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The Genus of Curves in $\mathbf{P}^4$ and $\mathbf{P}^5$

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The classification problem for algebraic space curves has been studied since the 19th century. There are two steps:

(1) Every algebraic curve has two basic discrete invariants, the degree d and the genus g. Determine all pairs (d, g) of integers that occur as degree and genus of such a curve.

(2) For each d, g describe the set of all curves of degree d and genus g by finding the irreducible components and the dimension of the parameter space that parametrizes the algebraic families of such curves.

An answer to (1) appeared already in Halphen's work [4], but his proof turned out to be incomplete. A satisfactory proof was given only much later by Gruson and Peskine [3]. The question (2) is still open in general, although a lot of partial results are known (see, e.g., [2]).

One usually considers curves in $\mathbf{P}^3$, since every smooth irreducible curve can be projected isomorphically into $\mathbf{P}^3$ without changing degree and genus.

However, for $n>3$ it makes sense to ask the analogous problems (1) and (2) for curves in $\mathbf{P}^n$ that are nondegenerate, i.e., not contained in a hyperplane. Corresponding to (1), we have the generalized question

(*) Fix an integer $n\geqq 4$. Determine all pairs of integers (d, g) such that there exists a nondegenerate smooth irreducible curve in $\mathbf{P}^n$ of degree d and genus g.

What is known about this problem? Actually very little: Castelnuovo determined the maximum genus for a fixed degree. Eisenbud and Harris [5] determined functions $\pi_i(d)$ (for $i=1, \ldots, n-1$) with the property that a curve of genus greater than $\pi_i(d)$ must lie on a surface of degree less than $n+i-2$. These functions are quadratic polynomials in d and their leading term is $\frac{1}{2(n+i-2)}d^2$.

In this paper we complete the picture for $n=4$ and 5. We show

Theorem 0.1. *Let C be a smooth irreducible curve of degree $d>0$ and genus g in $\mathbf{P}^4$.*

a) If C is nondegenerate, then $0\leqq g\leqq \frac{1}{6}d^2-\frac{5}{6}d+1$.

b) If C lies on a nondegenerate cubic surface, then there are integers b, c such that $d=b+2c$, $g=(b-1)(c-1)+\frac{1}{2}c(c-1)$. *For every* $b\geqq 0$, $c\geqq 1$ *or* $(b, c)=(-1, 1)$ *or* $(1, 0)$ *such a curve exists and is nondegenerate for* $d\geqq 4$.

c) If C is nondegenerate and not contained in a surface of degree at most 3, then $0\leqq g\leqq \frac{1}{8}d^2-\frac{1}{2}d+1$. *(For* $d<7$ *the bound of a) is stronger.)*

d) For $4\leqq d\leqq 6$, *g satisfying the inequality of a) and* $d\geqq 7$, *g satisfying the inequality of c) there exists a nondegenerate smooth irreducible curve in* $\mathbf{P}^4$ *having these values as invariants.*

Theorem 0.2. *Let C be a smooth irreducible curve of degree d and genus g in* $\mathbf{P}^5$.

a) If C is nondegenerate, then $0\leqq g\leqq \frac{1}{8}(d-3)^2$.

b) If C lies on a nondegenerate quartic surface, then either there are integers a, b such that $d=2a+b$, $g=(a-1)(b-1)$; *(for* $(a, b)=(0, 1)$ *or* $(1, 0)$ *or* $a>0$, $b>0$ *such a curve exists) or there is an integer* $a>0$ *such that* $d=2a$, $g=\frac{1}{2}(a-1)(a-2)$. *These curves are nondegenerate for* $d\geqq 5$.

c) If C is nondegenerate and not contained in a surface of degree at most 4, then $0\leqq g\leqq \frac{1}{10}d^2-\frac{1}{2}d+1$. *(For* $d<9$ *the bound of a) is stronger.)*

d) For $5\leqq d\leqq 8$, *g satisfying the inequality of a) and* $d\geqq 9$, *g satisfying the inequality of c) there exists a nondegenerate smooth irreducible curve in* $\mathbf{P}^5$ *having these values as invariants.*

Our proof is based on techniques of Gruson and Peskine to construct curves on certain surfaces.

Throughout this paper, we work over an algebraically closed field of arbitrary characteristic.

Independently, Pasarescu has been able to construct most of the curves in the ranges given by (0.1) and (0.2) [9]. He uses different surfaces for (0.1 d) and (0.2 d).

For information on what to expect for $n>5$, see the fourth chapter of [10]. Pasarescu's paper also gives some information in this case; however, the results of [10] are stronger. Further details, e.g., the genus of curves in $\mathbf{P}^6$, have been worked out in the meantime by Ciliberto.

This paper is part of my Ph.D. thesis. I would like to thank my advisor Robin Hartshorne for suggesting this problem and many very helpful discussions.

1. The Genus of Curves in $\mathbf{P}^4$

In this chapter we shall prove theorem (0.1). The bounds a) and c) are well known and proved in [5, 3.7] and [5, 3.15]. Eisenbud and Harris work in characteristic 0, but their bounds hold in any characteristic (see [11]).

(0.1 b) is also known [5, 3.16].

In Sect. 1 we examine curves on the Del Pezzo surface which is isomorphic to $\mathbf{P}^2$ blown up in 5 points and we show

Theorem 1.0.1. *For every* $d\geqq 5$ *and every g such that*

$$(d+12)\sqrt{d+9}-\tfrac{11}{2}d-35\leqq g\leqq \tfrac{1}{8}d^2-\tfrac{1}{2}d+1$$

there exists a nondegenerate smooth irreducible curve of degree d and genus g on the Del Pezzo surface in $\mathbf{P}^4$.

In Sect. 2 we examine curves on the Bordiga surface which is isomorphic to $\mathbf{P}^2$ blown up in 10 points. We show

Theorem 1.0.2. *For every* $d \geqq 7$ *and every g such that*

$$0 \leqq g \leqq \tfrac{1}{12} d^2 - \tfrac{1}{6} d - \tfrac{11}{12}$$

there exists a nondegenerate smooth irreducible curve of degree d and genus g on the Bordiga surface in $\mathbf{P}^4$.

Section 3 contains a technical result on a certain quadratic form which is needed for the analysis in the second section.

In Sect. 4 we discuss some additional arguments needed in nonzero characteristic.

1.1. *Curves on the Del Pezzo Surface in* $\mathbf{P}^4$

The Del Pezzo surface X in $\mathbf{P}^4$ can be described [8] as $\mathbf{P}^2$ blown up in 5 points in general position and embedded by the complete linear system of all cubic curves passing through them; it has degree 4. $\mathrm{Pic}(X)$ is a free abelian group with basis $l, -e_1, \ldots, -e_5$ where l is the pullback of the class of a line in $\mathbf{P}^2$ and $e_1, \ldots, e_5$ are the classes of the exceptional divisors corresponding to the points that have been blown up. Let $(a; b_1, \ldots, b_5)$ denote the class of $al - \sum_{i=1}^{5} b_i e_i$ in $\mathrm{Pic}(X)$. The cubic surface in $\mathbf{P}^3$ is isomorphic to $\mathbf{P}^2$ with 6 points blown up. So one can obtain X, as an abstract surface, by blowing down one exceptional line on the cubic surface. Since one knows which divisor classes on the cubic surface contain irreducible nonsingular curves [6, V Ex. 4.8] one can use this information to get [7, p. 303]:

If $a \geqq b_1 + b_2 + b_3$, $b_1 \geqq b_2 \geqq b_3 \geqq b_4 \geqq b_5 \geqq 0$ then the divisor class $(a; b_1, b_2, b_3, b_4, b_5)$ contains an irreducible nonsingular curve with degree $d = 3a - \sum_{i=1}^{5} b_i$ and genus $g = \tfrac{1}{2}\left(a^2 - \sum_{i=1}^{5} b_i^2 - d\right) + 1$ except if $(a; b_1, \ldots, b_5) = (n; n, 0, 0, 0, 0)$ with $n > 1$.

Now one proceeds as in [7].

Introduce new parameters

$$r = a - b_1,$$

$$\alpha_i = \tfrac{1}{2} r - b_i, \quad i = 2, 3, 4, 5.$$

We have $r \in \mathbf{Z}$, $\alpha_i \in \tfrac{1}{2}\mathbf{Z}$.

Then

$$a=\tfrac{1}{2}\left(d+r-\sum_{i=2}^{5}\alpha_i\right),$$

$$b_1=a-r,$$

$$b_i=\tfrac{1}{2}r-\alpha_i, \quad i=2,3,4,5.$$

In order to recover $a, b_i\in\mathbf{Z}$ it is necessary that

$$\alpha_i\equiv\tfrac{1}{2}r \bmod 1, \quad i=2,3,4,5.$$

$$d+r-\sum_{i=2}^{5}\alpha_i\equiv 0 \bmod 2.$$

The inequalities above are now equivalent to

$$|\alpha_2|\leqq\alpha_3\leqq\alpha_4\leqq\alpha_5\leqq\tfrac{1}{2}r,$$

$$-\alpha_2+\alpha_3+\alpha_4+\alpha_5\leqq d-2r,$$

and one computes

$$g=\tfrac{1}{2}\left(d(r-1)-r^2-\sum_{i=2}^{5}\alpha_i^2\right)+1=F_d(r)-\sum_{i=1}^{5}\alpha_i^2$$

with

$$F_d(r)=\tfrac{1}{2}(d(r-1)-r^2)+1.$$

The maximum of $F_d(r)$ is given for $\frac{1}{2}d-r=0$: $F_d(\frac{1}{2}d)=\frac{1}{2}(d(\frac{1}{2}d-1)-\frac{1}{4}d^2)+1$ $=\frac{1}{8}d^2-\frac{1}{2}d+1$. If d is odd, the maximum for $r\in\mathbf{Z}$ is $\frac{5}{8}$ less.

Lemma 1.1.1. *Let d, g, r be integers with $12\leqq d$, $2\sqrt{d+9}-6<r\leqq\frac{1}{2}d$, $F_d(r-1)$ $<g\leqq F_d(r)$. Then there exist $\alpha_i\in\frac{1}{2}\mathbf{Z}$, $i=2, 3, 4, 5$ solving the above equations and inequalities.*

Proof. $F_d(r)-F_d(r-1)=\frac{1}{2}(d-2r+1)$, therefore we must solve $F_d(r)-g=\frac{1}{2}\sum_{i=2}^{5}\alpha_i^2$ with $F_d(r)-g<\frac{1}{2}(d-2r+1)$.

Before we start on the proof we note that $2\sqrt{d+9}-6>\sqrt{2d+\frac{31}{4}}-\frac{5}{2}$ for $d\geqq 12$.

Case 1. r is even; then $F_d(r)\in\frac{1}{2}\mathbf{Z}$, $n=2(F_d(r)-g)\in\mathbf{Z}$, so we have to find $\alpha_i\in\mathbf{Z}$, $|\alpha_i|\leqq\frac{1}{2}r$ solving $n=\sum_{i=2}^{5}\alpha_i^2$.

By assumption $n<d-2r+1$, because of $r>2\sqrt{d+9}-6$, $d+9<\frac{1}{4}(r+6)^2$, $d<\frac{1}{4}r^2+3r$ it follows that $n<\frac{1}{4}r^2+r+1=(\frac{1}{2}r)^2+2(\frac{1}{2}r)+1$.

The well known theorem that every positive integer is a sum of four squares can be reformulated as follows: Let $k>0$ be an integer. For every $n<k^2+2k+1$ there exist nonnegative integers $x_1, \ldots, x_4$ such that $|x_i|\leqq k$ and $n=\sum_{i=1}^{4} x_i^2$.

Using this we find the α_i and compute a, b_i. We still have to verify the congruences and inequalities:

$|\alpha_2|\leqq\alpha_3\leqq\alpha_4\leqq\alpha_5$: take the right order of the α_i's.

$\alpha_i\equiv\frac{1}{2}r \bmod 1$: since r is even, $\alpha_i\in\mathbf{Z}$.

$d+r-\sum_{i=2}^{5}\alpha_i\equiv 0 \bmod 2$: $g\in\mathbf{Z}$, therefore $\frac{1}{2}\left(d(r-1)-r^2-\sum_{i=2}^{5}\alpha_i^2\right)\in\mathbf{Z}$, $d(r-1)-r^2-\sum_{i=2}^{5}\alpha_i^2\equiv 0 \bmod 2$, $d-\sum_{i=2}^{5}\alpha_i\equiv 0 \bmod 2$.

$d-2r\geqq-\alpha_2+\alpha_3+\alpha_4+\alpha_5$: $\sum_{i=2}^{5}\alpha_i^2=n<2(F_d(r)-F_d(r-1))=d-2r-1$, therefore $-\alpha_2+\alpha_3+\alpha_4+\alpha_5\leqq\sum_{i=2}^{5}\alpha_i^2\leqq d-2r$.

Case 2. r is odd; then $F_d(r)\in\frac{1}{2}\mathbf{Z}-\mathbf{Z}$, $n=8(F_d(r)-g)\in\mathbf{Z}$, so we have to find $\alpha_i\in\frac{1}{2}\mathbf{Z}-\mathbf{Z}$ such that $|\alpha_i|\leqq\frac{1}{2}r$, $n=\sum_{i=2}^{5}\alpha_i^2$.

Note that $n\equiv 4 \bmod 8$, so we need odd integers $\beta_i=2\alpha_i$ such that $\sum_{i=2}^{5}\beta_i^2=8(F_d(r)-g)$; by assumption $n<4(d-2r+1)$, because of $\sqrt{2d+\frac{31}{4}}-\frac{5}{2}<r$, $2d+\frac{31}{4}<(r+\frac{5}{2})^2$, $4d<2r^2+10r-3$ it follows that $n<2r^2+2r+1$.

Now imitate the proof of [7, 2.2] to show: Let $k>0$ be an odd integer. For every $n<2k^2+2k+1$, $n\equiv 4 \bmod 8$ there are nonnegative odd integers $x_1, \ldots, x_4$ such that $|x_i|\leqq k$, $n=\sum_{i=1}^{4}x_i^2$.

Using this we get the β_i and the α_i; we still have to verify the congruences and inequalities:

$|\alpha_2|\leqq\alpha_3\leqq\alpha_4\leqq\alpha_5$: take the right order of the α_i's;

$\alpha_i\equiv\frac{1}{2}r \bmod 1$: r is odd, $\beta_i=2\alpha_i$ also;

$d+r-\sum_{i=2}^{5}\alpha_i\equiv 0 \bmod 2$: if not, change the sign of α_2

$d-2r\geqq-\alpha_2+\alpha_3+\alpha_4+\alpha_5$: $\sum_{i=2}^{5}(2\alpha_i)^2=n<4(d-2r+1)$, $n\equiv 0 \bmod 4$, therefore $n\leqq 4(d-2r)$; $-\alpha_2+\alpha_3+\alpha_4+\alpha_5\leqq\sum_{i=2}^{5}\alpha_i^2+1\leqq d-2r+1$. The inequality is thus only violated if $(\alpha_2, \ldots, \alpha_5)=(-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$; in this case one computes $a=\frac{1}{2}(3r-1)$, $b_1=\frac{1}{2}(r-1)=b_3=b_4=b_5$, $b_2=\frac{1}{2}(r+1)$, so that by exchanging b_1 and b_2 one reaches a permitted solution with the same values of d and g.

Proof of (1.0.1). For $d<12$ use the following table

$(d, g)=$ (6, 2): (4; 2, 1, 1, 1, 1)
$(d, g)=$ (7, 3): (4; 1, 1, 1, 1, 1)
$(d, g)=$ (8, 4): (5; 2, 2, 1, 1, 1)
$(d, g)=$ (8, 5): (6; 2, 2, 2, 2, 2)
$(d, g)=$ (9, 5): (5; 2, 1, 1, 1, 1)
$(d, g)=$ (9, 6): (6; 2, 2, 2, 2, 1)
$(d, g)=$(10, 6): (5; 1, 1, 1, 1, 1)
$(d, g)=$(10, 7): (6; 2, 2, 2, 1, 1)
$(d, g)=$(10, 8): (7; 3, 2, 2, 2, 2)
$(d, g)=$(11, 8): (6; 2, 2, 1, 1, 1)
$(d, g)=$(11, 9): (7; 3, 2, 2, 2, 1)
$(d, g)=$(11, 10): (7; 2, 2, 2, 2, 2)

For $d\geqq 12$ choose r_0 as the smallest integer above $2\sqrt{d+9}-6$; according to (1.1.1), for every integer g with $F_d(r-1)<g\leqq \frac{1}{8}d^2-\frac{1}{2}d+1$ there exists a smooth irreducible curve of degree d and genus g on the Del Pezzo surface in $\mathbf{P}^4$.

Let $\rho=\lfloor\sqrt{d+9}\rfloor-5$, then $r_0<\rho$, therefore

$$\begin{aligned}F_d(r_0-1)<F_d(\rho-1)&=\tfrac{1}{2}(d(2\lfloor\sqrt{d+9}\rfloor-7)-(2\lfloor\sqrt{d+9}\rfloor-6)^2)+1\\&=\tfrac{1}{2}(2d\lfloor\sqrt{d+9}\rfloor-7d-(4(d+9)-24\lfloor\sqrt{d+9}\rfloor+36))+1\\&=\tfrac{1}{2}((2d+24)\lfloor\sqrt{d+9}\rfloor-7d-4d-72)+1\\&=(d+12)\lfloor\sqrt{d+9}\rfloor-\tfrac{11}{2}d-35.\end{aligned}$$

1.2. Curves on the Bordiga Surface

The next candidate of a surface where we should look for curves should be a surface of degree 5. From the list of [8] it is clear that the Castelnuovo surface must be chosen. The cubic surface in $\mathbf{P}^3$ can be viewed as the Del Pezzo surface in $\mathbf{P}^4$ under a projection from a point on the surface, and in the same way the Castelnuovo surface yields the singular quartic surface which Gruson and Peskine use. However, since we want to apply the methods of [3], we would prefer a rational surface with at least 9 points blown up from $\mathbf{P}^2$. Therefore we propose to study curves on the Bordiga surface S in $\mathbf{P}^4$. It can be described as $\mathbf{P}^2$ blown up in 10 points $P_1, \ldots, P_{10}$ in general position and embedded in $\mathbf{P}^4$ by the linear system of all quartics passing through those points. This surface has degree 6.

$\mathrm{Pic}(S)$ is a free abelian group with basis l (the pullback of the class of a line in $\mathbf{P}^2$) and $-e_1, \ldots, -e_{10}$ where $e_1, \ldots, e_{10}$ are the classes of the exceptional lines corresponding to the points that have been blown up. Let $(a; b_1, \ldots, b_{10})$ denote the class of $al-\sum_{i=1}^{10} b_i$. Let Γ_0 be the divisor class of the unique cubic passing through $P_1, \ldots, P_9$. This cubic has genus 1; the divisor class of its strict

transform Γ is $(3; 1, \ldots, 1)$. Let $A=(1; 0, \ldots, 0, 1)$, $D=(1; 0, \ldots, 0)$, $E=(0; -1, -1, -1, 0, \ldots, 0)$, $F=(1; 1, -1, 0, \ldots, 0)$. The hyperplane section is the divisor $H=\Gamma+A$ and the canonical divisor is $K_S=(-3; -1, \ldots, -1)$. The intersection form on $\mathrm{Pic}(S)$ gives $\Gamma^2=0$, $A^2=0$, $A.\Gamma=D.\Gamma=E.\Gamma=F.\Gamma=3$, $D^2=1$, $E^2=-3$, $F^2=-1$.

Let T be the surface obtained by blowing down e_{10}. T is isomorphic to $\mathbf{P}^2$ blown up in 9 points, $\mathrm{Pic}(T)$ is a free abelian group with basis l (the pullback of the class of a line in $\mathbf{P}^2$) and $-e_1, \ldots, -e_9$. By abuse of notation we denote $\Gamma=(3; 1, \ldots, 1)$, $D=(1; 0, \ldots, 0)$, $E=(0; -1, -1, -1, 0, \ldots, 0)$, $F=(1; 1, -1, 0, \ldots, 0)$ as on S. $K_T=-\Gamma$ is the canonical divisor of T.

Proposition 1.2.1. *Let T be as defined above.*

a) For each $n\geqq 0$ the linear system $|D+n\Gamma|$ contains an irreducible nonsingular curve of genus $g=3n$.

b) For each $n\geqq 1$ the linear system $|E+n\Gamma|$ contains an irreducible nonsingular curve of genus $g=3n-2$.

c) For each $n\geqq 1$ the linear system $|F+n\Gamma|$ contains an irreducible nonsingular curve of genus $g=3n-1$.

Proof. a) We show by induction on n that $|D+n\Gamma|$ has no base points and $h^1\mathcal{O}(D+n\Gamma)=0$:

$n=0$: $|D|$ has no base points, $H^0\mathcal{O}(D)=3$, $H^1\mathcal{O}(D)=0$ follows from Riemann-Roch [6, V 1.6]: $h^0\mathcal{O}(D)-h^1\mathcal{O}(D)+h^2\mathcal{O}(D)=\frac{1}{2}D.(D-K)+1+\chi\mathcal{O}_T$, where $h^2\mathcal{O}(D)=h^0\mathcal{O}(K-D)=0$, $\chi\mathcal{O}_T=\chi\mathcal{O}_{\mathbf{P}^2}=0$ by [6, V 3.5]. $D.(D-K)=D.D+D.\Gamma=1+3=4$, so $3-h^1\mathcal{O}(D)=2+1$, $h^1\mathcal{O}(D)=0$.

$n>0$: The short exact sequence

$$0\to\mathcal{O}(D+(n-1)\Gamma)\to\mathcal{O}(D+n\Gamma)\to\mathcal{O}_\Gamma(D+n\Gamma)\to 0$$

yields the exact sequence

$$\begin{aligned}0\to H^0\mathcal{O}(D+(n-1)\Gamma)&\to H^0\mathcal{O}(D+n\Gamma)\to H^0\mathcal{O}_\Gamma(D+n\Gamma)\\&\to H^1\mathcal{O}(D+(n-1)\Gamma)\to H^1\mathcal{O}(D+n\Gamma)\to H^1\mathcal{O}_\Gamma(D+n\Gamma).\end{aligned}$$

By [6, IV 1.3.4] the divisor $D+n\Gamma$ restricted to Γ is nonspecial since its degree is $(D+n\Gamma).\Gamma=3$, therefore $h^0\mathcal{O}_\Gamma(D+n\Gamma)=3$, $h^1\mathcal{O}_\Gamma(D+n\Gamma)=0$. By assumption $h^1\mathcal{O}(D+(n-1)\Gamma)=0$, so $h^1\mathcal{O}(D+n\Gamma)=0$. As $|D+(n-1)\Gamma|$ has no base points by assumption, any base point of $|D+n\Gamma|$ must lie on Γ. But by [6, IV 3.2] $D+n\Gamma$ is even very ample on Γ. In characteristic 0, Bertini's theorem [6, III 10.9] now implies that there is a nonsingular curve in $|D+n\Gamma|$. For nonzero characteristic the arguments are given in Sect. 4.

Let $\mathcal{I}_C=\mathcal{O}(-D-n\Gamma)$ be the ideal sheaf of C. If $h^1\mathcal{I}_C=0$, then the cohomology sequence of $0\to\mathcal{I}_C\to\mathcal{O}\to\mathcal{O}_C\to 0$ forces $h^0\mathcal{O}_C=1$ and C must be connected and hence irreducible.

This is done by Riemann-Roch: $h^0\mathcal{O}(-D-n\Gamma)-h^1\mathcal{O}(-D-n\Gamma)+h^2\mathcal{O}(-D-n\Gamma)=\frac{1}{2}(-D-n\Gamma).(-D-n\Gamma-K)+1$; now $h^0\mathcal{O}(-D-n\Gamma)=0$, $h^2\mathcal{O}(-D-n\Gamma)=h^0\mathcal{O}(D+(n+1)\Gamma)=3n$ follows from the induction above;

$(-D-n\Gamma).(-D-(n-1)\Gamma)=D^2+(2n-1)D.\Gamma+n(n-1)\Gamma^2=1+6n-3=6n-2$, so $-h^1\mathcal{O}(-D-n\Gamma)+3n=3n-1+1=3n$ and $h^1\mathcal{O}(-D-n\Gamma)=h^1\mathcal{I}_C=0$.

The genus of C is computed by the adjunction formula: $2g-2=(D+n\Gamma).(D+(n+1)\Gamma)=D+(2n-1)D.\Gamma=6n-2$; $g=3n$.

b) The proof proceeds analogously. For the induction to work we have to show that $|E+\Gamma|$ has no base points and $h^1\mathcal{O}(E+\Gamma)=0$:

$E+\Gamma$ has no basepoints since the image of the associated rational map is the cubic surface in $\mathbf{P}^3$; $h^0\mathcal{O}(E+\Gamma)-h^1\mathcal{O}(E+\Gamma)+h^2\mathcal{O}(E+\Gamma)=\frac{1}{2}(E+2\Gamma).(E+2\Gamma)+1$ where $h^0\mathcal{O}(E+\Gamma)=4$, $h^2\mathcal{O}(E+\Gamma)=h^0\mathcal{O}(-E-2\Gamma)=0$, $(E+\Gamma).(E+2\Gamma)=-3+9=6$, $4-h^1\mathcal{O}(E+\Gamma)=3+1=4$, therefore $h^1\mathcal{O}(E+\Gamma)=0$.

Then let C be a nonsingular curve in $|E+n\Gamma|$, $n\geqq 1$. $h^0\mathcal{O}(-n\Gamma-E)-h^1\mathcal{O}(-n\Gamma-E)+h^2\mathcal{O}(-n\Gamma-E)=\frac{1}{2}(-E-n\Gamma).(-E-n\Gamma+\Gamma)+1-h^1\mathcal{O}(-n\Gamma-E)+3n-2=\frac{1}{2}(-3+(2n-1)3)+1=3n-2$ so $h^1\mathcal{O}(-n\Gamma-E)=h^1\mathcal{I}_C=0$ and C is irreducible. $2g-2=(E+n\Gamma).(E+(n-1)\Gamma)=E^2+(2n-1)E.\Gamma=6n-6$, $g=3n-2$.

c) Works like b): $|F+\Gamma|$ has no base points since this linear system maps T onto the Castelnuovo surface in $\mathbf{P}^4$ [8];

$h^0\mathcal{O}(F+\Gamma)-h^1\mathcal{O}(F+\Gamma)+h^2\mathcal{O}(F+\Gamma)=\frac{1}{2}(F+\Gamma).(F+2\Gamma)+1$, where $h^0\mathcal{O}(F+\Gamma)=5$, $h^2\mathcal{O}(F+\Gamma)=h^0\mathcal{O}(-F-2\Gamma)=0$, $(F+\Gamma).(F+2\Gamma)=-1+9=8$, $5-h^1\mathcal{O}(E+\Gamma)=4+1=5$, therefore $h^1\mathcal{O}(F+\Gamma)=0$. Let C be a nonsingular curve in $|F+n\Gamma|$, $n\geqq 1$. $h^0\mathcal{O}(-n\Gamma-F)-h^1\mathcal{O}(-n\Gamma-F)+h^2\mathcal{O}(-n\Gamma-F)=\frac{1}{2}(-F-n\Gamma).(-F-n\Gamma+\Gamma)+1$, $-h^1\mathcal{O}(-n\Gamma-F)+3n-1=\frac{1}{2}(-1+(2n-1)3)+1=3n-1$; so $h^1\mathcal{O}(-n\Gamma-F=h^1\mathcal{I}_C=0$ and C is irreducible. $2g-2=(F+n\Gamma).(F+(n-1)\Gamma)=F^2+(2n-1)F.\Gamma=6n-4$, $g=3n-1$.

Theorem 1.2.2. *For every g with $0\leqq g\leqq d-5$ there exists an irreducible nonsingular curve of degree d and genus g on the Bordiga surface S.*

Proof. Let G be the group of isometries of $\mathrm{Pic}(T)$ that leave Γ fixed. As in the proof of [7, 3.5] it follows that for every $\sigma\in G$ the divisor classes $\sigma(D+n\Gamma)$, $\sigma(E+n\Gamma)$, $\sigma(F+n\Gamma)$ also contain irreducible nonsingular curves of genus $3n$, $3n-2$, $3n-1$. We can assume that these curves do not pass through P_{10}. After blowing up P_{10} and mapping S into $\mathbf{P}^4$ we can compute the degrees of the curves:

Case 1. $g=3n$, $n\geqq 0$: The degree of the curve C in $|\sigma(\mathrm{D}+\mathrm{n}\Gamma)|$ is

$$\begin{aligned} d&=H.\sigma(D+n\Gamma)=(A+\Gamma).(\sigma(D)+n\Gamma)\\ &=A.\sigma(D)+n(A.\Gamma)+\Gamma.\sigma(D)=A.\sigma(D)+3n+3\\ &=A.\sigma(D)+3+g\\ &=D.\sigma(D)+3+g. \end{aligned}$$

We can replace A by D because the last component of $\sigma(D)$ must be 0. The result now follows from (1.3.2a) below.

Case 2. $g=3n-2$, $n\geqq 1$:

$$d=H.\sigma(E+n\Gamma)=A.\sigma(E)+3n+3=D.\sigma(E)+5+g,$$

and now apply (1.3.2 b) below.

Case 3. $g=3n-1$, $n\geqq 1$:

$$d=H.\sigma(F+n\Gamma)=A.\sigma(F)+3n+3=D.\sigma(F)+4+g,$$

and now apply (1.3.2 c) below.

Note that in the first case $g=d-4$ can also be constructed.

Proof of (1.0.2). For integers d_0, g_0 with $0\leqq g_0\leqq d_0-5$ a curve Y_0 can be found according to (1.2.2).

For $r\geqq 1$ consider the linear system $|Y_0+r(A+\Gamma)|$ on S. It is very ample because $|A+\Gamma|$ is very ample and $|Y_0|$ has no base points. Therefore it contains an irreducible nonsingular curve [6, II 8.18]. One computes

$$\begin{aligned} d&=H.(Y_0+r(A+\Gamma))=d_0+(A+\Gamma).r(A+\Gamma)\\ &=d_0+6r,\end{aligned}$$

$$\begin{aligned} 2g-2&=(Y_0+r(A+\Gamma)).(Y_0+r(A+\Gamma)+K_S)\\ &=2g_0-2+2\,Y_0.r(A+\Gamma)+r^2(A+\Gamma)^2+r(A+\Gamma).K_S\\ &=2g_0-2+2rd_0+6r^2-2r\end{aligned}$$

$$\begin{aligned} g&=g_0+rd_0+3r^2-r\\ &=g_0+r(d-3r-1)\end{aligned}$$

and the inequality $0\leqq g_0\leqq d_0-5$ is equivalent to

$$\begin{aligned} r(d-3r-1)\leqq g_0+r(d-3r-1)&\leqq d-6r-5+r(d-3r-1)\\ &=(r+1)(d-3(r+1)-1)-1.\end{aligned}$$

If we define $F_d(r)=r(d-3r-1)$ this is equivalent to $F_d(r)\leqq g\leqq F_d(r+1)-1$. The maximum value of $F_d(r)$ for real r is attained for $d-6r-1=0$, $r=\dfrac{d-1}{6}$.

$$F_d\left(\frac{d-1}{6}\right)=\frac{d-1}{6}\left(d-\frac{d-1}{2}-1\right)=\frac{d-1}{6}\,\frac{d-1}{2}=\frac{1}{12}d^2-\frac{1}{6}d+\frac{1}{12},$$

and a short calculation shows that for integers r the maximum is the greatest integer not exceeding that number.

Remark 1.2.3. *The upper bound in* (1.0.2) *is not sharp; for example, if* $d\equiv 1 \bmod 6$, *then* $g_0=F_d(r+1)-F_d(r)\equiv 0 \bmod 3$ *and the proof of* (1.2.2, *case* 1) *shows that an irreducible nonsingular curve with* $g=\frac{1}{12}d^2-\frac{1}{6}d+\frac{1}{12}$ *exists on* S. *An irreducible nonsingular curve in the linear system* $|r(A+\Gamma)|$ *even has* $d=6r$, $g=3r^2-r+1$ $=\frac{1}{12}d^2-\frac{1}{6}d+1>\max F_d(r)$.

Proof of (0.1 d). For $4\leqq d\leqq 6$ one immediately finds the curves on the rational normal scroll.

We still have to verify that for $d \geqq 7$ every integer g between 0 and $\frac{1}{8}d^2 - \frac{1}{2}d + 1$ satisfies the hypotheses of (1.0.1) or (1.0.2).

For $7 \leqq d \leqq 11$ this shows a direct calculation of the bounds while for $d \geqq 12$ we prove that $(d+12)\sqrt{d+9} - \frac{11}{2}d - 35 < \frac{1}{12}d^2 - \frac{1}{6}d - \frac{11}{12}$.

One way is as follows: for $d=12$ this is certainly true, and we can prove the same inequality for the derivatives on both sides (as differentiable functions of d). The latter means

$$\sqrt{d+9} + (d+12)\frac{1}{\sqrt{d+9}} - \frac{11}{2} < \frac{1}{6}d - \frac{1}{6},$$

$$3d + 30 < (\tfrac{1}{3}d + \tfrac{32}{3})\sqrt{d+9}, \qquad 9d + 90 < (d+32)\sqrt{d+9}.$$

Repeating this argument we arrive at

$$9 < \sqrt{d+9} + (d+32)\frac{1}{2\sqrt{d+9}}, \qquad 18\sqrt{d+9} < 3d + 50.$$

In the next step we get $9\frac{1}{\sqrt{d+9}} < 3$, $3 < \sqrt{d+9}$, which is true for $d > 0$.

1.3. A Technical Result

Let $V = \mathbf{Z}^{10}$ with canonical basis $e_0, \ldots, e_9$, let Q be the quadratic form $x_0^2 - \sum_{i=1}^{9} x_i^2$ on V with "." the associated inner product; this means $e_0^2 = 1$, $e_i^2 = -1$ for $i = 1, \ldots, 9$ and $e_i, e_j = 0$ for $i \neq j$. Let $\omega = (-3; -1, \ldots, -1)$; $\omega^2 = 0$; let H be the hyperplane of elements orthogonal to ω. It is well known [7, Ch. 4] that there are elements $r_1 = (1, 1, 1, 0, \ldots, 0)$, $r_2 = (0, -1, 1, 0, \ldots, 0)$, $r_3 = (0, 0, -1, 1, 0, \ldots, 0)$, $\ldots$, $r_8 = (0, \ldots, 0, -1, 1, 0)$ such that:

$\omega, r_1, \ldots, r_8, e_9$ is a basis for V
$\omega, r_1, \ldots, r_8$ is a basis for H
$r_i^2 = -2$,
$r_i, r_j = 0$ except if $\{i, j\} \in \{\{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\}, \{6, 7\}, \{7, 8\}, \{1, 4\}\}$,

and in that case $r_i, r_j = 1$.

Let G be the group of isometries of V leaving ω fixed. It is well known that G is generated by the following elements [7, Ch. 4]:

$$f_i \quad i = 1, \ldots, 8: \qquad f_i(x) = x + (x.r_i)r_i$$

$$\tau_z, z \in H \qquad \tau_z(x) = x - (x.z)\omega + (x.\omega)z - \tfrac{1}{2}(z.z)(x.\omega)\omega.$$

Note that the full permutation group of the last nine components lies in G.

Let $d=(1, 0, \ldots, 0)$ and compute some elements of its orbit under G:

$$
\begin{aligned}
&f_1(1,0,0,0,0,0,0,0,0,0)=(2,1,1,1,0,0,0,0,0,0) && d_1\\
&f_1(2,1,0,0,1,1,0,0,0,0)=(3,2,1,1,1,1,0,0,0,0) && d_2\\
&f_1(3,1,1,0,2,1,1,0,0,0)=(4,2,2,1,2,1,1,0,0,0) && d_3\\
&f_1(4,1,1,1,2,2,2,0,0,0)=(5,2,2,2,2,2,2,0,0,0) && d_4\\
&f_1(3,2,0,0,1,1,1,1,0,0)=(4,3,1,1,1,1,1,1,0,0) && d_5\\
&f_1(5,2,2,0,2,2,2,2,0,0)=(6,3,3,1,2,2,2,2,0,0) && d_6\\
&f_1(4,1,1,0,3,1,1,1,1,0)=(6,3,3,2,3,1,1,1,1,0) && d_7\\
&f_1(3,1,0,0,2,1,1,1,0,0)=(5,3,2,2,2,1,1,1,0,0) && d_8\\
&f_1(5,2,1,0,3,2,2,1,1,0)=(7,4,3,2,3,2,2,1,1,0) && d_9\\
&f_1(6,2,2,1,3,3,2,2,0,0)=(7,3,3,2,3,3,2,2,0,0) && d_{10}\\
&f_1(7,3,3,0,3,3,2,2,2,0)=(8,4,4,1,3,3,2,2,2,0) && d_{11}\\
&f_1(8,3,3,1,2,4,4,2,2,0)=(9,4,4,2,2,4,4,2,2,0) && d_{12}
\end{aligned}
$$

Lemma 1.3.1. *Let $x, y\in V$ such that $x.\omega=y.\omega=-3$ and $y-x=u=\sum_{i=0}^{9} u_i e_i$ with $u_1=u_2$, $u_3=u_4$, $u_5=u_6$, $u_7=u_8$. Then $y.\sigma(x)$ assumes all integers $9n+x.y$, $n\geqq 0$ as σ varies in G.*

Proof. Let $z\in H$, $z=\lambda\omega+\sum_{i=1}^{8}\lambda_i r_i=\sum_{i=0}^{9} z_i e_i$

$$
\begin{aligned}
y.\tau_z(x)&=y.[x-(x.z)\omega+(x.\omega)z-\tfrac{1}{2}(z.z)(x.\omega)\omega]\\
&=(y.x)-(x.z)(y.\omega)+(x.\omega)(y.z)-\tfrac{1}{2}(z.z)(x.\omega)(y.\omega)\\
&=(y.x)+3(x-y).z-\tfrac{9}{2}(z.z)
\end{aligned}
$$

so we try to find $z\in H$ such that $(x-y).z=0$ and $-\frac{1}{2}(z.z)=n$.

We have

$$
\begin{aligned}
&(z_0, z_1, \ldots, z_9)\\
&=(-3\lambda+\lambda_1, -\lambda+\lambda_1-\lambda_2, -\lambda+\lambda_1+\lambda_2-\lambda_3, -\lambda+\lambda_1+\lambda_3-\lambda_4, -\lambda+\lambda_4-\lambda_5,\\
&\quad -\lambda+\lambda_5-\lambda_6, -\lambda+\lambda_6-\lambda_7, -\lambda+\lambda_7-\lambda_8, -\lambda+\lambda_8, -\lambda),
\end{aligned}
$$

so $u.z=0$ is equivalent to

$$
\begin{aligned}
0&=u_0 z_0-u_1(z_1+z_2)-u_3(z_3+z_4)-u_5(z_5+z_6)-u_7(z_7+z_8)-u_9 z_9\\
&=u_0(-3\lambda+\lambda_1)-u_1(-2\lambda+2\lambda_1-\lambda_3)-u_3(-2\lambda+\lambda_1+\lambda_3-\lambda_5)\\
&\quad -u_5(-2\lambda+\lambda_5-\lambda_7)-u_7(-2\lambda+\lambda_7)-u_9(-\lambda),
\end{aligned}
$$

therefore $\lambda_1=\lambda_3=\lambda_5=\lambda_7=\lambda=0$ solves $u.z=0$ and we have

$$
-\tfrac{1}{2}(\lambda_2 r_2+\lambda_4 r_4+\lambda_6 r_6+\lambda_8 r_8)^2=\lambda_2^2+\lambda_4^2+\lambda_6^2+\lambda_8^2.
$$

The conclusion follows because every positive integer is a sum of four squares.

Proposition 1.3.2. *Let* $d=(1, 0, \ldots, 0)$, $e=(0, -1, -1, -1, 0, \ldots, 0)$, *and let* $f=(1, -1, 1, 0, \ldots, 0)$.

a) $d.\sigma(d)$ *assumes all integers* $\geqq 1$ *as* σ *varies in G.*

b) $d.\sigma(e)$ *assumes all integers* $\geqq 0$ *as* σ *varies in G.*

c) $d.\sigma(f)$ *assumes all integers* $\geqq 1$ *as* σ *varies in G.*

Proof. a) It is sufficient to prove that for every integer $n \geqq 1$ there are $\sigma_1, \sigma_2 \in G$ such that $\sigma_1(d).\sigma_2(d)=n$. We distinguish 9 cases according to n mod 9 and apply (1.3.1) in each case. We use the elements $d_1, \ldots, d_{12}$ that we computed before (1.3.1). The notation following a given element $\sigma_1(d)$ indicates from which of the d_i (computed before (1.3.1)) it can be obtained as a permutation of the last nine components.

1.	$n=9k+1$:	$\sigma_1(d)=d$,	$\sigma_2(d)=d$
2.	$9k+2$:	$(2, 1, 1, 0, 0, 0, 0, 0, 0, 1)$	d_1, d
3.	$9k+3$:	$(3, 1, 1, 1, 1, 0, 0, 0, 0, 2)$	d_2, d
4.	$9k+4$:	$(4, 0, 0, 1, 1, 1, 1, 1, 1, 3)$	d_5, d
5.	$9k+5$:	$(5, 2, 2, 2, 2, 2, 2, 0, 0, 0)$	d_4, d
6.	$9k+6$:	$(6, 3, 3, 2, 2, 2, 2, 0, 0, 1)$	d_6, d
7.	$9k+7$:	$(7, 3, 3, 3, 3, 2, 2, 0, 0, 2)$	d_{10}, d
8.	$9k+8$:	$(8, 3, 3, 2, 1, 4, 4, 2, 2, 0)$	d_{11}, $(2, 1, 1, 1, 0, \ldots, 0)=d_1$
9.	$9k+9$:	$(9, 4, 4, 2, 2, 4, 4, 2, 2, 0)$	d_{12}, d

b) we proceed as in a)

1.	$9k$:	$(2, 0, 0, 0, 1, 1, 1, 0, 0, 0)$	d_1, e
2.	$9k+1$:	$(4, 0, 0, 1, 2, 2, 2, 1, 1, 0)$	d_3, e
3.	$9k+2$:	$(2, 1, 1, 0, 1, 0, 0, 0, 0, 0)$	d_1, e
4.	$9k+3$:	$(4, 1, 1, 1, 2, 2, 2, 0, 0, 0)$	d_3, e
5.	$9k+4$:	$(6, 1, 1, 2, 3, 3, 3, 1, 1, 0)$	d_7, e
6.	$9k+5$:	$(4, 2, 2, 1, 2, 1, 1, 0, 0, 0)$	d_3, e
7.	$9k+6$:	$(6, 3, 3, 0, 1, 2, 2, 2, 2, 0)$	d_6, e
8.	$9k+7$:	$(8, 3, 3, 1, 2, 4, 4, 2, 2, 0)$	d_{11}, e
9.	$9k+8$:	$(6, 3, 3, 2, 3, 1, 1, 1, 1, 0)$	d_7, e

c) we proceed as in a); first compute $e_2=f_1(1, 1, -1, 0, \ldots, 0)$ $=(2, 2, 0, 1, 0, \ldots, 0)$, $e_3=f_1(2, 1, 0, 0, 2, 0, \ldots, 0)=(3, 2, 1, 1, 2, 0, \ldots, 0)$.

1.	$9k+1$:	$(3, 2, 0, 1, 1, 1, 1, 0, 0, 0)$	d_2, f
2.	$9k+2$:	$(5, 3, 2, 0, 1, 2, 1, 1, 1, 0)$	d_8, $(1, 1, 0, -1, 0, \ldots, 0)=f_3(e_2)$
3.	$9k+3$:	$(3, 1, 0, 1, 2, 1, 1, 0, 0, 0)$	d_2, $f_3(f)=(1, 1, 0, -1, 0, \ldots, 0)$
4.	$9k+4$:	$(2, 1, 1, 0, 0, 0, 0, 0, 0, 1)$	d_1, e_3
5.	$9k+5$:	$(7, 4, 2, 3, 3, 2, 2, 1, 1, 0)$	d_9, f
6.	$9k+6$:	$(5, 1, 0, 2, 3, 1, 1, 2, 2, 0)$	d_8, $f_3(f)$
7.	$9k+7$:	$(7, 3, 2, 3, 4, 2, 2, 1, 1, 0)$	d_9, $f_3(f)$
8.	$9k+8$:	$(4, 1, 1, 1, 1, 1, 1, 0, 0, 3)$	d_5, e_3
9.	$9k+9$:	$(3, 0, 0, 0, 0, 1, 1, 1, 1, 2)$	d_2, e_3

1.4. The Case of Nonzero Characteristic

In (1.2.1) we used Bertini's theorem to find a nonsingular curve in a basepoint free linear system. This theorem is only valid over a field of characteristic 0. In this chapter we want to show that the linear systems we used for the construction of curves on the Bordiga surface also contain nonsingular curves, if the characteristic of the field k is not 0.

In [7] Hartshorne introduces the notion of an almost very ample linear system and shows that every almost very ample linear system contains a nonsingular curve regardless of char(k). Let us recall his definition:

For a smooth variety X and a morphism $f: X \to \mathbf{P}^n$ let $T_{f,x}$: $T_{X,x} \to T_{\mathbf{P}^n, f(x)}$ be the induced map of the tangent spaces and define $\Sigma_i = \{x \in X: \dim(\mathrm{Ker}(T_{f,x})) \geqq i\}$. Now a basepoint free linear system is called *almost very ample* if the associated morphism $X \to \mathbf{P}^n$ has the property that codim $(\Sigma_i, X) \geqq i$ for all $i > 0$.

We proceed to apply this in our case. Note that we only have to consider $|D+n\Gamma|$, $n \geqq 0$, $|E+n\Gamma|$, $n \geqq 1$, $|F+n\Gamma|$, $n \geqq 1$ on T.

a) $D+n\Gamma$: $n=0$: $|D|$ gives the map $T \to \mathbf{P}^2$ that blows down the exceptional lines $e_1, \ldots, e_9$, therefore $\Sigma_1 \subset e_1 \cup \ldots \cup e_9$, $\Sigma_2 = \varnothing$;

$n=1$: $|D+n\Gamma|$ is even very ample on T;

$n>1$: Since $|D+\Gamma|$ is very ample, we must have $\Sigma_1 \subset \Gamma$, since $D+n\Gamma$ restricted to Γ is very ample we have $\Sigma_2 = \varnothing$.

b) $|E+n\Gamma|$: $n=1$: $|E+\Gamma|$ gives the map $T \to \mathbf{P}^3$ that blows down the exceptional lines e_1, e_2, e_3 yielding the cubic surface in $\mathbf{P}^3$, therefore $\Sigma_1 \subset e_1 \cup e_2 \cup e_3$, $\Sigma_2 = \varnothing$.

$n>1$ follows as above.

c) $|F+n\Gamma|$: $n=1$: $|F+\Gamma|$ gives the map $T \to \mathbf{P}^4$ that blows down $\mathbf{P}^2$ yielding the Castelnuovo surface in $\mathbf{P}^4$, therefore $\Sigma_1 \subset e_2$, $\Sigma_2 = \varnothing$.

$n>1$ follows as above.

2. The Genus of Curves in $\mathbf{P}^5$

This chapter is devoted to a proof of theorem (0.2). The inequalities a), c) are proved in [5, 3.7] and [5, 3.15], b) can be found in [5, 3.16].

In order to establish d) we study curves on the Del Pezzo and the Bordiga surface in $\mathbf{P}^5$.

2.1. Curves on the Del Pezzo Surface in $\mathbf{P}^5$

The Del Pezzo surface X in $\mathbf{P}^5$ can be defined as $\mathbf{P}^2$ blown up in 4 points in general position and embedded by the complete linear system of all cubic passing through those points. It has degree 5. Pic(X) is a free abelian group with basis l, $-e_1$, $-e_2$, $-e_3$, $-e_4$ where l is the pullback of the class of a line in $\mathbf{P}^2$, e_1, e_2, e_3, e_4 are the classes of the exceptional divisors corresponding

to the points that have been blown up. Let $(a; b_1 \ldots b_4)$ denote the class of $al - \sum_{i=1}^{5} b_i e_i$ in $\mathrm{Pic}(X)$. Now it is well known that for $a \geqq b_1 + b_2 + b_3$, $b_1 \geqq b_2 \geqq b_3 \geqq b_4 \geqq 0$ the divisor class $(a; b_1 \ldots b_4)$ contains an irreducible nonsingular curve of degree $d = 3a - \sum_{i=1}^{4} b_i$ and genus $g = \frac{1}{2}\left(a^2 - \sum_{i=1}^{4} b_i^2 - d\right) + 1$ unless $(a; b_1 \ldots b_4) = (n; n, 0, 0, 0)$ for some $n > 1$.

Define

$$r = a - b_1$$

$$\alpha_i = \tfrac{1}{2} r - b_i, \qquad i = 2, 3, 4$$

We have $r \in \mathbf{Z}$, $\alpha_i \in \frac{1}{2}\mathbf{Z}$. Then

$$a = \tfrac{1}{2}\left(d + \tfrac{1}{2} r - \sum_{i=2}^{4} \alpha_i\right)$$

$$b_1 = a - r$$

$$b_i = \tfrac{1}{2} r - \alpha_i, \qquad i = 2, 3, 4$$

To have $a, b_i \in \mathbf{Z}$ it is necessary that

$$\alpha_i \equiv \tfrac{1}{2} r \bmod 1, \qquad i = 2, 3, 4$$

$$d + \tfrac{1}{2} r - \sum_{i=2}^{4} \alpha_i \equiv 0 \bmod 2.$$

The inequalities above are equivalent to

$$|\alpha_2| \leqq \alpha_3 \leqq \alpha_4 \leqq \tfrac{1}{2} r$$

$$-\alpha_2 + \alpha_3 + \alpha_4 \leqq d - \tfrac{5}{2} r$$

One computes

$$g = \tfrac{1}{2}\left[d(r-1) - \tfrac{5}{4} r^2 - \sum_{i=2}^{4} \alpha_i^2\right] + 1 = F_d(r) - \sum_{i=2}^{4} \alpha_i^2$$

with

$$F_d(r) = \tfrac{1}{2}\left[d(r-1) - \tfrac{5}{2} r^2\right] + 1.$$

The maximum of $F_d(r)$ is given for $d - \frac{5}{2} r = 0$. $F_d(\frac{2}{5} d) = \frac{1}{2}\left[d(\frac{2}{5} d - 1) - \frac{5}{4} \frac{4}{25} d^2\right] + 1 = \frac{1}{10} d^2 - \frac{1}{2} d + 1$. *For* $d = 1, 2, 3, 4 \bmod 5$ the maximum for $r \in \mathbf{Z}$, $r \leqq \frac{2}{5} d$ is $\frac{1}{5}$, $\frac{2}{5}$, $\frac{1}{40}$, $\frac{9}{40}$ less.

Lemma 2.1.1. *Let d, g, r be integers such that r is odd, $d \geqq 17$, $\sqrt{8d + 160} - 12 < r < \frac{2}{5} d$ and $F_d(r-2) < g \leqq F_d(r)$. Then there exist $\alpha_i \in \mathbf{Z}$, $i = 2, 3, 4$, solving the above equations and inequalities.*

Proof.

$$F_d(r) - F_d(r-2) = \tfrac{1}{2}\left[d(r-1) - \tfrac{5}{4} r^2\right] - \tfrac{1}{2}\left[d(r-3) - \tfrac{5}{2}(r-2)^2\right]$$

$$=d+\tfrac{5}{8}(-4r+4)=d-\tfrac{5}{2}r+\tfrac{5}{2},$$

so we must solve $F_d(r)-g=\frac{1}{2}\sum_{i=2}^{4}\alpha_i^2$ with $F_d(r)-g<d-\frac{5}{2}r+\frac{5}{2}$.

Now r is odd, so $F_d(r)\in\frac{1}{2}\mathbf{Z}-\mathbf{Z}$, $n=8(F_d(r)-g)\in\mathbf{Z}$ and we have to find $\alpha_i\in\frac{1}{2}\mathbf{Z}-\mathbf{Z}$, $|\alpha_i|\leqq\frac{1}{2}r$, $n=\sum_{i=2}^{4}\alpha_i^2$ with odd integers $\beta_i=2\alpha_i$; note that $n\equiv 3 \bmod 8$.

By assumption $n<8(d-\frac{5}{2}r+\frac{5}{2})$; because of $\sqrt{8d+160}-12<r$, $8d+160<(r+12)^2$, $8d<r^2+24r-16$ it follows that $n<8d-20r+20<r^2+4r+4=(r+2)^2$.

Now we know: For every $n\equiv 3 \bmod 8$, there are odd integers x_1, x_2, x_3 such that $n=\sum_{i=1}^{3}x_i^2$. If $n<(r+2)^2$, r odd, we also have $|x_i|\leqq r$.

Using this we get the β_i and α_i; we have to verify the congruences and inequalities:

$|\alpha_2|\leqq\alpha_3\leqq\alpha_4$: obvious by right order

$\alpha_i\equiv\frac{1}{2}r \bmod 1$: r is odd as well as $\beta_i=2\alpha_i$

$d+\frac{1}{2}r-\sum_{i=2}^{4}\alpha_i\equiv 0 \bmod 2$: if not, change the sign of α_2

$d-\frac{5}{2}r\geqq-\alpha_2+\alpha_3+\alpha_4$: we know $\sum_{i=2}^{4}(2\alpha_i)^2=n<8d-20r+20$, $n\equiv 3 \bmod 8$, r odd, so $n\leqq 8d-20r+15$. If $-\alpha_2+\alpha_3+\alpha_4\leqq\frac{1}{2}(\alpha_2^2+a_3^2+\alpha_4^2-2)$, then $-\alpha_2+\alpha_3+\alpha_4\leqq\frac{1}{8}n-1\leqq d-\frac{5}{2}r+\frac{15}{8}-1=d-\frac{5}{2}r+\frac{7}{8}$; and $\alpha_i\equiv\frac{1}{2}r \bmod 1$, r odd, implies $-\alpha_2+\alpha_3+\alpha_4\leqq d-\frac{5}{2}r$ and we are done. Otherwise $8(-\alpha_2+\alpha_3+\alpha_4)>4(a_2^2+\alpha_3^2+\alpha_4^2-2)$, $4(-\beta_2+\beta_3+\beta_4)>\beta_2^2+\beta_3^2+\beta_4^2-8$ with odd integers β_i and $|\beta_2|\leqq\beta_3\leqq\beta_4$. We can determine all those cases and find

$$\begin{aligned}(\beta_2,\beta_3,\beta_4)\in\{&(-1,1,1),(-1,1,3),(-1,3,3),(-3,3,3),\\&(-1,1,5),(-1,3,5),(-1,5,5),(-3,3,5),(-3,5,5),\\&(1,1,1),(1,1,3),(1,3,3),(1,1,5),(1,3,5)\}\end{aligned}$$

Even in those cases we have found integers $a, b_1, \ldots, b_4$ with $d=3a-\sum_{i=1}^{4}b_i$, $g=\frac{1}{2}\left(a^2-\sum_{i=1}^{4}b_i^2-d\right)+1$, $b_2\geqq b_3\geqq b_4\geqq 0$, $a\geqq b_1+b_2+b_3$. Reordering the b_i still gives a permitted solution as long as $b_1\geqq 0$ and $a\geqq$ sum of the three greatest of the b_i.

Now $a-b_2-b_3-b_4=\frac{1}{2}\left(d+\frac{1}{2}r-\sum_{i=2}^{4}\alpha_i\right)-\frac{3}{2}r+\sum_{i=2}^{4}\alpha_i=\frac{1}{2}d-\frac{5}{4}r+\frac{1}{2}\sum_{i=2}^{4}\alpha_i$ $=\frac{1}{2}(d-\frac{5}{2}r)+\frac{1}{2}\sum_{i=2}^{4}\alpha_i\geqq 0$ in any case.

$b_1\geqq 0$ means $a\geqq r$, $\frac{1}{2}\left(d+\frac{1}{2}-\sum_{i=2}^{4}\alpha_i\right)\geqq r$, $d+b_2+b_3+b_4-r\geqq 2r$, $(d-\frac{5}{2}r)+(b_2+b_3+b_4-\frac{1}{2}r)\geqq 0$. Now $b_2+b_3+b_4=\frac{3}{2}r-\sum_{i=2}^{4}\alpha_i\geqq\frac{3}{2}r-\frac{10}{2}$ (by the classifi-

cation of the exceptional cases) and $\frac{3}{2}r-5\geqq\frac{1}{2}r$ for $r\geqq 5$. But $\sqrt{8d+160}-12\geqq 5$ means $8d+160\geqq 289$, $8d\geqq 129$, $d\geqq 17$.

Theorem 2.1.2. *For each pair of integers (d, g) such that $d\geqq 20$ and*

$$\sqrt{2d+40}(d+30)-\tfrac{23}{2}d-189<g\leqq\tfrac{1}{10}d^2-\tfrac{1}{2}d+1$$

there exists a smooth irreducible curve of degree d and genus g on the Del Pezzo surface in $\mathbf{P}^5$.

Proof. For $d\geqq 20$ choose r_0 as the least integer above $\sqrt{8d+160}-12$. According to (2.1.1), for every g with $F_d(r_0-2)<g\leqq\max\{F_d(r): r \text{ odd}\}$ there exists a smooth irreducible curve of degree d and genus g on the Del Pezzo surface in $\mathbf{P}^5$.

Let $\rho=\sqrt{8+160}-10$; then $r_0<\rho(\leqq\frac{2}{5}d$ for $d\geqq 20)$, therefore

$$\begin{aligned}F_d(r_0-2)<F_d(\rho-2)&=F_d(\sqrt{8d+160}-12)\\&=\tfrac{1}{2}[d(2\sqrt{2d+40}-13)-\tfrac{5}{4}(2\sqrt{2d+40}-12)^2]+1\\&=\tfrac{1}{2}[\sqrt{2d+40}(2d+60)-13d-10d-200-180]+1\\&=\sqrt{2d+40}(d+30)-\tfrac{23}{2}d-189.\end{aligned}$$

In order to complete the proof we now have to consider the case that $F_d(r)$ might achieve its maximum (for integral $r\leqq\frac{2}{5}d$) at an even number r.

$$\begin{aligned}F_d(r)-F_d(r-1)&=\tfrac{1}{2}[d(r-1)-\tfrac{5}{4}r^2]-\tfrac{1}{2}[d(r-2)-\tfrac{5}{4}(r-1)^2]\\&=\tfrac{1}{2}[d-\tfrac{5}{4}(2r-1)]=\tfrac{1}{2}d=\tfrac{5}{4}r+\tfrac{5}{8}\end{aligned}$$

we have to solve $F_d(r)-g=\frac{1}{2}\sum_{i=2}^{4}\alpha_i^2<\frac{1}{2}d-\frac{5}{4}r+\frac{5}{8}$ with integers α_i. $2(F_d(r)-g)=\sum_{i=2}^{4}\alpha_i^2<d-\frac{5}{2}r+\frac{5}{4}$.

Since r is the largest integer not exceeding $\frac{2}{5}d$, we have $0\leqq\frac{2}{5}d-r\leqq\frac{4}{5}$, $0\leqq d-\frac{5}{2}r\leqq 2$, therefore $2(F_d(r)-g)=\sum_{i=2}^{4}\alpha_i^2\leqq 2+1=3$; which can certainly be solved with $\alpha_i\in\{-1, 0, 1\}$.

We have to check the congruences and inequalities:

$|\alpha_2|\leqq\alpha_3\leqq\alpha_4$: obvious by right order;

$\alpha_4\leqq\frac{1}{2}r$: works for $r\geqq 2$;

$\alpha_i\equiv\frac{1}{2}r \bmod 1$: is clear;

$d+\frac{1}{2}r-\sum_{i=2}^{4}\alpha_i\equiv 0 \bmod 2$: $g=\frac{1}{2}\left[d(r-1)-\frac{5}{4}r^2-\sum_{i=2}^{4}\alpha_i^2\right]+1\in\mathbf{Z}$, so $d(r-1)-\frac{5}{4}r^2-\sum_{i=2}^{4}\alpha_i^2\equiv 0 \bmod 2$, $d-5(\frac{r}{2})^2-\sum_{i=2}^{4}\alpha_i^2\equiv 0 \bmod 2$, $d+\frac{r}{2}-\sum_{i=2}^{4}\alpha_i\equiv 0 \bmod 2$.

$d-\frac{5}{2}r\geqq-\alpha_2+\alpha_3+\alpha_4$: we know $-\alpha_2+\alpha_3+\alpha_4\leqq\alpha_2^2+\alpha_3^2+\alpha_4^2<d-\frac{5}{2}r+\frac{5}{4}$ so $-\alpha_2+\alpha_3+\alpha_4\leqq d-\frac{5}{2}r+\frac{5}{4}$. Let $d\equiv f \bmod 5$ with $0\leqq f\leqq 4$. The assumption on r

above implies that $f\in\{0, 1, 2\}$. In each of these cases we have $\sum_{i=2}^{4}\alpha_i^2\equiv d(r-1)-\frac{5}{4}r^2 \bmod 2$ $(g\in\mathbf{Z})$, $\equiv d[\frac{2}{5}(d-f)-1]-\frac{5}{4}\frac{4}{25}(d-f)^2 \bmod 2\equiv\frac{2}{5}d(d-f)-d-\frac{1}{5}d(d-f)+\frac{1}{5}f(d-f) \bmod 2\equiv\frac{1}{5}(d+f)(d-f)-d \bmod 2\equiv d^2-f^2 \bmod 2\equiv f^2 \bmod 2\equiv f \bmod 2$, however $d-\frac{5}{2}r+1\equiv d-(d-f)+1\equiv f+1 \bmod 2$. Therefore we actually have $\alpha_2^2+\alpha_3^2+\alpha_4^2\leqq d-\frac{5}{2}r$, $-\alpha_2+\alpha_3+\alpha_4\leqq d-\frac{5}{2}r$ which is what we were looking for.

2.2. *Curves on the Bordiga Surface in* $\mathbf{P}^5$

The Bordiga surface T in $\mathbf{P}^5$ can be defined as $\mathbf{P}^2$ blown up in 9 points in general position and embedded by the complete linear system of all quartics passing through those points. It has degree 7. $\mathrm{Pic}(T)$ is a free abelian group with basis l (the pullback of the class of a line in $\mathbf{P}^2$), and $-e_1, \ldots, -e_9$. Let $D=(1; 0, \ldots, 0)$, $\Gamma=(3; 1, \ldots, 1)$; we use the notation of (1.2).

Proposition 2.2.1. *For every g with $0\leqq g\leqq d-5$ there exists an irreducible nonsingular curve of degree d and genus g on the Bordiga surface T in $\mathbf{P}^5$.*

Proof. The same arguments as in (1.2.2) apply, but now it is even simpler. We do not need to blow up another point and can use that $H=D+\Gamma$.

Theorem 2.2.2. *For every d, g such that $0\leqq g\leqq\frac{1}{14}d^2-\frac{3}{14}d+\frac{9}{56}$ there exists a smooth irreducible curve of degree d and genus g on the Bordiga surface in $\mathbf{P}^5$.*

Proof. For integers d_0, g_0 with $0\leqq g_0\leqq d_0-5$ a curve Y_0 can be found according to (1.2.1). For $r\geqq 1$ consider the linear system $|Y_0+r(D+\Gamma)|$ on T. It is very ample and therefore contains an irreducible nonsingular curve C. One computes

$$d=(D+\Gamma).(Y_0+r(D+\Gamma)=d_0+r(D+\Gamma)^2=d_0+7r$$

$$\begin{aligned}2g-2&=(Y_0+r(D+\Gamma)).(Y_0+r(D+\Gamma)-\Gamma)\\&=Y_0.(Y_0-\Gamma)+2rY_0.(D+\Gamma)+r^2(D+\Gamma)^2-r\Gamma.(D+\Gamma)\\&=2g_0-2+2rd_0+7r^2-3r\end{aligned}$$

$$\begin{aligned}g&=g_0+rd_0+\tfrac{1}{2}(7r-3)=g_0+r(d_0+\tfrac{7}{2}r-\tfrac{3}{2})\\&=g_0+r(d-\tfrac{7}{2}r-\tfrac{3}{2})\end{aligned}$$

and $0\leqq g_0\leqq d_0-5$ is equivalent to

$$\begin{aligned}r(d-\tfrac{7}{2}r-\tfrac{3}{2})\leqq g&\leqq d-7r-5+r(d-\tfrac{7}{2}r-\tfrac{3}{2})\\&=d-7r-5+rd-\tfrac{7}{2}r^2-\tfrac{3}{2}r\\&=(r+1)d-\tfrac{7}{2}(r+1)^2+\tfrac{7}{2}-\tfrac{3}{2}(r+1)+\tfrac{3}{2}-5\\&=(r+1)(d-\tfrac{7}{2}(r+1)-\tfrac{3}{2})\end{aligned}$$

If we define $F_d(r)=r(d-\frac{7}{2}r-\frac{3}{2})$, this is equivalent to $F_d(r)\leqq g\leqq F_d(r+1)$. The maximum of $F_d(r)$ for real r is attained for $d-\frac{7}{2}-\frac{3}{2}=0$, $r=\dfrac{2d-3}{14}$.

$$F_d\left(\frac{2d-3}{14}\right)=\frac{2d-3}{14}\left(d-\frac{2d-3}{4}-\frac{3}{2}\right)=\frac{2d-3}{14}\,\frac{2d-4}{3}=\frac{1}{14}d^2-\frac{3}{14}d+\frac{9}{56},$$

and for integers r the maximum is at most $\frac{1}{4}\frac{7}{2}=\frac{7}{8}$ less.

Proof of (0.2 d). For $5\leqq d\leqq 8$ one immediately finds all the curves on a scroll or the Bordiga surface.

We also know the existence of the curves in the following cases

(1) $0\leqq g\leqq\frac{1}{56}(2d-3)^2$ from (2.2.2)

(2) $d\geqq 20$ and $\sqrt{2d+40}(d+30)-\frac{23}{2}d-189<g\leqq\frac{1}{10}d^2-\frac{1}{2}d+1$ from (2.1.2).

Now for $d\geqq 87$ we have $\sqrt{2d+40}(d+30)-\frac{23}{2}d-189<\frac{1}{56}(2d-3)^2$:

For $d=87$ this is true, and we can prove the same inequality for the derivatives on both sides:

$$\sqrt{2d+40}+\frac{d+30}{\sqrt{2d+40}}-\frac{23}{2}<\frac{1}{7}d-\frac{3}{14},\qquad 7(3d+70)<(d+79)\sqrt{2d+40}.$$

Repeating this argument we arrive at

$$21<\sqrt{2d+40}+\frac{d+79}{\sqrt{2d+40}},\qquad 21\sqrt{2d+40}<3d+119.$$

In the next step we get $\dfrac{21}{\sqrt{2d+40}}<3$, $7<\sqrt{2d+40}$, $9<2d$ which is even true for $d\geqq 5$.

In the remaining cases (finite in number) it can be checked (with the help of a computer) that all these pairs (d, g) occur as degree and genus of a curve on the Del Pezzo surface. The list is rather long, it can be found as an appendix in [10]; we omit it here.

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