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Autor: SCHAEFER, Helmut H.; Ricker, Werner J.

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Kontakt/Contact

Digizeitschriften e.V.
SUB Göttingen
Platz der Göttinger Sieben 1
37073 Göttingen

info@digizeitschriften.de

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The Uniformly Closed Algebra Generated by a Complete Boolean Algebra of Projections

Werner J. Ricker [1] * and Helmut H. Schaefer [2]

[1] School of Mathematics, University of N.S.W., Kensington, N.S.W., 2033 Australia

[2] Mathematisches Institut, Universität Tübingen, Auf der Morgenstelle 10, D-7400 Tübingen, Federal Republic of Germany

1. Introduction and Main Result

The penetrating work of W.G. Bade [1–3] into the theory of Boolean algebras of projections in Banach spaces is by now well known; a comprehensive treatment of these results can be found in [10], for example. In attempting to extend Bade's results (and the reflexivity result of T.A. Gillespie [11]) to the setting of locally convex spaces many phenomena arise which cannot be overcome in a routine manner by simply replacing norms with seminorms and then using Banach space type arguments. Some of the earliest results for non-normable spaces can be found in [13–15, 21, 22, 24, 25]. In recent years there has been a revival of interest in this topic and most of Bade's program has been successfully extended to the setting of Boolean algebras of projections in locally convex spaces; see [4–9, 16, 17, 23], for example.

However, one of Bade's fundamental results, stating that the closed algebra generated by a complete Boolean algebra in the uniform operator topology is the same as the closed algebra that it generates in the weak operator topology, has remained somewhat resistant in attempts to extend it to locally convex spaces. Recently however, specific classes of Boolean algebras in non-normable spaces were exhibited for which the analogue of Bade's result is valid [18, 19]. Namely, for such Boolean algebras, the weakly closed algebra that they generate coincides with the closed algebra generated with respect to the topology of uniform convergence on bounded sets. However, such Boolean algebras are somewhat special and the question remains of whether Bade's result is valid for all "reasonable" Boolean algebras of projections in a large class of locally convex spaces? In this paper it is shown that this question has an affirmative answer if we interpret "reasonable" to mean that the Boolean algebra is complete and equicontinuous. The only restrictions on the locally convex space X in which the projections act is that it be quasicomplete and that the space $L(X)$

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of all continuous linear operators from X into itself is sequentially complete for the topology of pointwise convergence in X. These restrictions are very mild in practice and include most spaces of interest.

It is time to be more precise. Let X be a locally convex space (always assumed to be Hausdorff) with continuous dual space X'. Then $L_s(X)$ and $L_b(X)$ denote $L(X)$ equipped with the topology ρ_s of pointwise convergence in X and the topology ρ_b of uniform convergence on bounded sets in X, respectively. If X is a Banach space, then ρ_b is just the uniform operator topology. It will always be assumed that X is quasicomplete and $L_s(X)$ is sequentially complete. In this case $L_b(X)$ is also sequentially complete [18; Lemma 2.3].

The concept of a Boolean algebra of projections is not a priori connected with normability of the topology of the vector space on which the algebra acts; the definition usually given in Banach spaces can be extended to locally convex spaces in a straight-forward way. If $\mathscr{A} \subseteq L(X)$ is a Boolean algebra, then $\langle \mathscr{A} \rangle_s$ and $\langle \mathscr{A} \rangle_b$ denote the closed algebra generated by $\mathscr{A}$ in $L_s(X)$ and $L_b(X)$, respectively. Since $\langle \mathscr{A} \rangle_s$ is the closure in $L_s(X)$ of the linear hull of $\mathscr{A}$ (which is a convex subset of $L(X)$) it follows that $\langle \mathscr{A} \rangle_s$ is also the closed algebra generated by $\mathscr{A}$ with respect to the weak operator topology in $L(X)$. A Boolean algebra $\mathscr{A} \subseteq L(X)$ is called equicontinuous if it is an equicontinuous subset of $L(X)$. The notions of σ-completeness and completeness of a Boolean algebra $\mathscr{A}$ used by Bade [2] are topological and algebraic, and consequently extend themselves immediately to the locally convex setting. Namely, $\mathscr{A}$ is complete (σ-complete) if it is complete (σ-complete) as an abstract Boolean algebra and if, for every set (sequence) $\{A_\alpha\}$ contained in $\mathscr{A}$,

$$(\wedge_\alpha A_\alpha)(X) = \cap_\alpha A_\alpha(X) \quad \text{and} \quad (\vee_\alpha A_\alpha)(X) = \operatorname{sp}(\cup_\alpha A_\alpha(X)),$$

the closed subspace of X generated by $\cup_\alpha A_\alpha(X)$. We can now formulate the main result.

Theorem 1. *Let X be a quasicomplete locally convex space such that $L_s(X)$ is sequentially complete. Let $\mathscr{A} \subseteq L(X)$ be a complete, equicontinuous Boolean algebra. Then $\langle \mathscr{A} \rangle_b$ and $\langle \mathscr{A} \rangle_s$ are equal as linear subspaces of $L(X)$ and, in particular, $\langle \mathscr{A} \rangle_b$ coincides with the closed algebra generated by $\mathscr{A}$ with respect to the weak operator topology in $L(X)$.*

The standard argument used in Banach spaces [10; XVII Lemma 2.1] to show that $\langle \mathscr{A} \rangle_b$ is a full algebra (i.e. $T \in \langle \mathscr{A} \rangle_b$ invertible in $L(X)$ implies that $T^{-1} \in \langle \mathscr{A} \rangle_b$) does not apply in non-normable spaces. The reason is that the set of invertible elements in $L(X)$ may not be an open set with respect to ρ_b. For example, if X is the Fréchet space consisting of all sequences $x = (x_1, x_2, \ldots)$ equipped with the topology of co-ordinatewise convergence and $T_n \in L(X)$, $n = 1, 2, \ldots$, is the operator given by projecting onto the first n co-ordinates, then none of the operators $\{T_n\}$ is invertible, yet $\{T_n\}$ converges in $L_b(X)$ to the identity operator I (in X). Accordingly, the question of whether $\langle \mathscr{A} \rangle_b$ is a full algebra cannot be resolved as simply as in the Banach space setting. Nevertheless, as a consequence of Theorem 1 we have the following fact.

Corollary 1. *Let X be a quasicomplete locally convex space such that $L_s(X)$ is sequentially complete. If $\mathscr{A} \subseteq L(X)$ is a complete, equicontinuous Boolean algebra, then $\langle \mathscr{A} \rangle_b$ is a full subalgebra of $L(X)$.*

The following result is an analogue of the classical result of W. Bade [2; Theorem 4.3] describing the uniformly closed algebra generated by a complete Boolean algebra in terms of the invariance of certain closed subspaces of the underlying Banach space.

Corollary 2. *Let X be a quasicomplete locally convex space such that $L_s(X)$ is sequentially complete. If $\mathscr{A} \subseteq L(X)$ is a complete, equicontinuous Boolean algebra, then $\langle \mathscr{A} \rangle_b$ consists of precisely those continuous linear operators which leave invariant each closed, $\mathscr{A}$-invariant subspace of X.*

A Boolean algebra $\mathscr{A} \subseteq L(X)$ is said to be cyclic if there exists an element $x \in X$ such that the linear span of $\{Ax;\ A \in \mathscr{A}\}$ is dense in X. For such Boolean algebras Corollary 2 can be considerably strengthened; see [2; Theorem 4.2] for the classical result in the Banach space setting.

Corollary 3. *Let X be a quasicomplete locally convex space such that $L_s(X)$ is sequentially complete and let $\mathscr{A} \subseteq L(X)$ be a complete, equicontinuous Boolean algebra which is cyclic. Then a continuous linear operator in X belongs to $\langle \mathscr{A} \rangle_b$ if and only if it commutes with every element of $\mathscr{A}$.*

2. Preliminaries

The proof of Theorem 1 will be based on the theory of integration with respect to spectral measures and so some further notation and definitions are needed.

An $L_s(X)$-valued operator measure is a σ-additive map $P: \Sigma \to L_s(X)$ whose domain Σ is a σ-algebra of subsets of some set Ω. We remark that P is σ-additive if and only if the complex-valued set function

$$\langle Px, x' \rangle: E \to \langle P(E)x, x' \rangle, \quad E \in \Sigma,$$

is σ-additive for each $x \in X$ and $x' \in X'$, [5; p. 140]. The measure P is said to be equicontinuous if its range $\mathscr{R}(P) = \{P(E);\ E \in \Sigma\}$ is an equicontinuous subset of $L(X)$. If $P(\Omega) = I$ and $P(E \cap F) = P(E)\,P(F)$, for every $E, F \in \Sigma$ (i.e. P is multiplacative), then P is called a spectral measure.

Let $P: \Sigma \to L_s(X)$ be a spectral measure. A complex-valued Σ-measurable function f is said to be P-integrable if it is integrable with respect to each complex measure $\langle Px, x' \rangle$, $x \in X$ and $x' \in X'$, and if there exists an element $P(f) = \int_\Omega f\,dP$ in $L(X)$ such that

$$\langle P(f)x, x' \rangle = \int_\Omega f\,d\langle Px, x' \rangle,$$

for each $x \in X$ and $x' \in X'$. The indefinite integral of f with respect to P is then given by

$$\int_E f\,dP = P(f)\,P(E) = P(E)\,P(f),$$

for every $E\in\Sigma$. This agrees with the usual definition of integration with respect to arbitrary vector measures [12]; see [5; Proposition 1.2]. In this case a P-integrable function f is P-null or equal to zero P-a.e. (cf. [12; Ch. II, Section 2] for the general definition) if and only if $P(f)=0$. The space of all P-integrable functions is denoted by $L(P)$. We note that if f and g are P-integrable functions, then their pointwise product fg is also P-integrable and

$$\int_E fg\,dP = P(f)\,P(g)\,P(E) = P(g)\,P(f)\,P(E), \qquad E\in\Sigma; \tag{1}$$

see [5; Lemma 1.3]. It follows from the Orlicz-Pettis lemma that the indefinite integral of $f\in L(P)$ is an $L_s(X)$-valued operator measure. Bounded Σ-measurable functions are always P-integrable [12; II Lemma 3.1].

The topology of $L_s(X)$ is determined by the seminorms

$$q_F\colon\ T\to\max\{q(Tx), x\in F\}, \qquad T\in L(X),$$

where F is any finite subset of X and q any continuous seminorm in X. Each such seminorm q_F induces a seminorm $q_F(P)$ in $L(P)$ by the formula

$$q_F(P)(f)=\sup\{q_F(\int_E f\,dP); E\in\Sigma\}, \qquad f\in L(P). \tag{2}$$

The locally convex topology $\tau_s(P)$ so defined in $L(P)$ may not be Hausdorff. The quotient space of $L(P)$ with respect to the subspace of all P-null functions is denoted by $L^1(P)$. The resulting Hausdorff topology on $L^1(P)$ is again denoted by $\tau_s(P)$. It is clear from (2) that $\tau_s(P)$ is the topology of uniform convergence on Σ of indefinite integrals. For a more comprehensive treatment of the space $(L^1(P), \tau_s(P))$ we refer to [5; Section 1].

A spectral measure $P\colon \Sigma\to L_s(X)$ is called a closed measure if $L^1(P)$ is complete with respect to $\tau_s(P)$. Since $L_s(X)$ is assumed to be sequentially complete this agrees with the usual definition for vector measures [5; p. 139]. We remark that an equicontinuous spectral measure $P\colon \Sigma\to L_s(X)$ is a closed measure if and only if its range $\mathscr{R}(P)$ is a closed subset of $L_s(X)$, [16; Proposition 3]. In this case $\mathscr{R}(P)$ is a complete Boolean algebra in $L(X)$, [5; Section 2]. Furthermore, $L^1(P)$ is a complete, unital, commutative locally convex algebra with respect to pointwise multiplication (of equivalence classes), [5; Proposition 1.4], and the integration mapping

$$f\to P(f)=\int_\Omega f\,dP, \qquad f\in L^1(P), \tag{3}$$

is a bicontinuous isomorphism of the (complete) locally convex algebra $L^1(P)$ onto the operator algebra $\langle\mathscr{R}(P)\rangle_s$, [5; Proposition 1.5]. In particular, $\langle\mathscr{R}(P)\rangle_s$ is a complete subspace of $L_s(X)$.

Let $P\colon \Sigma\to L_s(X)$ be a spectral measure. A Σ-measurable function f is said to be P-essentially bounded if

$$|f|_P=\inf\{\sup\{|f(w)|; w\in E\}; E\in\Sigma, P(E)=I\}$$

is finite. The space of (equivalence classes of) P-essentially bounded functions is denoted by $L^\infty(P)$; it is a Banach algebra with respect to the norm $|\cdot|_P$. If $f\in L^\infty(P)$, then it follows from the σ-additivity of P that there is a set $E\in\Sigma$ with $P(E)=I$ such that

$$|f|_P=\|f\chi_E\|_\infty=\sup\{|f(w)|;\, w\in E\}.$$

This observation together with [25; Proposition 2.1] and [12; II Lemma 3.1] imply the following result.

Lemma 1. *Let $P\colon \Sigma\to L_s(X)$ be an equicontinuous spectral measure. Then*

$$\{\int_\Omega f\,dP,\, f\in L^\infty(P),\, |f|_P\leqq 1\} \tag{4}$$

is an equicontinuous subset of $L(X)$.

As an immediate consequence we have the following lemma.

Lemma 2. *Let X be a quasicomplete locally convex space such that $L_s(X)$ is sequentially complete. Let $P\colon \Sigma\to L_s(X)$ be an equicontinuous spectral measure. If f is a P-integrable function, then*

$$\{\int_\Omega g\,dP;\, g \text{ is } \Sigma\text{-measurable and } |g|\leqq|f|,\ P\text{-a. e.}\} \tag{5}$$

is an equicontinuous subset of $L(X)$.

Proof. If g is a Σ-measurable function satisfying $|g|\leqq|f|$, P-a. e., then g is also P-integrable [12; II Theorem 3.1]. Furthermore, $g=f\cdot(g/f)$ where f is P-integrable and $|g/f|_P\leqq 1$. It follows from (1) that

$$\int_\Omega g\,dP=(\int_\Omega f\,dP)\cdot(\int_\Omega (g/f)\,dP).$$

Accordingly, the set (5) is contained in

$$\{P(f)\,P(h);\, h\in L^\infty(P),\, |h|_P\leqq 1\}$$

and so (5) is equicontinuous (by Lemma 1). □

The topology of $L_b(X)$ is generated by seminorms of the form

$$q_B\colon\ T\to\sup\{q(Tx);\, x\in B\},\qquad T\in L(X), \tag{6}$$

where B is a bounded subset of X and q is a continuous seminorm in X. So, if $f\in L^\infty(P)$, then

$$\begin{aligned} q_B(\int_\Omega f\,dP) &= \sup\{q((\int_\Omega f\,dP)\,x);\, x\in B\} \\ &= |f|_P \sup\{q((\int_\Omega (f/|f|_P)\,dP)\,x;\, x\in B\} \\ &\leqq |f|_P \sup\{q((\int_\Omega h\,dP)\,x);\, x\in B,\, h\in L^\infty(P),\, |h|_P\leqq 1\}. \end{aligned}$$

Since B is a bounded set and (4) is an equicontinuous subset of $L(X)$ it follows that

$$\alpha(q, B, P) = \sup\{q((\int_\Omega h\,dP)\,x;\, x\in B,\, h\in L^\infty(P),\, |h|_P\leqq 1\}$$

is finite. Accordingly, for every continuous seminorm q_B of the form (6) there is a constant $\alpha(q, B, P)$ such that

$$q_B(\int_\Omega f\,dP) \leqq \alpha(q, B, P)\,|f|_P, \qquad f\in L^\infty(P).$$

So, we have established the following result.

Lemma 3. *Let $P\colon \Sigma\to L_s(X)$ be an equicontinuous spectral measure. Then the restriction of the integration map* (3) *to $L^\infty(P)$ is continuous mapping from $(L^\infty(P), |\cdot|_P)$ into $L_b(X)$.*

3. Proof of Theorem 1

The containment $\langle\mathscr{A}\rangle_b \subseteq \langle\mathscr{A}\rangle_s$ is clear since ρ_b is a stronger topology than ρ_s.

So, it suffices to establish the reverse inclusion. Just as in the Banach space situation, a σ-complete or complete, equicontinuous Boolean algebra may be realized as the range of an $L(X)$-valued spectral measure (for example, on the Baire or Borel sets of its Stone space [25; Proposition 1.3]). So, let $P\colon \Sigma\to L_s(X)$ be any spectral measure such that $\mathscr{R}(P)=\mathscr{A}$. Since $\mathscr{A}$ is a closed set in $L_s(X)$, [5; Proposition 4.2], it follows from earlier remarks that P is a closed, equicontinuous measure and hence the integration map (3) maps $L^1(P)$ onto $\langle\mathscr{A}\rangle_s = \langle\mathscr{R}(P)\rangle_s$. Accordingly, if $T\in\langle\mathscr{A}\rangle_s$, then there exists a P-integrable function f such that $T=\int_\Omega f\,dP$.

Define disjoint sets $\delta_n(f)=\{w\in\Omega;\, n^2\leqq|f(w)|<(n+1)^2\}$, for each $n=0, 1, \ldots$ If $\varphi_n=\sum_{k=0}^{n} k^2\chi_{\delta_k(f)}$, then $|\varphi_n(w)|\leqq|f(w)|$, for each $w\in\Omega$ and every $n=0, 1, \ldots$ Since $|f|$ is P-integrable [12; II Lemma 2.1], Lemma 2 implies that the sequence

$$Q_n=\int_\Omega \varphi_n\,dP=\sum_{k=0}^{n} k^2 P(\delta_k(f)), \qquad n=0, 1, \ldots, \tag{7}$$

is equicontinuous. Accordingly, the sequence of consecutive differences

$$Q_n - Q_{n-1} = n^2 P(\delta_n(f)), \qquad n = 1, 2, \ldots, \tag{8}$$

is also equicontinuous in $L(X)$.

Fix a seminorm q_B generating the topology of $L_b(X)$; see (6). The equicontinuity of the sequence of operators (8) guarantees that

$$B(f) = \bigcup_{n=0}^{\infty} [n^2 P(\delta_n(f))](B)$$

is a bounded set in X. Accordingly,

$$q_B(P(\delta_n(f))) = n^{-2} q_B(n^2 P(\delta_n(f))) \leqq n^{-2} \beta(B, f, q), \tag{9}$$

for every $n = 1, 2, \ldots$, where

$$\beta(B, f, q) = \sup\{q(x); x \in B(f)\} < \infty.$$

We have used here the fact that

$$[n^2 P(\delta_n(f))](B) \subseteq B(f),$$

for every $n = 0, 1, \ldots$ Defining the (disjoint) unions

$$E_n = \bigcup_{k=n}^{\infty} \delta_k(f) = \{w \in \Omega; n^2 \leqq |f(w)|\}, \qquad n = 0, 1, \ldots,$$

it follows from (9) and the σ-additivity of P in $L_s(X)$ that

$$q(P(E_n)x) = q\left(\sum_{k=n}^{\infty} P(\delta_k(f))x\right) \leqq \sum_{k=n}^{\infty} q(P(\delta_k(f))x)$$
$$\leqq \sum_{k=n}^{\infty} q_B(P(\delta_k(f))) \leqq \beta(B, f, q) \sum_{k=n}^{\infty} k^{-2},$$

for every $x \in B$ and every $n = 1, 2, \ldots$ That is,

$$q_B(P(E_n)) \leqq \beta(B, f, q) \sum_{k=n}^{\infty} k^{-2}, \qquad n = 1, 2, \ldots,$$

which shows that $q_B(P(E_n)) \to 0$ as $n \to \infty$. Since q_B is an arbitrary seminorm determining the topology ρ_b it follows that $P(E_n) \to 0$ in $L_b(X)$. Hence, if $H_n = \Omega \setminus E_n$, for each $n = 1, 2, \ldots$, then $P(H_n) \to I$ in $L_b(X)$ and so $TP(H_n) \to T$ in $L_b(X)$. But, (1) implies

$$TP(H_n) = \left(\int_\Omega f\,dP\right) P(H_n) = \int_\Omega (f\chi_{H_n})\,dP,$$

for every $n=1, 2, \ldots$, which shows that there exist bounded, Σ-measurable functions $f\chi_{H_n}$, $n=1, 2, \ldots$, (necessarily P-integrable [12; II Lemma 3.1]) such that $\int_\Omega (f\chi_{H_n})\,dP \to T$ in $L_b(X)$.

So, it remains to show that $\int_\Omega g\,dP$ belongs to $\langle\mathscr{A}\rangle_b$ whenever g is a bounded, Σ-measurable function. But, in this case there exist Σ-simple functions, say $\{g_n\}$, such that $g_n \to g$ in $L^\infty(P)$. Since $P(g_n)\in\langle\mathscr{A}\rangle_b$, for every $n=1, 2, \ldots$, it is clear from Lemma 3 that $P(g)=\int_\Omega g\,dP$ belongs to $\langle\mathscr{A}\rangle_b$.

This completes the proof of Theorem 1. □

Proof of Corollary 1. This follows immediately from Theorem 1, the fact that the integration map (3) is an isomorphism of $L^1(P)$ onto $\langle\mathscr{A}\rangle_s$ and [17; Lemma 3]. □

Proof of Corollary 2. This is a direct consequence of Theorem 1 and the Bade reflexivity theorem for locally convex spaces [5; Theorem 3.1]. □

Proof of Corollary 3. This follows from Theorem 1 and [18; Theorem 5.4] which states that Corollary 3 is valid with $\langle\mathscr{A}\rangle_s$ replacing $\langle\mathscr{A}\rangle_b$ in its formulation. □

4. Concluding Remarks

(1) If X is a Banach space, then in the notation of the proof of Theorem 1 it was shown that $P(E_n)\to 0$ with respect to the uniform operator topology, where $E_n=\{w\in\Omega;\ n^2\leqq|f(w)|\}$, for each $n=0, 1, \ldots$ Accordingly, there exists N such that $P(E_n)=0$ for all $n\geqq N$ and hence f is bounded P-a. e. This gives a direct proof of the well known fact [10; XVIII Theorem 2.11(c)] that the only P-integrable functions in a Banach space are the P-essentially bounded ones.

(2) For Fréchet spaces the equicontinuity hypothesis in the statement of Theorem 1 can be omitted; it follows from the completeness of the Boolean algebra, [25; Proposition 1.2].

(3) A somewhat simpler proof of Theorem 1 (based on the methods of [18]) is available for spaces X with the property that bounded sets are precompact; see [20]. Such spaces include all Montel spaces and Schwartz spaces (and hence, all nuclear spaces.)

(4) If the space X is barrelled, then the equicontinuity of the operators $\{Q_n\}$ in (7) follows directly (rather than appealing to Lemma 2) from the observation that $\varphi=\sum_{n=0}^{\infty} n^2\chi_{\delta_n}(f)$ is P-integrable (as $|\varphi|\leqq|f|$ pointwise on Ω). Indeed, the ρ_s-countable additivity of the indefinite integral of φ with respect to P implies that the series

$$\int_\Omega \varphi\,dP=\sum_{n=0}^{\infty} n^2 P(\delta_n(f))$$

converges (unconditionally) in $L_s(X)$ and hence, in particular, its sequence of partial sums Q_n, $n=0, 1, \ldots$ (cf. (7)) is convergent in $L_s(X)$. The barrelledness of X then ensures the equicontinuity of $\{Q_n\}$. For barrelled spaces X we note that quasicompleteness of X implies the quasicompleteness of $L_s(X)$.

(5) It was remarked earlier that under the hypotheses of Theorem 1 the algebra $\langle \mathscr{A} \rangle_s$ (equal to $\langle \mathscr{R}(P) \rangle_s$) is actually a complete subspace of $L_s(X)$. Since $L_b(X)$ has a basis of neighbourhoods of zero consisting of ρ_s-closed sets it follows that $\langle \mathscr{A} \rangle_b$ is a complete subspace of $L_b(X)$; see Proposition 5.2 of [18]. Actually, if ρ is the topology in $L(X)$ of uniform convergence on any saturated family of bounded, closed, convex and balanced subsets of X, then $\rho_s \leqq \rho \leqq \rho_b$ and ρ has a neighbourhood basis at zero consisting of ρ_s-closed sets. It follows that $\langle \mathscr{A} \rangle_\rho$ is a complete subspace of $L_\rho(X)$. Furthermore, Theorem 1 implies that $\langle \mathscr{A} \rangle_s = \langle \mathscr{A} \rangle_\rho = \langle \mathscr{A} \rangle_b$ as linear subspaces of $L(X)$ and it follows from Corollary 1 that $\langle \mathscr{A} \rangle_\rho$ is a full subalgebra of $L(X)$.

(6) Let $\mathscr{A}$ be a Boolean algebra as in Theorem 1 and let $P: \Sigma \to L_s(X)$ be any closed, equicontinuous spectral measure such that $\mathscr{R}(P) = \mathscr{A}$. Each seminorm q_B (c.f. (6)) determining the topology of $L_b(X)$ induces a seminorm $q_B(P)$ on the P-integrable functions by

$$q_B(P)(f) = \sup\{q_B(\int_E f\,dP); E \in \Sigma\}, \qquad f \in L^1(P).$$

This induces a locally convex Hausdorff topology in $L^1(P)$ which we denote by $\tau_b(P)$. It is clear that $\tau_b(P)$ is stronger than $\tau_s(P)$. An examination of the proof of Lemma 4.2 in [18] shows that the calculations made there guarantee the existence of another bounded set $\tilde{B}$ in X such that

$$q_B(\int_\Omega f\,dP) \leqq q_B(P)(f) \leqq q_{\tilde{B}}(\int_\Omega f\,dP),$$

for every $f \in L^1(P)$; this is valid without the additional assumption on P made in [18]. Accordingly, the integration map (3) is a bicontinuous isomorphism of $(L^1(P), \tau_b(P))$ onto its range in $L_b(X)$, namely the linear subspace $\langle \mathscr{A} \rangle_s$. But, by Theorem 1 this is precisely $\langle \mathscr{A} \rangle_b$. So, we have established the following representation theorem.

Theorem 2. *Let X be a quasicomplete locally convex space such that $L_s(X)$ is sequentially complete. Let $\mathscr{A} \subseteq L(X)$ be a complete, equicontinuous Boolean algebra of projections. Then $\langle \mathscr{A} \rangle_b$ is isomorphic to the complete, commutative, unital, locally convex algebra $(L^1(P), \tau_b(P))$ of (equivalence classes of) P-integrable functions for any spectral measure P such that $\mathscr{R}(P) = \mathscr{A}$.*

(7) Recall that a subset U of a topological space (Y, τ) is sequentially dense if, for every $y \in Y$, there is a sequence $\{u_n\} \subseteq U$ which τ-converges to y. If X is non-normable, then in general $\langle \mathscr{A} \rangle_b$ is not metrizable with respect to ρ_b and so its topology is not given by a countable family of seminorms. Nevertheless, an examination of the proof of Theorem 1 shows that $\{\int_\Omega h\,dP;\ h \in L^\infty(P)\}$ is sequentially dense in $\langle \mathscr{A} \rangle_b$. Combining this observation with Lemma 3 it

follows that, for any $T\in\langle\mathscr{A}\rangle_b$, there exists a countable subset $G(T)$ of the linear span $\mathrm{sp}(\mathscr{A})$, of $\mathscr{A}$, with the property that for any ρ_b-neighbourhood of zero, say V, there is $S\in G(T)$ such that $(S-T)\in V$. It cannot be concluded in general that $\mathrm{sp}(\mathscr{A})$ is sequentially dense in $\langle\mathscr{A}\rangle_b$. This is in contrast to the sequential density of $\mathrm{sp}(\mathscr{A})$ in $\langle\mathscr{A}\rangle_s$; see [5; Proposition 1.5] and [12; II Lemma 3.1]. However, there are examples where $\mathrm{sp}(\mathscr{A})$ is sequentially dense in $\langle\mathscr{A}\rangle_b$. Of course, if X is a Banach space, then $L_b(X)$ is also a Banach space and so $\langle\mathscr{A}\rangle_b$ is just the sequential closure of $\mathrm{sp}(\mathscr{A})$. For examples in non-normable spaces X, let $\mathscr{A}$ be boundedly σ-complete [18; Section 5]. Then $\mathrm{sp}(\mathscr{A})$ is known to be sequentially dense in $\langle\mathscr{A}\rangle_b$; see Theorem 5.1 (and its proof) in [18]. By Theorem 2 these comments can be translated into a statement about the sequential density of $L^\infty(P)$ in $(L^1(P), \tau_b(P))$ and a similar statement (as for $\mathrm{sp}(\mathscr{A})$) concerning the density of the Σ-simple functions with respect to $\tau_b(P)$ is valid.

(8) It is known [6; Section 2] that $(L^1(P), \tau_s(P))$ has the structure of a Dedekind complete, complex Riesz space and $\tau_s(P)$ has the Lebesgue property (i.e. if $\{f_\alpha\}\subseteq L^1(P)$ is downwards filtering to 0 in the order sense, then $f_\alpha\to 0$ with respect to $\tau_s(P)$). It is essential here that P is σ-additive with respect to ρ_s. The same order applies in $L^1(P)$ considered with respect to the topology $\tau_b(P)$.

If $\mathscr{A}$ is boundedly σ-complete, then P is σ-additive with respect to $\tau_b(P)$ and P is a closed measure with respect to $\tau_s(P)$ if and only if it is a closed measure with respect to $\tau_b(P)$, [18; Corollary 6.3.1]. Accordingly, in this case $(L^1(P), \tau_b(P))$ also has the structure of a Dedekind complete, complex Riesz space with Lebesgue topology. However, for general $\mathscr{A}$ the order structure of $(L^1(P), \tau_b(P))$ cannot be expected to have such strong topological properties. Indeed, if X is a Banach space, then $L^1(P)$ consists of the P-essentially bounded functions and $\tau_b(P)$ is equivalent to the norm topology $|\cdot|_P$. In this case Theorem 2 is the classical representation of the uniformly closed algebra generated by $\mathscr{A}$ as an L^∞-space which, except in trivial cases, does not have the Lebesgue property.

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