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Configurations of I_n Fibers on Elliptic K 3 Surfaces

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1. Introduction

In this article we continue a series of papers devoted to understanding the possibilities for configurations of singular fibers on elliptic surfaces. In [MP], we classified the rational elliptic surfaces with a finite group of sections; except for one case (with J constant) they are all determined by their singular fibers, and there are a total of only 16 cases. In [P], the second author went on to classify all possible configurations of singular fibers on rational elliptic surfaces, giving constructions via double covers: there are close to 300 in number. The first author, in [M], has given a more combinatorial and less geometric analysis of the same problem.

In this article we begin an analysis of similar questions on elliptic K 3 surfaces. The a priori possibilities for configurations of singular fibers on elliptic K 3 surfaces run into the tens of thousands, so we have made a simplifying assumption in this work, namely that the elliptic fibration has only I_n fibers; this is the "semi-stable" case. If X is an elliptic K 3 surface and $\pi: X \to C$ is the elliptic fibration, then C has genus 0 and the induced J-map from C to $\mathbf{P}^1$ has degree at most 24; our assumption that the fibers of π are all of type I_n is equivalent to the J-map having degree exactly 24, which is the "general" case.

Note that we have $\sum n_i = 24$ with this assumption, where the s singular fibers are of type $I_{n_1}, \ldots, I_{n_s}$.

Assume that the fibration π has a section. If ρ is the Picard number of the K 3 surface, then by the formula of Shioda and Tate [S] we have

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$$\rho=\sum(n_i-1)+\text{rank}(\text{MW})+2=26+\text{rank}(\text{MW})-s,$$

where MW is the Mordell-Weil group of sections of π. Since $\rho \leqq 20$ and rank(MW)$\geqq 0$, this implies that $s \geqq 6$: there are at least six singular fibers.

The assumption of a section is harmless; if π does not, we simply consider the Jacobian surface, which has the identical configuration of singular fibers and is again a K 3 surface. Henceforward we will assume that a section exists.

Note that in the case of exactly six singular fibers, the formula of Shioda and Tate implies that the Mordell-Weil group MW has rank 0, i.e., it is a finite abelian group. Since this case occupies a special place in the constructions, we will refer to these fibrations as *extremal*. This agrees with the notation of [MP], where extremal is used to refer to a fibration $\pi: X \to C$ with $2+\sum(m_c - 1)=h^{1,1}(X)$ (here m_c is the number of components of the fiber over $c \in C$; the above formula states that the zero-section and components of fibers of π fill up $H^{1,1}$ over $\mathbb{R}$).

Our main problem can now be stated combinatorially as follows: we ask for which s-tuples $[n_1, \ldots, n_s]$ does there exist an elliptic K 3 surface $\pi: X \to \mathbb{P}^1$ with exactly s singular fibers of type $\mathrm{I}_{n_1}, \ldots, \mathrm{I}_{n_s}$. If such a fibration exists, we will say that π *realizes* $[n_1, \ldots, n_s]$. As an abuse of language, we will say that "$[n_1, \ldots, n_s]$ exists" or "$[n_1, \ldots, n_s]$ does not exist" to mean that such an elliptic K 3 surfaces does or does not exist.

The restrictions imposed by the Euler characteristic ($=24$) and the rank of the Neron-Severi group ($\leqq 20$) leave us with the number of partitions of 24 into at least 6 summands, which incidentally turn out to be 1242 s-tuples, of which 199 are those with exactly 6 summands. As a comparison it may be instructive to note that the corresponding numbers for rational surfaces are 58 and 15, respectively, out of which 46 actually occur. For the next case where $p_g=2$, there are 1801 possible "extremal" cases (with 8 singular fibers), and a total of 13966 apriori possibilities for the semi-stable ones!

Our result is that exactly 135 s-tuples in the K 3 case do not exist.

The technique for constructing those that exist is to remark that an s-tuple exists if and only if an appropriate J-map exists from $\mathbb{P}^1$ to $\mathbb{P}^1$. The J-maps for the 112 extremal 6-tuples which exist are then constructed via permutation arguments. The s-tuples which exist for $s \geqq 7$ are obtained essentially by deforming those with $s=6$. This is explained in Sects. 2 and 3.

The technique for eliminating the 135 is to use discriminant-form arguments on the Neron-Severi group of the K 3 surface to deduce the existence of torsion sections of the elliptic fibration; then to analyze the fixed points of the automorphism obtained by translating by the torsion section to arrive at a contradiction. This is explained in Sects. 4–6.

The main drawback of our construction of the 1107 cases is that it is not very geometric. In a sequel to this paper we intend in particular to present the 112 extremal cases in more loving detail, giving the possible Mordell-Weil groups for the fibrations. (Incidentally the techniques of this paper suffice to pin those down uniquely in about 90 of the 112 case.) We will also give more geometric constructions for many of them. Furthermore we intend to give a more or less complete answer to these more precise questions: