Werk

Titel: § 3. Zagier's Method.

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and

$$K_\Gamma^\tau f_j = h^\tau(\lambda_j^{(1)}, \lambda_j^{(2)}) f_j = h(\lambda_j^{(2)}, \lambda_j^{(1)}) f_j \,. \tag{2.51}$$

Denote the operator norm of K_Γ to be given by

$$\|K_\Gamma\| := \left\{ \int_{F\times F} |k_\Gamma(z,z')|^2 \, dz dz' \right\}^{1/2},$$

so that

$$\|K_\Gamma^\tau\| = \|K_\Gamma\| \,.$$

Theorem 2.52. (*a*) (*Zograf*) *Considered as operators in the usual way, the limits*

$$\lim_{J\to\infty} \sum_{j=1}^{J} h(\lambda_j^{(1)}, \lambda_j^{(2)}) f_j(z)\overline{f_j(z')} \,,$$

and

$$\lim_{T,T'\to\infty} \frac{1}{16\pi\sqrt{D}\log\varepsilon} \sum_{j=1}^{h} \sum_{-T'<k<T'} \int_{-T}^{T} h\left(r+\frac{\pi k}{2\log\varepsilon}, r-\frac{\pi k}{2\log\varepsilon}\right) \times$$
$$\times E_j(z, \tfrac{1}{2}+ir, k) E_j(z', \tfrac{1}{2}-ir, -k) dr \,,$$

exist in the operator norm and these limits satisfy

$$k_\Gamma(z,z') = \sum_{j=1}^{\infty} h(\lambda_j^{(1)}, \lambda_j^{(2)}) f_j(z)\overline{f_j(z')}$$
$$+ \frac{1}{16\pi\sqrt{D}\log\varepsilon} \sum_{j=1}^{h} \sum_{-\infty<k<\infty} \int_{-\infty}^{\infty} h\left(r+\frac{\pi k}{2\log\varepsilon}, r-\frac{\pi k}{2\log\varepsilon}\right) \times$$
$$\times E_j(z, \tfrac{1}{2}+ir, k) E_j(z', \tfrac{1}{2}-ir, -k) dr \,.$$

(*b*) *In the twisted case, we have*

$$k_\Gamma^\tau(z,z') = \sum_{j=1}^{\infty} h^\tau(\lambda_j^{(2)}, \lambda_j^{(1)}) f_j(z)\overline{f_j(z')}$$
$$+ \frac{1}{16\pi\sqrt{D}\log\varepsilon} \sum_{j=1}^{h} \sum_{-\infty<k<\infty} \int_{-\infty}^{\infty} h\left(r-\frac{\pi k}{2\log\varepsilon}, r+\frac{\pi k}{2\log\varepsilon}\right) \times$$
$$\times E_j(z, \tfrac{1}{2}+ir, k) E_j(z', \tfrac{1}{2}-ir, k) dr \,.$$

Proof. This follows from the results above; for more details, see [Kub; Ef]. The fact that the "twisting" T_τ exchanges $E(z', \frac{1}{2}-ir, -k)$ with $E(z', \frac{1}{2}-ir, k)$ is a trivial consequence of the definition of the Eisenstein series. □

§ 3. Zagier's Method

3.1. *Introducing the Kloostermann Sums*

As a simplifying assumption, let us assume that K has class number one. This implies that $\mathcal{O}_K$ is a principle ideal domain and that $\Gamma\backslash H^2$ has only one cusp. As

$D \equiv 2, 3 \pmod 4$, we also have $\mathcal{O}_K = \mathbb{Z} + \sqrt{D}\mathbb{Z}$ and its dual lattice is given by

$$\mathrm{Diff}^{-1} = \mathcal{O}_K{}^{\wedge} = \mathbb{Z} + \frac{1}{\sqrt{D}}\mathbb{Z} .$$

In Zagier's method one of the steps is to transform

$$\frac{1}{4D} \int_{\mathcal{D}} \int_{\mathcal{D}} k_\Gamma(x+iy, x'+iy') \overline{e_\nu(x)}\, e_{\nu'}(x') dx dx' \tag{3.1}$$

or, in the twisted case,

$$\frac{1}{4D} \int_{\mathcal{D}} \int_{\mathcal{D}} k_\Gamma^\tau(x+iy, x'+iy') \overline{e_\nu(x)}\, e_{\nu'}(x') dx dx' , \tag{3.1}$$

into an "abelian sum" (i.e., one over ideals in $\mathcal{O}_K$, not over elements of Γ) involving Kloosterman sums. This subsection will be devoted to carrying out this first step.

First, write K_Γ, K_Γ^τ in (3.1–1′) as a sum over (c) and d (mod c), as follows.

Lemma 3.2. *Suppose $f(\gamma z, \gamma z') = f(z, z')$, for all $\gamma \in \Gamma$. We have*

$$\int_{\mathcal{D}} \int_{\mathcal{D}} \sum_{\gamma \in \Gamma} f(z, \gamma z') dx dx' = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \sum_{c,d} f(z, \gamma_{c,d} z') dx dx' ,$$

where the sum runs over (c), with $c = 0$ and d an arbitrary unit, and representatives $(c) \neq 0$, $(d) \pmod c$, with $(c, d) = 1$, and

$$\gamma_{c,d} = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma$$

(we will show that the right-hand integral is well-defined, independent of the choice of $\gamma_{c,d}$).

Proof. Since

$$\Gamma_\infty = \bigcup_{k=-\infty}^{\infty} \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} | a \in \mathcal{O}_K \right\} \cdot \begin{pmatrix} \varepsilon^k & 0 \\ 0 & \varepsilon^{-k} \end{pmatrix} , \tag{3.3}$$

the coset Γ / Γ_∞ can be expressed as the union of all possible pairs

$$(c, d) \in (A \times A)/\mathcal{O}_k{}^\times , \quad (c, d) \neq (0, 0) , \tag{3.4}$$

with (c), (d) having no common factors, A being any ideal of (the p.i.d.) $\mathcal{O}_K$. Here $\mathcal{O}_K^\times$-equivalence means

$$(c, d) \sim (c', d') \Leftrightarrow c' = uc, d' = ud, \text{ for some } u \in \mathcal{O}_K^\times .$$

Phrased in another way: $\gamma_{c,d} \sim \gamma_{c',d'}$ (i.e., they belong to the same coset modulo Γ_∞) if and only if

$$\begin{pmatrix} * & * \\ c' & d' \end{pmatrix} = \begin{pmatrix} \varepsilon^k & 0 \\ 0 & \varepsilon^{-k} \end{pmatrix} \begin{pmatrix} * & * \\ c & d \end{pmatrix} ,$$

for some $k \in \mathbb{Z}$. If $ad-bc=1$ and $a'd-b'c=1$, so

$$\gamma_1 := \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \gamma_2 := \begin{pmatrix} a' & b' \\ c & d \end{pmatrix}$$

are two possible $\gamma_{c,d}$, we have to show $f(z, \gamma_{c,d} z')$ makes sense. We have $b = b' + nd$, $a = a' + nc$, for some $n \in \mathcal{O}_K$, if and only if γ_1, γ_2 both represent $\gamma_{c,d}$. A simple calculation shows that

$$\begin{pmatrix} a' & b' \\ c & d \end{pmatrix} z' = \begin{pmatrix} 1 & -n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} z' .$$

On the other hand, our hypothesis on f implies that

$$\int_{\mathbb{R}^2} f\left(z, \begin{pmatrix} a & b \\ c & d \end{pmatrix} z'\right) dx = \int_{\mathbb{R}^2} f\left(\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} z, \begin{pmatrix} a' & b' \\ c & d \end{pmatrix} z'\right) dx$$

$$= \int_{\mathbb{R}^2} f\left(z, \begin{pmatrix} a' & b' \\ c & d \end{pmatrix} z'\right) dx .$$

The lemma follows. □

From (3.2), it follows that

$$\int_{\mathcal{D}} \int_{\mathcal{D}} k_T(z, z') \overline{e_\nu(x)} e_{\nu'}(x') dx dx' = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left(\sum_{c,d} k(z, \gamma_{c,d} z') \right) \overline{e_\nu(x)} e_{\nu'}(x') dx dx' , \tag{3.5}$$

and

$$\int_{\mathcal{D}} \int_{\mathcal{D}} k_T^\tau(z, z') \overline{e_\nu(x)} e_{\nu'}(x') dx dx' = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left(\sum_{c,d} k(z, \gamma_{c,d} T_\tau z') \right) \overline{e_\nu(x)} e_{\nu'}(x') dx dx' . \tag{3.6}$$

As in section 1, we want to treat the "$c=0$ term" in this expression separately. For $c \neq 0$, in the right-hand side of (3.5), replace x_1 by $x_1 + a/c$, x_2 by $x_2 + \bar{a}/\bar{c}$, x_1' by $x_1' - d/c$, x_2' by $x_2' - \bar{d}/\bar{c}$, i.e., replace $\gamma_{c,d}$ by

$$\begin{pmatrix} 1 & -a/c \\ 0 & 1 \end{pmatrix} \gamma_{c,d} \begin{pmatrix} 1 & -d/c \\ 0 & 1 \end{pmatrix},$$

where $\gamma_{c,d}$ is represented by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Neglecting the "$c=0$ term", this yields for (3.15) the expression

$$\sum_{(c) \neq 0} \left(\sum_{\substack{(d) (\mathrm{mod}\,(c)) \\ ad \equiv 1 \,(\mathrm{mod}\,(c)) \\ (c,d) \in (A \times A)/\mathcal{O}_K}} \overline{e_\nu(a/c, \bar{a}/\bar{c})}\, \overline{e_{\nu'}(d/c, \bar{d}/\bar{c})} \right) \times \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left(\sum_{c,d} k\left(z, \begin{pmatrix} 0 & -1/c \\ c & 0 \end{pmatrix} z'\right) \right) \overline{e_\nu(x)} e_{\nu'}(x') dx dx' . \tag{3.7a}$$

The "$c=0$ term" of $k(z, \gamma_{c,d}z')$ in (3.5) is

$$\sum_{\gamma\in\Gamma_\infty} k(z,\gamma z')$$
$$= \sum_{b\in\mathcal{O}_K} k_0\left(\frac{(x_1-x_1')^2+(y_1-y_1')^2}{2y_1y_1'}, \frac{(x_2-x_2')^2+(y_2-y_2')^2}{2y_2y_2'}\right)$$
$$= \sum_{\mu\in\mathcal{O}_K\hat{}} c_\mu(y,y')e_\mu(x-x') ,$$

where

$$c_\mu(y,y') := \int_{\mathbb{R}^2} e_\mu(u)k_0\left(\frac{u_1^2+(y_1-y_1')^2}{2y_1y_1'}, \frac{u_2^2+(y_2-y_2')^2}{2y_2y_2'}\right)du .$$

Thus the "$c=0$ term" of (3.5) is

(3.7b) $$\delta_{v,v'}c_v(y,y') .$$

For $c\neq 0$, in (3.6), replace x_1 by x_1+a/c, x_2 by $x_2+\bar{a}/\bar{c}$, by $x_1'-\bar{d}/\bar{c}$, x_2' by $x_2'-d/c$. Neglecting the $c=0$ term, this yields the analogous expression

(3.8a)
$$\sum_{(c)\neq 0}\left(\sum_{\substack{(d)(\mathrm{mod}\,(c))\\ ad\equiv 1\,(\mathrm{mod}\,(c))\\ (c,d)\in(A\times A)/\mathcal{O}_K}} \overline{e_v(a/c,\bar{a}/\bar{c})e_{v'}(\bar{d}/\bar{c},d/c)}\right)\times$$
$$\times\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left(\sum_{c,d} k\left(z,\begin{pmatrix}0 & -1/c\\ c & 0\end{pmatrix}T_\tau z'\right)\right)\overline{e_v(x)}\,e_{v'}(x')dxdx' .$$

The "$c=0$ term" of $k^\tau(z,\gamma_{c,d}z')$ in (3.6) is

$$\sum_{\gamma\in\Gamma_\infty} k^\tau(z,\gamma z')$$
$$= \sum_{b\in\mathcal{O}_K} k_0\left(\frac{(x_1-x_2')^2+(y_1-y_2')^2}{2y_1y_2'}, \frac{(x_2-x_1')^2+(y_2-y_1')^2}{2y_2y_1'}\right)$$
$$= \sum_{\mu\in\mathcal{O}_K\hat{}} c_\mu^\tau(y,y')e_\mu(x-T_\tau x') ,$$

where

$$c_\mu^\tau(y,y') := \int_{\mathbb{R}^2} e_\mu(u)k_0\left(\frac{u_1^2+(y_1-y_2')^2}{2y_1y_2'}, \frac{u_2^2+(y_2-y_1')^2}{2y_2y_1'}\right)du .$$

Thus the "$c=0$ term" of (3.6) is

(3.8b) $$\delta_{v,\bar{v}'}c_v^\tau(y,y') .$$

Here

(3.9)
$$S(v,v':c) := \sum_{\substack{(d)(\mathrm{mod}\,(c))\\ ad\equiv 1\,(\mathrm{mod}\,(c))\\ (c,d)\in(A\times A)/\mathcal{O}_K}} \overline{e_v(a/c,\bar{a}/\bar{c})e_{v'}(d/c,\bar{d}/\bar{c})} ,$$

and its twisted analog in (3.8), which we denote by S^{τ}, can be shown to be ("real quadratic") Kloostermann sums, so Weil's estimate is applicable as in the classical case. Indeed, if $(c)=(p)\subset \mathcal{O}_K$ is a prime ideal of K then $d \pmod{p}$ runs through a field of $N(p)$ elements and, moreover, we have an additive character of the field $\mathcal{O}_K/(p)$,

$$\psi_{\nu,\nu'}(a):=e_{\nu}(a/p,\bar{a}/\bar{p})e_{\nu'}(d/p,\bar{d}/\bar{p})=\exp\left(\pi i\mathrm{tr}\left(\frac{a\nu+d\nu'}{p}\right)\right),\quad (ad\equiv 1 \pmod{(p)}),$$

which yields the usual expression for a Kloostermann sum over a finite field. The twisted sum S^{τ} is treated similarly, except that ν' gets replaced by $\overline{\nu'}$.

3.2. *A Change of Variables*

Next, we need a clearer understanding of the expression

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left(\sum_{c\neq 0,\,d} k\left(z,\begin{pmatrix}0 & -1/c\\ c & 0\end{pmatrix}z'\right)\right)\overline{e_{\nu}(x)}\,e_{\nu'}(x')\,dx\,dx' \tag{3.10}$$

and its twisted analog

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left(\sum_{c\neq 0,\,d} k\left(z,\begin{pmatrix}0 & -1/c\\ c & 0\end{pmatrix}T_{\tau}z'\right)\right)\overline{e_{\nu}(x)}\,e_{\nu'}(x')\,dx\,dx' . \tag{3.11}$$

This involves a rather lengthy sequence of change-of-variables, analogous to one carried out in the previous section, but nothing more. Once this is accomplished, we can turn to the second step of Zagier's method.

For brevity, let

$$\begin{aligned}\nu&:=m+n/\sqrt{D},\quad \bar{\nu}:=m-n/\sqrt{D}\,,\\ \nu'&:=m'+n'/\sqrt{D},\quad \bar{\nu}':=m'-n'/\sqrt{D}\,.\end{aligned} \tag{3.12}$$

As in the previous section, we assume that $\nu\neq 0$, $\nu'\neq 0$. In the notation of (3.12), we have

$$\overline{e_{\nu}(x)}\,e_{\nu'}(x')=\exp[-\pi i(x_1\nu-x_1'\nu'+x_2\bar{\nu}-x_2'\bar{\nu}')]\,. \tag{3.13}$$

To reduce (3.10) to something which can be more easily understood, it is useful to change variables in such a way that x_i and x_i' then have the same coefficients in the exponent of (3.13). This amounts to replacing

$$\begin{aligned}x_1&\mapsto\frac{1}{c}\sqrt{\nu'/\nu}\cdot x_1,\quad x_2\mapsto\frac{1}{\bar{c}}\sqrt{\bar{\nu}'/\bar{\nu}}\cdot x_2\\ x_1'&\mapsto\frac{1}{c}\sqrt{\nu/\nu'}\cdot x_1',\quad x_2'\mapsto\frac{1}{c}\sqrt{\bar{\nu}/\bar{\nu}'}\cdot x_2'\,,\end{aligned} \tag{3.14}$$

provided $\nu'/\nu\in K^{\times}$ is totally positive (we take the positive value of the square root). Then $\bar{e}_{\nu}e_{\nu'}$ is replaced by

$$E_{\nu,\nu',c}(x,x'):=\exp\left[-\frac{i\pi}{c}\sqrt{\nu\nu'}\cdot(x_1-x_1')-\frac{i\pi}{c}\sqrt{\bar{\nu}\bar{\nu}'}\cdot(x_2-x_2')\right], \tag{3.15}$$

and $k\left(z, \begin{pmatrix} 0 & -1/c \\ c & 0 \end{pmatrix} z'\right)$ is replaced by

$$k\left(\frac{\sqrt{v'/v}}{c}\cdot x_1 + iy_1, \frac{\sqrt{\bar{v}'/\bar{v}}}{\bar{c}}\cdot x_2 + iy_2, \frac{-1/c^2}{\frac{\sqrt{v/\bar{v}'}}{c}\cdot x_1' + iy_1'}, \frac{-1/\bar{c}^{-2}}{\frac{\sqrt{\bar{v}/\bar{v}'}}{\bar{c}}\cdot x_2' + iy_2'}\right). \tag{3.16}$$

Now set $v_1 := vy_1$, $v_2 := \bar{v}y_2$, $v_1' := v'y_1'$, $v_2' := \bar{v}'y_2'$, and use the fact that $k(\lambda z, \lambda z') = k(z, z')$ to see that (3.16) is equal to

(3.17)

$$k\left(x_1 + \frac{ic}{\sqrt{vv'}}v_1, x_2 + \frac{i\bar{c}}{\sqrt{\bar{v}\bar{v}'}}v_2, -\left(x_1' + \frac{ic}{\sqrt{vv'}}v_1'\right)^{-1}, -\left(x_2' + \frac{i\bar{c}}{\sqrt{\bar{v}\bar{v}'}}v_2'\right)^{-1}\right).$$

Combining (3.14–17), we arrive at the following

Lemma 3.18A. *If $vv' \in K^\times$ is totally positive then*

$$\int_{vy_1 = v'y_1' > 0} \int_{\bar{v}y_2 = \bar{v}'y_2' > 0} \frac{1}{y_1 y_1' y_2 y_2'} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} k\left(z, \begin{pmatrix} 0 & -1/c \\ c & 0 \end{pmatrix} z'\right) \overline{e_v(x)}\, e_{v'}(x')\, dx\, dx'\, dv_1\, dv_2$$

$$= N(c)^{-2} \int_{v_1 = v_1' > 0} \int_{v_2 = v_2' > 0} \frac{1}{\sqrt{y_1 y_1' y_2 y_2'}}$$

$$\times \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} E_{v, v', c}(x, x') k\left(w_1, w_2, \frac{-1}{w_1'}, \frac{-1}{w_2'}\right) dx\, dx'\, dv_1\, dv_2,$$

where $E_{v, v', c}$ is as in (3.15) *and*

$$w_1 := x_1 + \frac{ic}{\sqrt{vv'}}v_1, \quad w_1' := x_1' + \frac{ic}{\sqrt{vv'}}v_1',$$

$$w_2 := x_2 + \frac{i\bar{c}}{\sqrt{\bar{v}\bar{v}'}}v_2, \quad w_2' := x_2' + \frac{i\bar{c}}{\sqrt{\bar{v}\bar{v}'}}v_2'.$$

Lemma 3.18B. *If $vv' \in K^\times$ is totally positive then*

$$\delta_{v, v'} \int_{vy_1 = v'y_1' > 0} \int_{\bar{v}y_2 = \bar{v}'y_2' > 0} \frac{1}{\sqrt{y_1 y_1' y_2 y_2'}} c_v(y, y')\, dv_1\, dv_2$$

$$= \int_{\mathbb{R}_+^2} \int_{\mathbb{R}^2} e_v(u_1 y_1, u_2 y_2) k_0(u_1^2/2, u_2^2/2)\, du\, dy$$

$$= \delta_{v, v'} 64 k_0(0, 0).$$

Remarks. (a) The integrals over $vy_1 = v'y_1'$, etc., are with respect to $dv_1\, dv_2$, not $dy_1\, dy_2$, so there is a suppressed constant factor which will appear explicitly later. (b) Our assumptions on $k(z, z')$ in §1 easily imply absolute convergence of the integrals.

We need the twisted analog of (3.18). In this case, we want to do the same as in (3.14), except that the roles of x_1' and x_2' are exchanged. This amounts to replacing

$$x_1 \mapsto \frac{1}{c}\sqrt{\bar{v}'/v}\cdot x_1, \quad x_2 \mapsto \frac{1}{\bar{c}}\sqrt{v'/\bar{v}}\cdot x_2 \tag{3.19}$$

$$x_1' \mapsto \frac{1}{\bar{c}}\sqrt{\bar{v}/v'}\cdot x_1', \quad x_2' \mapsto \frac{1}{\bar{c}}\sqrt{v/\bar{v}'}\cdot x_2' ,$$

provided $v'\bar{v}\in K^\times$ is totally positive. Then $\bar{e}_v e_{v'}$ is replaced by

$$E^\tau_{v,v',c}(x,x') := \exp\left[-\frac{i\pi}{c}\sqrt{v\bar{v}'}\cdot(x_1-x_2')-\frac{i\pi}{c}\sqrt{\bar{v}v'}\cdot(x_2-x_1')\right], \tag{3.20}$$

and $k\left(z, \begin{pmatrix} 0 & -1/c \\ c & 0 \end{pmatrix} T_\tau z'\right)$ is replaced by

$$k\left(\frac{\sqrt{\bar{v}'/v}}{c}\cdot x_1 + iy_1, \frac{\sqrt{v'/\bar{v}}}{\bar{c}}\cdot x_2 + iy_2, \frac{-1/c^2}{\frac{\sqrt{v/\bar{v}'}}{c}\cdot x_2' + iy_2'}, \frac{-1/\bar{c}^2}{\frac{\sqrt{\bar{v}/v'}}{\bar{c}}\cdot x_1' + iy_1'}\right). \tag{3.21}$$

Now set $v_1 := vy_1$, $v_2 := \bar{v}y_2$, $v_1' := v'y_1'$, $v_2' := \bar{v}'y_2'$, as before, and use the fact that $k(\lambda z, \lambda T_\tau z') = k(z, T_\tau z')$ to see that (3.21) is equal to

$$k\left(x_1 + \frac{ic}{\sqrt{v\bar{v}'}}v_1, x_2 + \frac{i\bar{c}}{\sqrt{v\bar{v}'}}v_2, \frac{-1}{x_2' + \frac{ic}{\sqrt{v\bar{v}'}}v_2'}, \frac{-1}{x_1' + \frac{i\bar{c}}{\sqrt{\bar{v}v'}}v_1'}\right). \tag{3.22}$$

Whereas in the untwisted case, Zagier's method dictated that we integrate over $v_1 = v_1'$, $v_2 = v_2'$, in the twisted case we must integrate over $v_1 = v_2'$, $v_2 = v_1'$, i.e., $vy_1 = \bar{v}'y_2'$, $\bar{v}y_2 = v'y_1'$. Summarizing (3.19–22), we obtain

Lemma 3.23A. *If $v'\bar{v}\in K^\times$ is totally positive then*

$$\int_{vy_1=\bar{v}'y_2'>0}\int_{\bar{v}y_2=\bar{v}'y_1'>0} \frac{1}{\sqrt{y_1y_1'y_2y_2'}} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} k\left(z, \begin{pmatrix} 0 & -1/c \\ c & 0 \end{pmatrix} T_\tau z'\right)$$

$$\times \overline{e_v(x)}\, e_{v'}(x')\, dx\, dx'\, dv_1\, dv_2$$

$$= N(c)^{-2} \int_{v_1=v_2'>0}\int_{v_2=v_1'>0} \frac{1}{\sqrt{y_1y_1'y_2y_2'}}$$

$$\times \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} E^\tau_{v,v',c}(x,x')\, k\left(\zeta_1, \zeta_2, \frac{-1}{\zeta_1'}, \frac{-1}{\zeta_2'}\right) dx\, dx'\, dv_1\, dv_2 ,$$

where $E^\tau_{v,v',c}$ is as in (3.20) *and*

$$\zeta_1 := x_1 + \frac{ic}{\sqrt{v\bar{v}'}}v_1, \quad \zeta_1' := x_2' + \frac{ic}{\sqrt{v\bar{v}'}}v_2' ,$$

$$\zeta_2 := x_2 + \frac{i\bar{c}}{\sqrt{\bar{v}v'}}v_2, \quad \zeta_2' := x_1' + \frac{i\bar{c}}{\sqrt{\bar{v}v'}}v_1' .$$

Lemma 3.23B. *If $v'\bar{v} \in K^\times$ is totally positive then*

$$\delta_{v,\bar{v}'} \int_{vy_1=\bar{v}'y_2'>0} \int_{\bar{v}y_2=\bar{v}'y_1'>0} \frac{1}{\sqrt{y_1y_1'y_2y_2'}} c_v^\tau(y, y')dv_1dv_2$$

$$= \int_{\mathbb{R}_+^2} \int_{\mathbb{R}^2} e_v(u_1y_1, u_2y_2)k_0(u_1^2/2, u_2^2/2)du\,dy$$

$$= \delta_{v,\bar{v}'} 64k_0(0, 0).$$

Now only one more transformation remains to be performed. Define the Zagier transform by

$$(Zk)(t_1, t_2) := \int_{H^2} k(z_1+t_1, z_2+t_2, -1/z_1, -1/z_2)y_1y_2dz_1dz_2\,, \tag{3.24}$$

and let

$$(Zk)^\wedge(r_1, r_2) := \int_{\mathbb{R}^2} e^{i(r_1t_1+r_2t_2)/2}(Zk)(t_1, t_2)dt_1dt_2\,. \tag{3.25}$$

This Fourier transform differs slightly from the Fourier transform in (2.19). From (3.18), we obtain

$$\sqrt{vv'\bar{v}\bar{v}'}(Zk)^\wedge\left(2\pi\frac{\sqrt{vv'}}{c}, 2\pi\frac{\sqrt{\bar{v}\bar{v}'}}{\bar{c}}\right) \tag{3.26}$$

$$= \int_{vy_1=v'y_1'>0} \int_{\bar{v}y_2=\bar{v}'y_2'>0} \frac{1}{\sqrt{y_1y_1'y_2y_2'}} \times$$

$$\times \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} k\left(z, \begin{pmatrix} 0 & -1/c \\ c & 0 \end{pmatrix} z'\right) \overline{e_{v'}(x)} e_v(x')dxdx'dy_1dy_2\,,$$

if $vv' \in K^\times$ is totally positive.

In the twisted situation, define the twisted Zagier transform by

$$(Z^\tau k)(t_1, t_2) := \int_{H^2} k(z_1+t_1, z_2+t_2, -1/z_2, -1/z_1)y_1y_2dz_1dz_2, \tag{3.27}$$

and let

$$(Z^\tau k)^\wedge(r_1, r_2) := \int_{\mathbb{R}^2} e^{i(r_1t_1+r_2t_2)/2}(Z^\tau k)(t_1, t_2)dt_1dt_2\,. \tag{3.28}$$

From (3.18), we obtain

$$\sqrt{vv'\bar{v}\bar{v}'}(Z^\tau k)^\wedge\left(2\pi\frac{\sqrt{v\bar{v}'}}{c}, 2\pi\frac{\sqrt{\bar{v}v'}}{\bar{c}}\right) \tag{3.29}$$

$$= \int_{vy_1=\bar{v}'y_2'>0} \int_{\bar{v}y_2=v'y_1'>0} \frac{1}{\sqrt{y_1y_1'y_2y_2'}} \times$$

$$\times \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} k\left(z, \begin{pmatrix} 0 & -1/c \\ c & 0 \end{pmatrix} z'\right) \overline{e_v(x)}\, e_{v'}(x')dxdx'dy_1dy_2\,,$$

if $v'\bar{v} \in K^\times$ is totally positive.

This concludes the first step.

3.3. *Contribution of the Eisenstein Series and the f_j*

Here we examine the terms

$$\sum_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\lambda_1(r,k), \lambda_2(r,k)) E(z, \tfrac{1}{2}+ir, k) E(z', \tfrac{1}{2}-ir.-k) dr \tag{3.30}$$

and

$$\sum_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\lambda_1(r,k), \lambda_2(r,k)) E(z, \tfrac{1}{2}+ir, k) E(z', \tfrac{1}{2}-ir, k) dr \tag{3.31}$$

from the spectral decomposition of the kernel function, where

$$\lambda_1 = \lambda_1(r,k) := -\left(s+\frac{i\pi k}{2\log\varepsilon}\right)\left(s+\frac{i\pi k}{2\log\varepsilon}-1\right),$$

$$\lambda_2 = \lambda_2(r,k) := -\left(s-\frac{i\pi k}{2\log\varepsilon}\right)\left(s-\frac{i\pi k}{2\log\varepsilon}-1\right).$$

However, we are only interested in the term arising from the ν^{th} and ν'^{th} Fourier coefficients of the Eisenstein series. In other words, we consider

$$\sum_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\lambda_1, \lambda_2) a_\nu(z, \tfrac{1}{2}+ir, k) a_{\nu'}(z', \tfrac{1}{2}-ir, -k) dr \tag{3.32}$$

and, in the twisted case,

$$\sum_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\lambda_1, \lambda_2) a_\nu(z, \tfrac{1}{2}+ir, k) a_{\nu'}(z', \tfrac{1}{2}-ir, k) dr\ . \tag{3.33}$$

Since these Fourier coefficients involve K-Bessel functions, which converge very rapidly, there are no convergence problems here.

Let us begin with (3.32). For brevity, denote

$$\alpha := \tfrac{1}{2}+ir+\frac{i\pi k}{2\log\varepsilon}$$

$$\beta := \bar{\alpha} = \tfrac{1}{2}-ir-\frac{i\pi k}{2\log\varepsilon}\ ,$$

and recall that, for some constant $C = C(k, r, \nu, \nu') \in \mathbb{C}^\times$ not depending on y, we have, by (2.37),

(3.34)

$$\begin{aligned} &a_\nu(y, \tfrac{1}{2}+ir, k) a_{\nu'}(y', \tfrac{1}{2}-ir, -k) \\ &= C\sqrt{y_1 y_2 y_1' y_2'} \cdot K_{\alpha-1/2}(\pi y_1|\nu|) K_{\alpha-1/2}(\pi y_1'|\nu'|) K_{\beta-1/2}(\pi y_2|\bar{\nu}|) K_{\beta-1/2}(\pi y_2'|\bar{\nu}'|) \\ &= 4C\sqrt{\frac{\mu_1\mu_2\mu_1'\mu_2'}{\nu\nu'\bar{\nu}\bar{\nu}'}} \cdot K_{\alpha-1/2}(2\pi\mu_1) K_{\alpha-1/2}(2\pi\mu_1') K_{\beta-1/2}(2\pi\mu_2) K_{\beta-1/2}(2\pi\mu_2')\ , \end{aligned}$$

where

$$\mu_1 := |\nu| y_1/2, \quad \mu_2 := |\bar{\nu}| y_2/2, \quad \mu_1' := |\nu'| y_1', \quad \mu_2' := |\bar{\nu}'| y_2'\ .$$

By the formula following (1.7), it is easy to see that

$$(3.35)\qquad \int_0^\infty \int_0^\infty K_{\alpha-1/2}(2\pi x)^2 K_{\beta-1/2}(2\pi y)^2\,dxdy = \frac{(\pi/8)^2}{\cosh\left[\pi\left(r+\frac{\pi k}{2\log\varepsilon}\right)\right]^2}\,.$$

This fact will be used later.

In the twisted case, one has, for some constant $C'=C'(k,r,v,v')\in\mathbb{C}^\times$ independent of y,

(3.36)

$$\begin{aligned} &a_v(y,\tfrac{1}{2}+ir,k)a_{v'}(y',\tfrac{1}{2}-ir,k)\\ &=C'\sqrt{y_1y_2y_1'y_2'}\cdot K_{\alpha-1/2}(\pi y_1|v|)K_{\alpha-1/2}(\pi y_1'|v'|)K_{\beta-1/2}(\pi y_2|\bar v|)K_{\beta-1/2}(\pi y_2'|\bar v'|)\\ &=4C'\sqrt{\frac{\mu_1\mu_2\mu_1'\mu_2'}{vv'\bar v\bar v'}}\cdot K_{\alpha-1/2}(2\pi\mu_1)K_{\alpha-1/2}(2\pi\mu_1')K_{\beta-1/2}(2\pi\mu_2)K_{\beta-1/2}(2\pi\mu_2')\,, \end{aligned}$$

where, as above,

$$\mu_1:=|v|y_1/2,\quad \mu_2:=|\bar v|y_2/2,\quad \mu_1':=|v'|y_1',\quad \mu_2':=|\bar v'|y_2'\,,$$

Now we turn to the f_j. Those $f_j\in L^2_{\mathrm{res}}(F,dz)$ may be ignored, as they must be constant and we have assumed $v\neq 0$, $v'\neq 0$. For those $f_j\in L^2_{\mathrm{cusp}}(F,dz)$, we have

$$(3.37)\qquad f_j(z)=\sum_v A_v(y,j)e_v(x)\,,$$

where A_v is given by (2.49). By the results of section 2, we have

(3.38)

$$\begin{aligned} &A_v(y,j)\overline{A_{v'}(y',j)}\\ &=\rho_j(v)\overline{\rho_j(v')}\sqrt{y_1y_2y_1'y_2'}\\ &\quad\times K_{ir_{1j}}(\pi y_1|v|)K_{ir_{1j}}(\pi y_1'|v'|)K_{ir_{2j}}(\pi y_2|\bar v|)K_{ir_{2j}}(\pi y_2'|\bar v'|)\\ &=4\rho_j(v)\overline{\rho_j(v')}\sqrt{\frac{\mu_1\mu_2\mu_1'\mu_2'}{vv'\bar v\bar v'}}\\ &\quad\times K_{ir_{1j}}(2\pi\mu_1)K_{ir_{1j}}(2\pi\mu_1')K_{ir_{2j}}(2\pi\mu_2)K_{ir_{2j}}(2\pi\mu_2')\,, \end{aligned}$$

in the notation above. Combining the spectral decomposition with (3.35, 3.38), we find that the $f_j\in L^2_{\mathrm{cusp}}(F,dz)$ contribute (in the untwisted case)

$$(3.39)\qquad \sum_j h(\lambda_j^{(1)},\lambda_j^{(2)})\rho_j(v)\overline{\rho_j(v')}K_{ir_{1j}}(2\pi\mu_1)K_{ir_{1j}}(2\pi\mu_1')K_{ir_{2j}}(2\pi\mu_2)K_{ir_{2j}}(2\pi\mu_2')\,,$$

and the Eisenstein–Maass series contribute

$$\begin{aligned}(3.40)\qquad &(4\pi\sqrt{D}\log\varepsilon)^{-1}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} C'_{v,v'}(k,r)h(\lambda_1(r,k),\lambda_2(r,k))\times\\ &\qquad\times K_{ir}(2\pi\mu_1)K_{ir}(2\pi\mu_1')K_{ir}(2\pi\mu_2)K_{ir}(2\pi\mu_2')dr\,,\end{aligned}$$

where

$$\alpha := \tfrac{1}{2} + ir + \frac{i\pi k}{2\log\varepsilon}$$

$$\beta := \bar{\alpha} = \tfrac{1}{2} - ir - \frac{i\pi k}{2\log\varepsilon},$$

$$\lambda_1 := -\left(s + \frac{i\pi k}{2\log\varepsilon}\right)\left(s + \frac{i\pi k}{2\log\varepsilon} - 1\right),$$

$$\lambda_2 := -\left(s - \frac{i\pi k}{2\log\varepsilon}\right)\left(s - \frac{i\pi k}{2\log\varepsilon} - 1\right).$$

Finally, (3.7) contributes to the untwisted case

(3.41)

$$\frac{1}{16D}\sqrt{\frac{\nu\nu'\bar{\nu}'\bar{\nu}}{\mu_1\mu_2\mu_1'\mu_2'}}\sum_{(c)} S(\nu, \nu'; c)\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} k\left(z, \begin{pmatrix} 0 & -1/c \\ c & 0 \end{pmatrix} z'\right)\overline{e_\nu(x)}\, e_{\nu'}(x')\,dx\,dx'$$
$$+ \delta_{\nu,\nu'}(y_1y_2y_1'y_2')^{-1}c_\nu(y, y').$$

Here we have cancelled $4|\nu\nu'\bar{\nu}'\bar{\nu}|^{-1/2}(\mu_1\mu_2\mu_1'\mu_2')^{-1/2}$ from both sides of the spectral decomposition and the usual assumptions on ν, ν' are in force.

In the twisted case the terms are

(3.42) $$\sum_j h(\lambda_j^{(2)}, \lambda_j^{(1)})\rho_j(\nu)\overline{\rho_j(\nu')}K_{ir_{1j}}(2\pi\mu_1)K_{ir_{1j}}(2\pi\mu_1')K_{ir_{2j}}(2\pi\mu_2)K_{ir_{2j}}(2\pi\mu_2');$$

(3.43) $$\frac{1}{4\pi\sqrt{D}\log\varepsilon}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} C'_{\nu,\nu'}(k, r)h^\tau(\lambda_1(r, k), \lambda_2(r, k))\times$$
$$\times K_{ir}(2\pi\mu_1)K_{ir}(2\pi\mu_1')K_{ir}(2\pi\mu_2)K_{ir}(2\pi\mu_2')dr;$$

and,

(3.44)

$$\frac{1}{16D}\sqrt{\frac{\nu\nu'\bar{\nu}\bar{\nu}'}{\mu_1\mu_2\mu_1'\mu_2'}}\sum_{(c)} S^\tau(\nu, \nu'; c)\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} k\left(z, \begin{pmatrix} 0 & -1/c \\ c & 0 \end{pmatrix} T_\tau z'\right)\overline{e_\nu(x)}\, e_{\nu'}(x')\,dx\,dx'$$
$$+ \delta_{\nu,\bar{\nu}'}(y_1y_2y_1'y_2')^{-1}c_\nu^\tau(y, y').$$

Here again, the usual assumptions on ν, ν' remain in force.

3.4. *The Real Quadratic Kuznetsov–Bruggeman Formula*

In the untwisted case, one must simply integrate

$$(3.39) + (3.40) = (3.41) + (3.7b)$$

over the quarter-plane $\mu_1=\mu_1'>0$, $\mu_2=\mu_2'>0$, assuming that the interchange order of integration is justified for our fixed v, $v'\in\mathcal{O}_K^{\wedge}$, and using the two-dimensional analog of the result in the appendix to section 1 to express the formula in its final form. The interchange of orders of integration can be verified just as in the $SL(2,\mathbb{Z})$ case (see the appendix to section 1). For (3.39), we obtain

$$\left(\frac{\pi}{8}\right)^2\sum_j h(\lambda_j^{(1)},\lambda_j^{(2)})\frac{\rho_j(v)\overline{\rho_j(v')}}{\cosh(\pi r_{1j})\cosh(\pi r_{2j})} \tag{3.45}$$

whereas (3.40) yields

(3.46)

$$\left(\frac{\pi}{8}\right)^2(4\pi\sqrt{D}\log\varepsilon)^{-1}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}C_{v,v'}(k,r)h(\lambda_1(r,k),\lambda_2(r,k))\frac{dr}{\cosh\left[\pi\left(r+\dfrac{\pi k}{2\log\varepsilon}\right)\right]^2},$$

and finally, if vv' is totally positive then (3.41) formally yields (by (3.26))

$$\frac{1}{8D}\sqrt{vv'\bar{v}\bar{v}'}\cdot\sum_{(c)}\frac{S(v,v';c)}{N(c)^2}(Zk)^{\wedge}\left(\frac{2\pi\sqrt{vv'}}{c}\cdot\frac{2\pi\sqrt{\bar{v}'\bar{v}}}{\bar{c}}\right).$$

Let $\phi_J(r_1,r_2)$ and $\phi_H(x_1,x_2)$ be the obvious two-variable generalizations of the transforms defined in (1.10). Since, for $h(r_1,r_2)=\phi_J(r_1,r_2)$, the appendix to section 1 yields the identity

$$\frac{4x_1x_2}{\pi^4}(Zk)^{\wedge}(x_1,x_2)=\phi_H(x_1,x_2),$$

(see (3.25) for the definitions), the contribution of (3.41) may be rewritten

$$\frac{\pi^4}{32D}\cdot\sum_{(c)}\frac{S(v,v';c)}{N(c)}\phi_H\left(\frac{2\pi\sqrt{vv'}}{c},\frac{2\pi\sqrt{\bar{v}'\bar{v}}}{c}\right). \tag{3.47}$$

The real quadratic Kuznetsov formula is the identity

$$(3.45)+(3.46)=(3.47)+\delta_{v,v'}k_0(0,0),$$

assuming vv' is totally positive.

In the twisted case, we integrate over the quarter-plane $\mu_1=\mu_2'>0$, $\mu_2=\mu_1'>0$, formally interchanging orders of integration. Again, the validity of this can be verified by the method in the appendix to section 1. The contribution of (3.42) is

$$\sum_j h(\lambda_j^{(2)},\lambda_j^{(1)})\rho_j(v)\overline{\rho_j(v')}R_j^2, \tag{3.48}$$

where

$$R_j:=\int_0^{\infty}K_{ir_{1j}}(2\pi x)K_{ir_{2j}}(2\pi x)dx.$$

If $|\mathrm{Re}\,r_{1j}|+|\mathrm{Re}\,r_{2j}|<1$, this has an expression in terms of Γ-functions and Gauss'