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Autor: Ohshika, Ken'ichi

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Kontakt/Contact

Digizeitschriften e.V.
SUB Göttingen
Platz der Göttinger Sieben 1
37073 Göttingen

info@digizeitschriften.de

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On Limits of Quasi-Conformal Deformations of Kleinian Groups

Ken'ichi Ohshika
Department of Mathematics, Tokyo Metropolitan University, Fukasawa, Tokyo 158, Japan

Introduction

Let Γ be a finitely generated non-elementary Kleinian group (of the second kind). Let Λ_Γ denote its limit set and let Ω_Γ denote its region of discontinuity. Assume that all components of Ω_Γ are simply connected. The work of Bers [4] and Sullivan [21], among others, has shown that there is a homeomorphism between the space of quasi-conformal deformations of Γ and the Teichmüller space of Ω_Γ/Γ denoted by $\mathcal{T}(\Omega_\Gamma/\Gamma)$. A simple example is the case when Γ is a torsion free Fuchsian group of first kind with genus g. Then Ω_Γ is a union of two simply connected components. Therefore the space of quasi-Fuchsian groups which are isomorphic to Γ is homeomorphic to $\mathcal{T}_g \times \mathcal{T}_g$.

It is a natural question to ask what happens when a sequence of Kleinian groups goes to infinity in $\mathcal{T}(\Omega_\Gamma/\Gamma)$ through the homeomorphism above. The first example is that of quasi-Fuchsian groups studied by Bers ([3]). There he has shown that a sequence of quasi-Fuchsian groups corresponding to $\{m_0 \times m_i'\} \subset \mathcal{T}_g \times \mathcal{T}_g$, where m_0 is fixed and m_i' goes to infinity in $\mathcal{T}_g$, has a subsequence converging algebraically to a Kleinian group which has only one invariant component of Ω_Γ. The second example is "the double limit theorem" of Thurston ([24]). It states that, for two measured laminations λ, λ' satisfying a certain condition, a sequence of quasi-Fuchsian groups corresponding to a sequence converging to (λ, λ') regarded as elements of the Thurston boundary of Teichmüller space has a subsequence converging algebraically to a Kleinian group (of the first kind). On the other hand Thurston proved in [23] that deformation spaces of acylindrical hyperbolic 3-manifolds with the algebraic topology are compact. Hence in this case every sequence has a subsequence converging algebraically to a Kleinian group.

In this paper we will prove a theorem explaining these phenomena. It is based on a result of Thurston ([25], [24]). Our main theorem in the case of geometrically finite groups is as follows.

Theorem 3.5. *Let $\{\Gamma_i\}$ be a sequence of quasi-conformal deformations of a geometrically finite group Γ with $\mathbf{H}^3/\Gamma = M$. Suppose that for any essential annulus A in $(\bar{M}, \partial\bar{M})$ which is not homotopic into the cuspidal part of M, there exists a*

measured lamination γ on $\partial\bar{M}$ intersecting A essentially and a sequence of measured lamination $\{\gamma_i\}$ on S converging to γ such that the sequence $\{\text{length}_{\Omega_{\Gamma_i}/\Gamma_i}(\gamma_i)\}$ is bounded. Suppose moreover that γ is a simple closed curve when $\bar{M}$ is an I-bundle over a surface. Then $\{\Gamma_i\}$ has a subsequence which converges algebraically to a Kleinian group isomorphic to Γ.

This theorem implies the following theorem which is an analogue of the double limit theorem for geometrically finite groups.

Theorem 3.7. *Let Γ be a geometrically finite group which is not a quasi-Fuchsian group with $\mathbf{H}^3/\Gamma = M$. Let $\{\Gamma_i\}$ be a sequence of quasi-conformal deformations of Γ. Let $S_1, \ldots, S_n$ be the components of $\partial\bar{M}$. Let $\gamma_{i_1}, \ldots, \gamma_{i_k}$ be measured laminations on $S_{i_1}, \ldots, S_{i_k}$ where $S_{i_1}, \ldots, S_{i_k}$ are disjoint. Suppose that for each essential annulus A, the boundary ∂A intersects some γ_{i_j} essentially. Let $\{(g_m^1, \ldots, g_m^n)\}$ be a sequence in $\mathcal{T}(S_1) \times \ldots \times \mathcal{T}(S_n)$ such that $\{g_m^{i_j}\}$ converges to γ_{i_j} in $\overline{\mathcal{T}(S_{i_j})}$ as m goes to infinity. Then the sequence $\{qc(g_i^1, \ldots, g_i^n)\}$ has a subsequence which converges to a Kleinian group isomorphic to Γ.*

The main theorem also implies the following corollary which generalizes the theorem by Bers in [3] showing the compactness of his completion of Teichmüller space.

Corollary 3.3. *Let Γ be a geometrically finite group with $M = \mathbf{H}^3/\Gamma$. Let $S_1, \ldots, S_n$ be the components of $\partial\bar{M}$. Suppose that there are no essential annuli in $(\bar{M}, \partial\bar{M})$ both of whose boundary components are on S_1. Let $\{m_i^1\}$ be an arbitrary sequence of $\mathcal{T}(S_1)$. Let $\{m_i^2\}, \ldots, \{m_i^n\}$ be bounded sequences of $\mathcal{T}(S_2), \ldots, \mathcal{T}(S_n)$ respectively. Then $\{qc(m_i^1, \ldots, m_i^n)\}$ has a subsequence converging algebraically to a Kleinian group isomorphic to Γ.*

In the last section we generalize this theorem to geometrically infinite tame groups under some assumptions.

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1. Preliminaries

A discrete subgroup of $PSL_2(\mathbf{C})$ is called a Kleinian group. In this paper we always assume that Kleinian groups are torsion free. This assumption is not essential and is only made for the sake of simplicity. A Kleinian group Γ acts on $\mathbf{H}^3$ by isometries and on S^2_∞ by conformal maps. The minimal nonempty closed set invariant by Γ is called the limit set of Γ and denoted by Λ_Γ. Its complement $S^2_\infty - \Lambda_\Gamma$ is called the domain of discontinuity of Γ and denoted by Ω_Γ. When Λ_Γ is a finite set, Γ is said to be elementary. In this paper we always assume that a Kleinian group Γ is non-elementary.

Let C_Γ denote the convex hull of Λ_Γ in $\mathbf{H}^3$. The quotient space C_Γ/Γ is a convex hyperbolic 3-manifold. When C_Γ/Γ has finite volume, the group Γ is said to be geometrically finite.

For a complete hyperbolic 3-manifold M with a holonomy $r: \pi_1(M) \to PSL_2(\mathbf{C})$, let $AH(M)$ denote the space of all discrete faithful representations

from $\pi_1(M)$ to $PSL_2(\mathbf{C})$ which send parabolic elements of $\pi_1(M)$ to those of $PSL_2(\mathbf{C})$. The topology of $AH(M)$ is the algebraic topology where the convergence means point-wise convergence with respect to a generator system of $\pi_1(M)$. Let $QH_0(M)$ denote the space of quasi-conformal deformations of r with the quasi-conformal topology. There is a canonical continuous inclusion map i from $QH_0(M)$ to $AH(M)$. See Thurston's lecture notes [22] for details of the above definitions.

In this paper we always assume that Γ or $\pi_1(M)$ is finitely generated. Then Ω_Γ/Γ is a finite union of Riemann surfaces of finite type by Ahlfors finiteness theorem ([2]). Moreover in this paper we always assume that Γ satisfies the following condition (*).

(*) If $\Gamma \cong A * B$, then there exists a parabolic element whose conjugate can be contained neither A nor B.

By Bonahon [5] it is known that, if Γ satisfies (*), then Γ is geometrically tame. Let M_0 be the non-cuspidal part of M. If M is geometrically tame, M_0 has a topological compactification $\overline{M}_0$. This compactification induces a natural topological completion $\bar{M}$ of M such that $\partial \bar{M}$ is incompressible. In $\bar{M}$, parabolic loci are left to be open cusps. When Γ is geometrically tame, every component of Ω_Γ is simply connected. Conversely a geometrically finite group such that every component of its domain of discontinuity is simply connected satisfies (*).

Then we can define a map ρ: $QH_0(\mathbf{H}^3/\Gamma) \to \mathcal{T}(\Omega_\Gamma/\Gamma)$ where $\mathcal{T}$ denotes the Teichmüller space by corresponding conformal structure of Ω_Γ/Γ. It is shown by Bers [4], Sullivan [21] and others that ρ is a homeomorphism. We let $qc: \mathcal{T}(\Omega_\Gamma/\Gamma) \to QH_0(\mathbf{H}^3/\Gamma)$ denote the inverse of ρ. It is shown by Marden [19] that when Γ is geometrically finite, $i(QH_0(M))$ forms a neighbourhood of Γ, where i is the canonical inclusion from $QH_0(M)$ to $AH(M)$.

Let M be a geometrically tame hyperbolic 3-manifold. For each geometrically infinite tame end, there exists a unique maximal geodesic lamination which cannot be realized in M (see Thurston [22] and Bonahon [5]). We call it the ending lamination of the end.

2. Lengths of Geodesic Laminations

In this section we prove some propositions which will be used afterwards. The first proposition is a generalized version of a result of Bers in [3]. We need a purely topological lemma for the proposition.

Lemma 2.1. *Let $(M, \partial M)$ be a compact irreducible ∂-irreducible atoroidal 3-manifold. Let S be a component of ∂M. Then for any element $\alpha \in \pi_1(S)$ there exist only finitely many free homotopy classes in ∂M which are freely homotopic to α in M.*

Proof. By Jaco-Shalen-Johannson theory ([10], [11]), there exist finite characteristic Seifert pairs $(\Sigma_1, \Phi_1), \ldots, (\Sigma_n, \Phi_n)$ in $(M, \partial M)$. As M is atoroidal, all of the pairs are I-pairs or solid tori. Let c_1 and c_2 be loops on ∂M which are freely homotopic in M. Then there exists a singular annulus A such that $\partial A = c_1 \cup c_2$. The annulus $(A, \partial A)$ can be homotoped into some (Σ_i, Φ_i). Hence if Σ_i is not

a solid torus, for some loop γ in Φ_i, $[c_1]=[\gamma\times\{0\}]$ and $[c_2]=[\gamma\times\{1\}]$ with respect to some I-bundle structure of Σ_i. If Σ_i is a solid torus, the annulus $(A,\partial A)$ is homotopic to a singular annulus bounded by two closed curves γ_1 and γ_2 which are contained in different components of Φ_i, where γ_j is homotopic to c_2 for $j=1,2$. It is obvious that if we fix free homotopy class of c_1 in ∂M, there exist only finitely many homotopy classes of such singular annuli in $(\Sigma_1,\Phi_1),\ldots,(\Sigma_n,\Phi_n)$. □

Proposition 2.2. *Let Γ be a geometrically finite Kleinian group. Let γ be a loxodromic element of Γ. Let $\Delta_1,\ldots,\Delta_n$ be the components of Ω_Γ which are invariant by γ. (There are only finitely many such Δ_i's by Lemma 2.1.) The element γ corresponds to an element γ_i in $PSL_2(\mathbf{R})$ through a conformal homeomorphism from Δ_i to the upper half plane. Then the following inequality holds.*

$$\frac{2n}{\mathrm{length}_{\mathbf{H}^3/\Gamma}(\gamma)}\geqq\sum_{i=1}^{n}\frac{1}{\mathrm{length}_{\Delta_i}(\gamma_i)}$$

where $\mathrm{length}_{\Delta_i}$ *denotes the translation length function with respect to the hyperbolic structure compatible with the conformal structure.*

Proof. The proof of this proposition is essentially the same as the proof of the quasi-Fuchsian case in [3]. We assume that the reader is familiar with [3]. Let Γ_{Δ_i} be the subgroup of Γ consisting of all elements which leave Δ_i invariant. By Proposition 7.1 in [15], Γ_{Δ_i} is geometrically finite. As Γ_{Δ_i} is isomorphic to a surface group, it is a quasi-Fuchsian group or a regular b-group. Hence $\partial\overline{\Delta_i}$ is a quasi-circle or a countable union of quasi-circles when Γ is a regular b-group. In each case a conformal homeomorphism from the upper half plane to Δ_i extends to its closure continuously as shown in Abikoff [1].

Thus the proof of Theorem 3 in [3] can be applied only restricting σ so that $\mathrm{supp}(\sigma)\subset\overline{\bigcup_{i=1}^{n}\Delta_i}$. Notice that the inequality $M\geqq 2\pi/\mathrm{length}_{\mathbf{H}^3/\Gamma}(\gamma)$ in [3] no longer holds. But this is not needed to obtain the inequality which we want. □

The second proposition states the continuity of the length function in the set of realized laminations. This enables us to generalize Proposition 2.2 to the inequality for lengths of geodesic laminations. Thurston announced in [24] that the length function is continuous in the set of realizable laminations in $AH(S\times I)\times\mathcal{ML}(S)$ for a hyperbolic surface S.

Proposition 2.3. *Let S be a component of $\partial\bar{M}$, where $M=\mathbf{H}^3/\Gamma$. Let R be a subset of $\mathcal{ML}(S)$ consisting of all realizable laminations. Let $L:R\to\mathbf{R}$ be the length function in M. Then L is continuous. If we define $L(\lambda)=0$ when λ is not realizable, L is continuous on $\mathcal{ML}(S)$.*

Proof. The last statement is proved in § 9 of [22]. Hence we only prove the proposition only for R. Let μ be a measured lamination in R, and let $\{\mu_i\}$ be a sequence of measured laminations on S converging to μ. There exists a pleated surface P_μ in M realizing μ. By taking a subsequence $\{\mu_j\}$ of $\{\mu_i\}$, there exists a sequence of pleated surfaces $\{P_{\mu_j}\}$ realizing $\{\mu_j\}$ such that the sequence of the marked hyperbolic structures of the P_{μ_j}'s converges to the marked hyper-

bolic structure of P_μ as shown in Thurston [22]. The length of a lamination on a surface is a continuous function of both the lamination and the hyperbolic structure on the surface, as proved in Kerckhoff [14]. Hence the realized lengths of the μ_j's on the pleated surfaces P_{μ_j}'s converge to the realized length of μ on P_μ. □

Corollary 2.4. *Let Γ be a geometrically finite Kleinian group with $M=\mathbf{H}^3/\Gamma$. Let L be a geodesic measured lamination in $S\subset\partial\bar{M}$ and let λ be its realization in M. Then there is a constant N depending only on the topological type of M such that $\text{length}_M(\lambda)\leqq N\,\text{length}_S(L)$ where we let S have hyperbolic structure induced from the conformal structure on the corresponding component of Ω_Γ/Γ.*

Proof. By Proposition 2.2 the inequality holds when L is a weighted simple closed curve. By Proposition 2.3 both length_M and length_S are continuous functions on the set of measured laminations $\mathcal{ML}(S)$. As the set of weighted simple closed curves is dense in $\mathcal{ML}(S)$, the inequality holds for general measured laminations. □

3. Limits of Geometrically Finite Groups

In this section we prove the main theorem in the case of geometrically finite group. Our main theorem deeply depends on the following theorem proved by Thurston in [25].

Theorem 3.1. *Let M be a geometrically tame complete hyperbolic 3-manifold such that $\partial\bar{M}$ is incompressible. Let (Σ,Φ) be the union of characteristic pairs in $(\bar{M},\partial\bar{M})$. Let $\{g_i\}\subset AH(M)$ be a sequence which does not have a subsequence converging in $AH(M)$. Then there exists a subsequence $\{g_i'\}\subset\{g_i\}$ and an essential incompressible flat measured lamination (L,μ) in (Σ,Φ) such that $(\lim_{i\to\infty}\text{length}_{g_i}(\sigma))_{\sigma\in\mathscr{C}}=(\inf_{s\in\sigma}\mu(s))_{\sigma\in\mathscr{C}}$ as an element of $P\mathbf{R}_+^{\mathscr{C}}$. In the case when $\partial\bar{M}$ has a parabolic locus, by restricting $AH(M)$ to the set preserving parabolicity, L can be made disjoint from parabolic locus.*

Now we can prove our main theorem in weak form in the case of geometrically finite groups.

Theorem 3.2. *Let $\{\Gamma_i\}$ be a sequence of quasi-conformal deformations of a geometrically finite group Γ with $M=\mathbf{H}^3/\Gamma$. Suppose that, for any essential annulus A in $(\bar{M},\partial\bar{M})$ which is not homotopic into the cuspidal part of M, there exists a simple closed curve γ on $\partial\bar{M}$ intersecting essentially with ∂A such that $\text{length}_{\Omega_{\Gamma_i}/\Gamma_i}(\Gamma)$ is bounded as i goes to infinity. Then $\{\Gamma_i\}$ has a subsequence which converges algebraically to a Kleinian group isomorphic to Γ.*

Proof. Suppose that $\{\Gamma_i\}$ does not have a subsequence which converges in $AH(M)$. Then by Theorem 3.1 there exists a subsequence $\{\Gamma_i'\}$ and an essential incompressible measured lamination (L,μ) such that $(\lim_{i\to\infty}\text{length}_{\Gamma_i}(\sigma))=(\inf_{s\in\sigma}\mu(s))$ in $\mathbf{R}_+^{\mathscr{C}}$. By assumption for every essential annulus A there exists a simple closed curve γ on $\partial\bar{M}$ which intersects A essentially such that the sequence $\{\text{length}_{\Omega_{\Gamma_i}/\Gamma_i}(\gamma)\}$ is bounded. First we will prove that there exists a simple closed

curve β on $\partial\bar{M}$ which intersects (L, μ) essentially and such that $\{\text{length}_{\Omega_{\Gamma_i}/\Gamma_i}(\beta)\}$ is bounded. We can assume that there is a component (L_0, μ_0) of (L, μ) which is contained in an I-bundle component Σ of the characteristic submanifold of M. Let $p:\Sigma \to B$ be the bundle projection. Then we can assume L_0 is a saturated set over a measured lamination $p(L_0)$ on B. Each component of $B-p(L_0)$ is homeomorphic to the interior of a bounded surface. Now suppose that the whole of M is not an I-bundle over a surface. Then some component C of $B-p(L_0)$ is not simply-connected. Let α be a simple closed curve in C which is homotopic to an end of C which is made by $p(L_0)$. Then every lamination intersecting α essentially intersects $p(L_0)$ essentially. Let A be the saturated annulus over α. We can choose a simple closed curve α so that the annulus A is not homotopic into the cuspidal part of M because we supposed that M is not an I-bundle over a surface. By assumption there exists a simple closed curve γ intersecting A essentially such that $\{\text{length}_{\Omega_{\Gamma_i}/\Gamma_i}(\gamma)\}$ is bounded. This simple closed γ also intersects L_0 essentially. Hence we can let this simple closed curve γ be β. Suppose that M itself is an I-bundle over a surface. Then the above argument cannot be applied only when each component of $B-p(L_0)$ is simply connected or an annulus containing an cusp end of B. But in this case every essential simple closed curve on B intersects $p(L_0)$ essentially. Hence for any essential annulus A, the simple closed curve γ intersecting A essentially can be made to be β.

Let $[\beta]$ denote a class of $\mathscr{C}$ represented by β. The fact $\mu([\beta])>0$ implies that $\text{length}_{\Gamma_i}([\beta])$ goes to infinity as i goes to infinity. On the other hand by Proposition 2.2 $\text{length}_{\Gamma_i}([\beta])$ is also bounded as i goes to infinity. This is a contradiction. Hence $\{\Gamma_i\}$ has a subsequence converging in $AH(M)$. By Jørgensen's inequality, limits of faithful discrete representations are also faithful discrete. Therefore the subsequence converges to a Kleinian group isomorphic to Γ. □

The following corollary is a generalization of a result by Bers in [3].

Corollary 3.3. *Let Γ be a geometrically finite group with $M=\mathbf{H}^3/\Gamma$. Let $S_1, \ldots, S_n$ be the components of $\partial\bar{M}$. Suppose that there are no essential annuli in $(\bar{M}, \partial\bar{M})$ both of whose boundary components are on S_1. Let $\{m_i^1\}$ be an arbitrary sequence of $\mathscr{T}(S_1)$. Let $\{m_i^2\}, \ldots, \{m_i^n\}$ be bounded sequences of $\mathscr{T}(S_2), \ldots, \mathscr{T}(S_n)$ respectively. Then $\{qc(m_i^1, \ldots, m_i^n)\}$ has a subsequence converging algebraically to a Kleinian group isomorphic to Γ.*

Proof. By assumption every essential annulus A must intersect one of $S_2, \ldots, S_n$. Then there is a simple closed curve γ on $S_j (2 \leqq j \leqq n)$ which intersects ∂A essentially. As $\{m_i^j\}$ is bounded, $\text{length}_{\Omega_{\Gamma_i}/\Gamma_i}(\gamma)$ is bounded. Hence Theorem 3.2 implies Corollary 3.3. □

Before stating our main theorem in generalized form, we prove a lemma which is essential in the proof of the theorem.

Lemma 3.4. *Let S be an underlying surface of a complete hyperbolic surface of finite type S_0. Let μ be a measured lamination on S and let $\{\mu_j\}$ be a sequence in $\mathscr{ML}(S)$ converging to μ. Let $\{g_i\}$ be a sequence in $\mathscr{T}(S)$ converging to a projective measured lamination $[\lambda] \in \mathscr{PL}(S)$ represented by a measured lamination λ with $i(\mu, \lambda)>0$. Then $\{\text{length}_{g_j}(\mu_j)\}$ goes to infinity.*

Proof. Let $\mathcal{SML}(S)$ be a sphere in $\mathcal{ML}(S)$ consisting of all measured laminations λ satisfying $\text{length}_{S_0}(\lambda)=1$. We can assume that λ is contained in $\mathcal{SML}(S)$. Let $q:\mathcal{T}(S)\to\mathcal{ML}(S)\cong\mathcal{MF}(S)$ be the function defined in exposé 8 of Fathi et al. [8] for some pants decomposition of S by a family of simple closed curves $\{K_n\}$ with $\text{length}_{g_j}(K_n)>\varepsilon$ for all j,n. (Although the surface S is assumed to be compact in [8] it is easy to extend the results there to the case when S is with cusps. In the non-compact case measured laminations are assumed to have compact support. For a non-compact surface some boundary component of pants may go to the circle at infinity i.e. some pants may be an annulus with a cusp or a disk with two cusps. From now on we will omit such remarks for the case when the surface is non-compact.) As is shown there, the inequality $\text{length}_{g_j}(\mu_j)\geqq i(q(g_j),\mu_j)$ holds for each j, and there exists a sequence of positive real number $\{t_j\}$ such that $q(g_j)=t_j\lambda_j$ for a sequence of measured laminations $\{\lambda_j\}\subset\mathcal{SML}(S)$ converging to λ. (More precisely the inequality above is proved only for simple closed curves in [8], the inequality for general measured laminations follows from the continuity of the intersection number and of the length function.) It is obvious that the sequence $\{t_i\}$ goes to infinity as $i\to\infty$. Then $i(q(g_i),\mu_i)=t_i i(\lambda_i,\mu_i)$, where the right hand goes to infinity because $i(\lambda_i,\mu_i)\to i(\lambda,\mu)>0$ as $i\to\infty$. (The continuity of the intersection number as a function from $\mathcal{ML}(S)\times\mathcal{ML}(S)$ to $\mathbf{R}$ is proved by Rees in [20].) □

Theorem 3.5. *Let $\{\Gamma_i\}$ be a sequence of quasi-conformal deformations of a geometrically finite group Γ with $\mathbf{H}^3/\Gamma=M$. Suppose that for any essential annulus A in $(\bar{M},\partial\bar{M})$ which is not homotopic into the cuspidal part of M, there exists a measured lamination γ on $\partial\bar{M}$ intersecting A essentially and a sequence of measured lamination $\{\gamma_i\}$ on S converging to γ such that the sequence $\{\text{length}_{\Omega_{\Gamma_i}/\Gamma_i}(\gamma_i)\}$* is bounded. *Suppose moreover that γ is a simple closed curve when $\bar{M}$ is an I-bundle over a surface. Then $\{\Gamma_i\}$ has a subsequence which converges algebraically to a Kleinian group isomorphic to Γ.*

Proof. Suppose that $\{\Gamma_i\}$ does not have a subsequence which converges in $AH(M)$. We will show that this implies a contradiction as in the proof of Theorem 3.2. There exists a subsequence $\{\Gamma_i'\}$ of $\{\Gamma_i\}$ which goes to infinity in $AH(M)$ in a way described by a measured lamination (L,μ) as in Theorem 3.1. Let M_i be the hyperbolic manifold $\mathbf{H}^3/\Gamma_i'$. The assumption implies, as in the proof of Theorem 3.2, that there exists a measured lamination γ on $\partial\bar{M}$ and a sequence of measured laminations $\{\gamma_i\}$ converging to γ such that γ intersects (L,μ) essentially and such that $\{\text{length}_{\Omega_{\Gamma_i'}/\Gamma_i}(\gamma_i)\}$ is bounded as $i\to\infty$. Let λ be a maximal lamination on the component S of $\partial\bar{M}$ containing γ and let λ_i be its realization in M_i such that all closed leaves of λ_i have length greater than ε for a fixed constant ε and such that for each closed leaf, all leaves limiting to this leaf spiral in the same direction. Such λ exists by taking a suitable subsequence of $\{\Gamma_i'\}$ as is shown in [24].

First assume that the component S of $\partial\bar{M}$ containing γ is doubly incompressible. Let $f_i:P_i\to M_i$ be a pleated surface with homeomorphisms $\{\phi_i:S\to P_i\}$ such that $f_i\circ\phi_i$ is homotopic to the inclusion $\iota:S\to\bar{M}$, whose pleating locus is equal to $\phi_i(\lambda)$. Then Theorem 3.2 in [24] can be applied, so that there exist a continuous function $w_\lambda:\mathcal{ML}(S)\to\mathbf{R}$ and a constant C such that $\text{length}_{M_i}(f_i\circ\phi_i(\nu))$

$\leqq \text{length}_{P_i}(\phi_i(\nu)) \leqq \text{length}_{M_i}(f_i \circ \phi_i(\nu)) + C w_\lambda(\nu)$ for every measured lamination ν on S, where length means the geodesic length. Hence the sequence $\{m_i\}$ of the marked hyperbolic structures of the P_i's converges in $\overline{\mathcal{T}(S)}$ to the projective lamination represented by $(L, \mu) \cap S$. As $i(\gamma, (L, \mu)) > 0$, the sequence $\{\text{length}_{P_i}(f_i(\gamma_i))\}$ goes to infinity as $i \to \infty$ by Lemma 3.4, which implies that the sequence $\{\text{length}_{M_i}(f_i \circ \phi_i(\gamma_i))\}$ also goes to infinity. This is a contradiction because $\text{length}_{M_i}(f_i \circ \phi_i(\gamma_i)) \leqq \text{length}_{\Omega_{\Gamma_i}/\Gamma_i}(\gamma_i)$ by Lemma 2.4.

In the general case let $p: \tilde{M} \to M$ and $p_i: \tilde{M}_i \to M_i$ be the coverings associated with $\pi_1(S)$. Then the $\tilde{M} \to M$ and $p_i: \tilde{M} \to M_i$ be the coverings associated with $\pi_1(S)$. Then the $\tilde{M}_i$'s are geometrically finite and the surface S lifts homeomorphically to $\tilde{S}$, which is a component of $\partial \overline{\tilde{M}}$. As $\tilde{S}$ is doubly incompressible in $\overline{\tilde{M}}$, there is a pleated surface $\tilde{f}_i: P_i \to \tilde{M}_i$ satisfying the same conditions as in the case when S is doubly incompressible. Since every realization of a lamination on S in M_i lifts preserving the length to a realization of the corresponding lamination on $\tilde{S}$ in $\tilde{M}_i$, using the pleated surfaces $\{p_i \circ \tilde{f}_i: P_i \to \tilde{M}_i\}$ the same argument as in the case when S is doubly incompressible can be applied. □

Theorem 3.5 implies an analogue of the double limit theorem. For proving it we need the following lemma which is a special case of Proposition 2.4 in Thurston [24]. As a proof of this lemma is not given in [24], we give a sketch of proof below.

Lemma 3.6. *Let S be an underlying surface of complete hyperbolic surface of finite type. Let $\{g_j\} \subset \mathcal{T}(S)$ be a sequence which converges in $\overline{\mathcal{T}(S)} = \mathcal{T}(S) \cup \mathcal{PL}(S)$ to a measured lamination γ. Then there exists a sequence $\{\lambda_j\} \subset \mathcal{ML}(S)$ such that λ_j converges to γ in $\mathcal{ML}(S)$ and such that $\{\text{length}_{g_j}(\lambda_j)\}$ is bounded.*

Proof. Fix a system of simple closed curves $\{K_n\}$ which decompose S into plants. Let $q: \mathcal{T}(S) \to \mathcal{ML}(S) (= \mathcal{MF}(S))$ be the function defined in exposé 8 of Fathi et al. [8]. Then as is shown in [8], for every hyperbolic metric m satisfying the inequality $\text{length}_m(K_n) > \varepsilon$ and a simple closed curve α the inequality $\text{length}_m(\alpha) \leqq i(q(m), \alpha) + A(\alpha)$ holds, where $A(\alpha)$ is a constant depending only on α. As is shown in Lemma II.2 in [8], $A(\alpha)$ is bounded by $C' \sum_n i(\alpha, K_n)$ where C' is a constant depending only on $\{K_n\}$. Therefore for a general measured lamination λ the same inequality holds where $A(\lambda) \leqq C' \sum_n i(\lambda, K_n)$. There exists a neighbourhood V of γ in $\overline{\mathcal{T}(S)}$ such that for any $m \in V \cap \mathcal{T}(S)$ the condition $\text{length}_m(K_n) > \varepsilon$ is satisfied (see III.2 in exposé 8 of [8]). Hence we can assume that all of $\{g_j\}$ satisfy the inequality above. Let $\{\mu_j\}$ be a sequence of positive real numbers such that $\{\mu_j q(g_j)\}$ converges to γ in $\mathcal{ML}(S)$. Such a sequence exists because a sequence of the projective classes $\{[q(g_j)]\}$ converges to the projective class $[\gamma]$ in $\mathcal{PL}(S)$. We let $\{\lambda_j\}$ be $\{\mu_j q(g_j)\}$. Then $\text{length}_{g_j}(\lambda_j) \leqq A(\lambda_j)$. As $\sum_n i(\lambda_j, K_n)$ converges to $\sum_n i(\gamma, K_n)$, the sequence $\{A(\lambda_j)\}$ is bounded. □

Combining Theorem 3.5 and the lemma above, we obtain the next theorem.

Theorem 3.7. *Let Γ be a geometrically finite group which is not a quasi-Fuchsian group with $\mathbf{H}^3/\Gamma = M$. Let $\{\Gamma_i\}$ be a sequence of quasi-conformal deformations*

of Γ. Let $S_1, \ldots, S_n$ be the components of $\partial \bar{M}$. Let $\gamma_{i_1}, \ldots, \gamma_{i_k}$ be measured lamination on $S_{i_1}, \ldots, S_{i_k}$ where $S_{i_1}, \ldots, S_{i_k}$ are disjoint. Suppose that for each essential annulus A, the boundary ∂A intersects some γ_{i_j} essentially. Let $\{(g_m^1, \ldots, g_m^n)\}$ be a sequence in $\mathcal{T}(S_1) \times \ldots \times \mathcal{T}(S_n)$ such that $\{g_m^{i_j}\}$ converges to γ_{i_j} in $\overline{\mathcal{T}(S_{i_j})}$ as m goes to infinity. Then the sequence $\{qc(g_i^1, \ldots, g_i^n)\}$ has a subsequence which converges to a Kleinian group isomorphic to Γ. □

4. Geometrically Infinite Groups

In the last section we proved the theorems under the assumption that Γ is geometrically finite. In this section we attempt to extend the theorems to the geometrically infinite (tame) case under some additional assumptions. Let Γ be a geometrically tame group with $M = \mathbf{H}^3/\Gamma$. We define an element $\gamma \in \Gamma$ to be an accidental parabolic element when γ is parabolic and there is a closed curve γ' in Ω_Γ/Γ homotopic to γ in $\bar{M}$, which represents a non-cuspidal class in $\pi_1(\Omega_\Gamma/\Gamma)$.

Lemma 4.1. *Let Γ be a geometrically infinite tame group. Let λ be the ending lamination on a component S of $\partial \bar{M}$. Then every region of $S - \lambda$ is either simply connected or an open annulus containing a cusp generated by a parabolic element of $\pi_1(S)$.*

Proof. Suppose that there is a complementary region V of λ in S which is neither simply connected nor an open annulus containing a cusp generated by a parabolic element. Then there is a simple closed geodesic γ in V such that $\lambda \cup \gamma$ is a geodesic lamination. As is shown in Proposition 8.8.5 in [22], the set of all pleated surfaces containing γ as a pleating locus is compact. But as λ is an ending lamination, a pleated surface whose pleating locus is near $\lambda \cup \gamma$ tends to the end. This is a contradiction. □

In the process of proving Theorem 3.5, we use the assumption that Γ is geometrically finite only when we prove Lemma 2.2. Hence for proving the main theorem for geometrically infinite groups, we have only to generalize Lemma 2.2.

Lemma 4.2. *Let Γ be a geometrically infinite (tame) group without accidental parabolic elements such that Ω_Γ is nonempty and disconnected. Then the inequality in Lemma 2.2 holds for Γ.*

Proof. In the proof of Lemma 2.2, we used geometrical finiteness of Γ to prove that $\mathbf{H}^3/\Gamma_{\Delta_i}$ is geometrically finite, where Γ_{Δ_i} denotes the subgroup of Γ consisting of all elements which leave Δ_i invariant. We will prove that this also holds for our Γ. Observe that Δ_i is a component of $\Omega_{\Gamma_{\Delta_i}}$. As $S_\infty^2 - \overline{\Delta_i}$ is not empty, there is another component of $\Omega_{\Gamma_{\Delta_i}}$. Since Γ_{Δ_i} has no accidental parabolic elements, Γ_{Δ_i} cannot degenerate by Lemma 4.1. Hence Γ_{Δ_i} is quasi-Fuchsian. Thus the proof of Lemma 2.2 can be applied to Γ. □

Theorem 4.3. *Theorem 3.5 holds for a geometrically infinite (tame) group Γ without accidental parabolic elements such that Ω_Γ has at least two components.* □

Remark. An example of Kleinian groups which satisfy the assumption of Lemma 4.2 and Theorem 4.3 can be constructed as follows. Let Γ be a geometrically finite group which satisfies the assumption of Corollary 3.3 and assume $n \geqq 3$. Let $\{m_i^1\}$ be a sequence in $\mathcal{T}(S_1)$ which converges in $\overline{\mathcal{T}(S_1)}$ to a projective lamination whose complement regions are simply connected. Let $m^2, \ldots, m^n$ be fixed elements in $\mathcal{T}(S_2), \ldots, \mathcal{T}(S_n)$ respectively. Then by Corollary 3.3 $qc(m_i^1, m^2, \ldots, m^n)$ converges to a Kleinian group G. This Kleinian group G is an example what we wanted. G has $n-1$ geometrically finite ends and one geometrically infinite end.

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