta),\ \omega(u; \delta) \to 0 \quad \text{for } \delta \to 0 . \quad \square$$

7. Concluding Remarks

In [1], pp. 249–250 subsolutions are considered, too. If $V \in L^1_{\text{loc}}(G)$, a measurable function u: $G \to \mathbb{R}$ is called a subsolution of $-\Delta + V$, if and only if $u \geqq 0$, u is locally bounded and upper semi-continuous in G and $-\Delta u + Vu \leqq 0$ holds in the sense of distributions. If we assume $V_- \in K^N_{\text{loc}}(G)$, $V_-(x) := \min(V(x), 0)$, then it follows immediately that (2.3) holds with u if the equality sign is replaced by $\leqq$ and V by V_-. The same holds true for Lemma 4.3 and the Corollary. We readily see the validity of Lemma 2.3 and Theorems 2.4, 4.5–4.7. If we put estimate (4.14) with V replaced by V_- into the right hand side of (4.9), written for u and $g(y) := -V_-(y)u(y)$ we end with

$$u(x) \leqq \frac{1}{\omega_N}\,(R^{1-N} + 8R\varphi_{v_-}(R)) \int\limits_{\partial K_R(x)} u(y)\,d\omega_y$$

being the result of [1], p. 249 for subsolutions.

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