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Titel: Towards a "Matrix Theory" for Unbounded Operator Matrices.

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Jahr: 1989

PURL: https://resolver.sub.uni-goettingen.de/purl?266833020_0201|log12

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Math. Z. 201, 57–68 (1989)

Mathematische Zeitschrift

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Towards a "Matrix Theory" for Unbounded Operator Matrices

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1. Introduction

Many linear evolution equations such as the wave equation [12], retarded and neutral differential equations [7, 22], Volterra integrodifferential equations [6, 10, 17], equations from thermoelasticity [15, 16] or fluid dynamics [19, 24] can be written formally as a first order Cauchy problem

$$\dot{u}(t)=A\,u(t), \qquad u(0)=u_0, \tag{1.1}$$

where $u(\cdot)$ is a function on $\mathbb{R}_+$ with values in a product $E_1\times\ldots\times E_n$ of different Banach spaces E_i and where $\mathscr{A}=(A_{ij})_{n\times n}$ is a matrix whose entries A_{ij} are (unbounded) linear oprators between the spaces E_j and E_i. The following example seems to indicate that this notation is only of formal nature not allowing any general and systematic "matrix theory".

On the product space $\mathbb{R}\times L^2(\mathbb{R}_+)$ we consider the operator $\mathscr{A}:=\begin{pmatrix} a & \delta_0 \\ c & D\end{pmatrix}$, where $D=\dfrac{d}{dx}$ is defined on $D(A)=W^1(\mathbb{R}_+)$, δ_0 is the Dirac measure in 0, $c\in L^2(\mathbb{R}_+)$ is understood as an operator from $\mathbb{R}$ into $L^2(\mathbb{R}_+)$ and $\mathscr{A}$ has domain $D(\mathscr{A})=\mathbb{R}\times W^1(\mathbb{R}_+)$. Such operators occur in many places (see [5, 8, 10]) and seem to have properties contradicting our expectations how a "matrix" should behave:

– It is not possible to multiply all the entries of $\mathscr{A}$. Therefore it is unclear how to define a "determinant" analogous to $\det\alpha=a_{11}\cdot a_{22}-a_{12}\cdot a_{21}$ for matrices $\alpha=(a_{ij})_{2\times 2}$ over $\mathbb{C}$ (or any commutative ring). In addition, the spectrum $\sigma(\mathscr{A})$ of $\mathscr{A}$ as computed in [17] contains the set

$$\left\{\lambda\in\mathbb{C}\colon \int_0^\infty e^{-\lambda s}c(s)\,ds=\lambda-a\right\} \tag{1.2}$$

* This work was done during a sabbatical stay at the Tulane University in New Orleans, USA. I gratefully acknowledge the wonderful hospitality and helpful cooperation of J.A. Goldstein

a formula bearing no resemblance to

$$\lambda \in \sigma(\alpha) \Leftrightarrow \det(\lambda - \alpha) \text{ not invertible.} \tag{1.3}$$

– The adjoint of $\mathscr{A}$ cannot be obtained by taking adjoints of the entries A_{ij}. Desch-Grimmer-Schappacher [8] computed the adjoint of $\mathscr{A} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{d}{dx} \end{pmatrix}$ as $\mathscr{A}' = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{d}{dx} \end{pmatrix}$ on the domain $D(\mathscr{A}') = \left\{ \begin{pmatrix} x \\ f \end{pmatrix} : f \in W^1(\mathbb{R}_+),\ f(0) = x \right\}$. Evenworse, the inverse of $\lambda - \mathscr{A}' = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda + \frac{d}{dx} \end{pmatrix}$ is not a diagonal but an upper triangular matrix.

As a consequence we note that naive matrix computations are misleading. This is not only due to the fact that our product space is made up of different Banach spaces and that the matrix entries do not commute.

– Take an unbounded semigroup generator A on some Banach space E. In [11] we studied matrices

$$\mathscr{A} = (a_{ij} \cdot A + b_{ij} \cdot \mathrm{Id})_{2 \times 2}$$

on $E \times E$ for $a_{ij}, b_{ij} \in \mathbb{C}$ and observed, e.g., that for $\mathscr{A} = \begin{pmatrix} A & \mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix}$ the "determinant" $\det \mathscr{A} = -\mathrm{Id}$ is invertible while $\mathscr{A}$ is not. In addition such simple matrices as $\begin{pmatrix} A & A^n \\ 0 & A \end{pmatrix}$ are semigroup generators only if A is an analytic generator (for $n = 1$) or if A is bounded (for $n \geqq 2$).

Not deterred by the above examples we nevertheless search for general results confirming the standard matrix rules even for matrices of unbounded operators on products of different Banach spaces. In this paper we present the first results of our investigation and

– introduce a non-commutative characteristic function yielding an efficient way of computing the spectrum of $\mathscr{A}$,
– obtain results on the generation of semigroups by operator matrices, and
– discuss a concrete example appearing in [24].

For preliminary or related results we refer to [11, 18] and [25]. Matrices with "non-diagonal" domains as appearing in [16] and [20] will be treated in a subsequent paper.

2. A Non-Commutative Characteristic Function

It is well known that a matrix over a commutative ring R is invertible if and only if its determinant is invertible in R. In addition it turned out that commuta-

tivity of R is quite essential for a general result of this type. For matrices of operators on a Hilbert space Halmos ([13], Chap. 7) discusses some particular cases in which the invertibility of the formal determinant $AD-BC$ characterizes the invertibility of the matrix $\mathscr{A}=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ on the product space (see also [4]).

In the even more "non-commutative" situation in which A, B, C and D are linear operators between two different Banach spaces E and F such that $\mathscr{A}$ yields an operator on $E\times F$ the above determinant cannot be defined and the search for a simple and general invertibility criterion seems to be hopeless. Not being discouraged we assume in a first step that all entries A, B, C, D are bounded (and everywhere defined). Since we want to use the determinant in order to invert $\mathscr{A}$ (and $\lambda-\mathscr{A}$ for $\lambda\in\mathbb{C}$) and in view of the applications it is not too restrictive to assume that the diagonal elements A and D already are invertible. All of a sudden, the problem of inverting $\mathscr{A}$ becomes easy.

Lemma 2.1. *Let E, F be Banach spaces and $A\in\mathscr{L}(E)$, $D\in\mathscr{L}(F)$, $B\in\mathscr{L}(F,E)$, $C\in\mathscr{L}(E,F)$ and consider $\mathscr{A}=\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in\mathscr{L}(E\times F)$. If A and D are invertible then the following assertions are equivalent:*

(a) *$\mathscr{A}$ is invertible in $\mathscr{L}(E\times F)$.*
(b) *$A-BD^{-1}C$, hence $\mathrm{Id}-BD^{-1}CA^{-1}$ is invertible in $\mathscr{L}(E)$.*
(c) *$D-CA^{-1}B$, hence $\mathrm{Id}-CA^{-1}BD^{-1}$ is invertible in $\mathscr{L}(F)$.*

Proof. By symmetry it suffices to show the equivalence of (a) and (b). If $\Delta:=A-BD^{-1}C$ is invertible then it is easy to verify that the matrix

$$\begin{pmatrix} \Delta^{-1} & -\Delta^{-1}BD^{-1} \\ -D^{-1}C\Delta^{-1} & D^{-1}(\mathrm{Id}+C\Delta^{-1}BD^{-1}) \end{pmatrix} \tag{2.1}$$

is the inverse of $\mathscr{A}$. Conversely, suppose that $\mathscr{A}$, A and D are invertible. Then it follows from the identity

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} A^{-1} & 0 \\ 0 & -D^{-1} \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}=\begin{pmatrix} A-BD^{-1}C & 0 \\ 0 & CA^{-1}B-D \end{pmatrix}$$

that Δ (and $D-CA^{-1}B$) is invertible. □

Remark. For finite dimensional Banach spaces E and F the expressions "$A-BD^{-1}C$" and "$D-CA^{-1}B$" are called *Schur complements* (see e.g. [23]).

This lemma indicates that – under suitable assumptions – one could take either $A-BD^{-1}C$ or $D-CA^{-1}B$ as a substitute for the determinant of $\mathscr{A}$. The spectral values $\lambda\in\sigma(\mathscr{A})$ for $\lambda\notin\sigma(A)\cup\sigma(D)$ can then be characterized by a condition as in (1.3).

But our interest is in matrices with unbounded entries and it is clear that we will need some assumptions on the domains of the operators involved. The following is satisfied in many of the concrete examples.

Assumption 2.2. Let E, F be Banach spaces and assume that

(i) the operator A with domain $D(A)$ has non-empty resolvent set $\rho(A)$ in E,

(ii) the operator D with domain $D(D)$ has non-empty resolvent set $\rho(D)$ in F,

(iii) the operator B with domain $D(B)$ is relatively D-bounded, i.e., $D(D)\subset D(B)$ and $BR(\lambda, D)\in\mathscr{L}(F, E)$ for $\lambda\in\rho(D)$,

(iv) the operator C with domain $D(C)$ is relatively A-bounded, i.e., $D(A)\subset D(C)$ and $CR(\lambda, A)\in\mathscr{L}(E, F)$ for $\lambda\in\rho(A)$,

(v) the operator $\mathscr{A}=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with domain $D(\mathscr{A})=D(A)\times D(D)$ is closed in $E\times F$.

While conditions (i) to (iv) are quite natural and express the fact that the diagonal elements of $\mathscr{A}$ determine the "degree of unboundedness" condition (v) could seem to be artificial. In fact it excludes such a simple matrix as $\mathscr{A}=\begin{pmatrix} A & A \\ A & A \end{pmatrix}$ which is closed on the domain $D(\mathscr{A})=\left\{\begin{pmatrix} x \\ y \end{pmatrix}: x+y\in D(A)\right\}$. As we will show in a later paper arbitrary matrices may have quite complicated "natural" domains. But at this stage it seems to be appropriate to restrict the generality.

Under the Assumption 2.2 we are now ready to define an operator theoretical analogue to the "characteristic polynomial" of a scalar 2×2-matrix. This will be valid for those $\lambda\in\mathbb{C}$ which are not in the spectrum of the diagonal operators, i.e. where the resolvents $R(\lambda, A):=(\lambda-A)^{-1}$ and $R(\lambda, D):=(\lambda-D)^{-1}$ are well-defined.

Definition 2.3. Under the Assumption 2.2 and for $\lambda\notin\sigma(A)\cup\sigma(D)$ we consider the operators

$$\Delta_E(\lambda):=\lambda-A-BR(\lambda, D)C \tag{2.2}$$

and

$$\Delta_F(\lambda):=\lambda-D-CR(\lambda, A)B \tag{2.3}$$

with domain $D(A)$ in E, resp. domain $D(D)$ in F. The function $\lambda\mapsto\Delta_E(\lambda)$, resp. $\Delta_F(\lambda)$ will be called the *E – characteristic*, resp. *F – characteristic operator function* of the matrix $\mathscr{A}=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

With these characteristic operator functions we are able to determine those spectral values of $\mathscr{A}$ which are not contained in $\sigma(A)\cup\sigma(D)$.

Theorem 2.4. *Consider an operator matrix $\mathscr{A}=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ on $E\times F$ for which the Assumption 2.2 holds. For $\lambda\notin\sigma(A)\cup\sigma(D)$ the following assertions are equivalent:*

(a) $\lambda\in\sigma(\mathscr{A})$,

(b) $0\in\sigma(\Delta_E(\lambda))$,

(c) $0\in\sigma(\Delta_F(\lambda))$.

Moreover, for $\lambda \notin \sigma(A) \cup \sigma(D) \cup \sigma(\mathscr{A})$ *the resolvent of* $\mathscr{A}$ *is given by*

$$R(\lambda, \mathscr{A}) = \begin{pmatrix} \Delta_E^{-1}(\lambda) & \Delta_E^{-1}(\lambda) B R(\lambda, D) \\ R(\lambda, D) C \Delta_E^{-1}(\lambda) & R(\lambda, D)(\mathrm{Id} + C \Delta_E^{-1}(\lambda) B R(\lambda, D)) \end{pmatrix} \tag{2.4}$$

or the analogous expression using $\Delta_F(\lambda)$.

Proof. For $\lambda \in \mathbb{C} \setminus (\sigma(A) \cup \sigma(D)) = \rho(A) \cap \rho(D)$ the matrix $\lambda - \mathscr{A}$ can be written as

$$\lambda - \mathscr{A} = \begin{pmatrix} \lambda - A & -B \\ -C & \lambda - D \end{pmatrix} = \begin{pmatrix} \mathrm{Id} & -BR(\lambda, D) \\ -CR(\lambda, A) & \mathrm{Id} \end{pmatrix} \begin{pmatrix} \lambda - A & 0 \\ 0 & \lambda - D \end{pmatrix} =: \mathscr{B}_\lambda \circ \mathscr{A}_\lambda .$$

Here $\mathscr{A}_\lambda$ is a bijection from $D(\mathscr{A})$ onto $E \times F$ while $\mathscr{B}_\lambda$ is bounded on $E \times F$. Therefore $\lambda - \mathscr{A}$ is invertible if and only if $\mathscr{B}_\lambda$ is invertible. Since $\mathscr{B}_\lambda$ has invertible diagonal entries the assertions follow from Lemma 2.1 and the observation that $\mathrm{Id} - BR(\lambda, D) CR(\lambda, A)$ is invertible if and only if $\Delta_E(\lambda)$ is invertible. The matrix representation of $R(\lambda, \mathscr{A})$ is a consequence of (2.1). □

Initially the fact that $\mathscr{A}$ acts on a product of two different Banach spaces E and F seemed to be a handicap. Now the above characterization of $\sigma(\mathscr{A})$ through one characteristic operator function on E and one on F offers a surprising advantage: If E and F are of different size we clearly compute $\sigma(\mathscr{A})$ through the characteristic operator function in the smaller space. In particular, if $\dim E < \infty$ then $\Delta_E(\lambda)$ becomes a matrix and $0 \in \sigma(\Delta_E(\lambda))$ becomes the "characteristic equation"

$$\det(\Delta_E(\lambda)) = 0. \tag{2.5}$$

We derive a well-known result (see [17]) from these considerations.

Example 2.5. Take $\mathbb{R} \times L^2(\mathbb{R}_+)$ and $\mathscr{A} = \begin{pmatrix} a & \delta_0 \\ c & D \end{pmatrix}$ as in Sect. 1. The resolvent of $D = \dfrac{d}{dx}$ is $R(\lambda, D) f(t) = \int_t^\infty e^{-\lambda(s-t)} f(s)\, ds$ for $\mathrm{Re}\,\lambda > 0$. Therefore $\Delta_{\mathbb{R}}(\lambda) = \lambda - a - \delta_0 R(\lambda, D) c(\cdot) = \lambda - a - \int_0^\infty e^{-\lambda s} c(s)\, ds$ for $\mathrm{Re}\,\lambda > 0$, $\lambda \neq a$. This yields by the above theorem and for $\mathrm{Re}\,\lambda > 0$, $\lambda \neq a$:

$$\lambda \in \sigma(\mathscr{A}) \Leftrightarrow \int_0^\infty e^{-\lambda s} c(s)\, ds = \lambda - a, \tag{2.6}$$

which is formula (1.2) but now obtained through an analogue of (1.3).

3. Operator Matrices as Semigroup Generators[1]

The original motivation for investigating operator matrices stems from systems of linear evolution equations. By standard semigroup theory it is well known that such systems are well-posed if and only if the corresponding operator matrix is the generator of a strongly continuous semigroup on the product space. Therefore we ask the following question:

Let E, F be two Banach spaces and consider an operator matrix $\mathscr{A}=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ on $E \times F$ satisfying Assumption 2.2. Under which conditions does $\mathscr{A}$ generate a strongly continuous semigroup?

If we assume again that the diagonal operators A and D behave well, i.e., are generators on E, resp. on F, then $\mathscr{A}_0 := \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$ is a generator on $E \times F$ and $\mathscr{B} := \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$ is a relatively bounded perturbation of $\mathscr{A}_0$. Therefore one may use any of the standard hypotheses (e.g., B and C bounded) implying that the sum $\mathscr{A}=\mathscr{A}_0+\mathscr{B}$ is a generator. Clearly, such results are not very interesting. Instead we are looking at perturbation results which are truly based on the matrix structure of $\mathscr{A}_0$ and $\mathscr{B}$. Here is a prototype of what we have in mind.

Proposition 3.1. *Assume that A and D generate strongly continuous semigroups $(e^{tA})_{t\geq 0}$ on E and $(e^{tD})_{t\geq 0}$ on F. For a D-bounded operator B: $D(D)\to E$ the following assertions are equivalent:*

(a) *The matrix $\mathscr{A}=\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ with domain $D(\mathscr{A})=D(A)\times D(D)$ is a generator on $E\times F$.*

(b) *The operators*

$$R(t) := \int_0^t e^{A(t-s)} B e^{Ds}\, ds \tag{3.1}$$

are uniformly bounded as $t\downarrow 0$.

In that case, the semigroup generated by $\mathscr{A}$ is given by the matrices

$$T(t) := \begin{pmatrix} e^{At} & R(t) \\ 0 & e^{Dt} \end{pmatrix}, \quad t\geq 0. \tag{3.2}$$

Proof. If (b) is satisfied it follows that the matrices (3.2) define a strongly continuous semigroup. Denote its generator by $\tilde{\mathscr{A}}$ and take $y\in D(D)$. If we can show that $t\to R(t)y$ is differentiable at 0 with derivative By we obtain that $\tilde{\mathscr{A}}$ is an extension of $\mathscr{A}$.

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[1] For this section I benefitted from some useful conversations with J. Prüss who considerably simplified some of my original proofs

Consider

$$\frac{1}{t}R(t)y-By=\frac{1}{t}\int_0^t (e^{A(t-s)}Be^{Ds}y-By)\,ds$$

$$=\frac{1}{t}\int_0^t ((e^{A(t-s)}-\mathrm{Id})Be^{Ds}y+B(e^{Ds}y-y))\,ds.$$

Since $y\in D(D)$ it follows that $\lim_{s\to 0} e^{Ds}y=y$ for the graph norm on $D(D)$. By assumption B is continuous for that norm, hence $B(e^{Ds}y-y)$ converges to zero and for each $\varepsilon>0$ there exists $t_1>0$ such that $\|B(e^{Ds}y-y)\|\leqq\varepsilon$ for $s\in[0,t_1]$. Similarly one concludes that $\{Be^{Ds}y\colon s\in[0,t]\}$ is a relatively compact set. Since $(e^{At})_{t\geqq 0}$ is strongly continuous one finds $t_2>0$ such that $\|(e^{A(t-s)}-\mathrm{Id})Be^{Ds}y\|\leqq\varepsilon$ for $0\leqq s\leqq t\leqq t_2$. Combining both estimates we obtain that $\lim_{t\to 0}\frac{1}{t}R(t)y=By$.

In order to show $\mathscr{A}=\tilde{\mathscr{A}}$ we observe that $\lambda-\mathscr{A}$ is an invertible operator for large λ having inverse $\begin{pmatrix} R(\lambda,A) & R(\lambda,A)BR(\lambda,D)\\ 0 & R(\lambda,D)\end{pmatrix}$, hence it has no proper invertible extension.

Conversely, if we assume that $\mathscr{A}$ is a generator then the Cauchy problem

$$\dot{u}(t)=\mathscr{A}u(t),\qquad u(0)=\begin{pmatrix}x\\ y\end{pmatrix},$$

is well-posed. For initial values $\begin{pmatrix}x\\ y\end{pmatrix}$ in the dense subspace $D(A)\times D(D)$ its solution is given by

$$u(t)=\begin{pmatrix} e^{At}x+\int_0^t e^{A(t-s)}Be^{Ds}y\,ds\\ e^{Dt}y\end{pmatrix}. \tag{3.3}$$

Therefore the integral in (3.3) can be extended to a bounded linear operator $R(t)$ from F to E. As the semigroup $(e^{\mathscr{A}t})_{t\geqq 0}$ is uniformly bounded as $t\downarrow 0$ the same holds for $(R(t))_{t\geqq 0}$. □

Remark. If (b) is not satisfied then $\mathscr{A}$ still generates an integrated semigroup in the sense of Arendt-Kellermann-Neubrander (e.g. [14]) and the associated Cauchy problem has unique solutions for all initial values in $D(\mathscr{A}^2)$.

Example 1. Take a Banach space E and $F:=L^1(\mathbb{R}_+,E)$. In the semigroup approach to integrodifferential equations initiated by Miller [17] (see Desch-Schappacher [10]) one ends up with an operator matrix

$$\mathscr{A}=\begin{pmatrix} A & \delta_0\\ c & \dfrac{d}{dx}\end{pmatrix}$$

for some semigroup generator A on E and a function $c(\cdot)$ from $\mathbb{R}_+$ into $\mathscr{L}(E)$. If $c(\cdot)$ is bounded then $\mathscr{A}$ is a generator if and only if $\begin{pmatrix} A & \delta_0 \\ 0 & \frac{d}{dx} \end{pmatrix}$ is a generator. This holds by the above proposition since

$$f \mapsto R(t)f = \int_0^t e^{A(t-s)} f(s)\, ds, \qquad t \geqq 0,$$

is uniformly bounded from $L^1(\mathbb{R}_+, E)$ into E.

Corollary 3.2. *Assume that A and D generate semigroups on E, resp. on F. If B is bounded from $D(D)$ into $D(A)$ then $\mathscr{A} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ is a generator on $E \times F$.*

Proof. We may assume that D is invertible. Then $\tilde{B} := ABD^{-1} \in \mathscr{L}(F, E)$ by assumption and

$$R(t) = \int_0^t e^{A(t-s)}(BD^{-1})\, D e^{Ds}\, ds = BD^{-1} e^{Dt} - e^{At} BD^{-1} + \int_0^t e^{A(t-s)} \tilde{B} e^{Ds}\, ds$$

is uniformly bounded for $t \in [0, 1]$. □

Example 2. Consider a matrix $\mathscr{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ satisfying the Assumption 2.2. If D is a generator and A is bounded then $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ is a generator by the above corollary. Since C is automatically bounded we obtain that $\mathscr{A}$ is a generator.

Corollary 3.3. *Assume that A and D generate analytic semigroups on E, resp. on F. If B is D-bounded then $\mathscr{A} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ is the generator of an analytic semigroup.*

Proof. By assumption and [21], Chap. 2, Thm. 5.2 there exists $M > 0$ such that $\|A e^{tA}\| \leqq \frac{M}{t}$ and $\|D e^{tD}\| \leqq \frac{M}{t}$ for $t > 0$. Moreover, $\tilde{B} := BD^{-1}$ is bounded from F into E. Therefore we obtain

$$\begin{aligned} R(t) &= \int_0^{t/2} e^{A(t-s)} \tilde{B} D e^{Ds}\, ds + \int_{t/2}^t e^{A(t-s)} \tilde{B} D e^{Ds}\, ds \\ &= e^{At/2} \tilde{B} e^{Dt/2} - e^{At} \tilde{B} + \int_0^{t/2} A e^{A(t-s)} \tilde{B} e^{Ds}\, ds \\ &\quad + \int_{t/2}^t e^{A(t-s)} \tilde{B} D e^{Ds}\, ds, \end{aligned}$$

and therefore $\|R(t)\| \leqq m_1 + m_2 + m_3 \int_0^{t/2} \frac{M}{t-s}\, ds + m_4 \int_{t/2}^{t} \frac{M}{s}\, ds \leqq c$ for $t \in [0, 1]$ and suitable constants. Hence $\mathscr{A}$ generates the semigroup $(T(t))_{t \geqq 0}$ given in (3.2). In fact the semigroup generated by $\begin{pmatrix} A & \varepsilon B \\ 0 & D \end{pmatrix}$ is analytic for small $\varepsilon > 0$ (use [12], Chap. 1, Thm. 6.6). Since the growth rate of $\|R(t)\|$ as $t \to 0$ does not depend on ε we conclude that the same holds for $\varepsilon = 1$ (use [21], Chap. 2, Thm. 5.2). □

Example 3. Take $\mathscr{A} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ as in Corollary 3.3 and let C be A-bounded. Again by [12], Chap. 1, Thm. 6.6 it follows that $= \begin{pmatrix} A & B \\ \varepsilon C & D \end{pmatrix}$ is a generator for $\varepsilon > 0$ sufficiently small. This is not true for arbitrary $\varepsilon > 0$: Take the matrix $\begin{pmatrix} A & -A \\ A & A \end{pmatrix}$ on $E \times E$ which is a generator if and only if $(1 \pm i)A$ is a generator (see [18]), a property which does not hold for all generators A of analytic semigroups.

Corollary 3.4. *Assume that A and D generate semigroups on E, resp. on F such that $(e^{tA})_{t \geqq 0}$ is analytic. If $(\lambda - A^\alpha)^{-1} B$ has a continuous closure from F into E for some large λ and some fractional power A^α, $0 \leqq \alpha < 1$, then $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ is a generator on $E \times F$.*

Proof. We may assume that A is invertible and $\overline{A^{-\alpha} B} \in \mathscr{L}(F, E)$. It follows from [21], Chap. 2, Thm. 6.13 that

$$\|A^\alpha e^{At}\| \leqq M/t^\alpha \tag{3.4}$$

for some constant M and all $t \in [0, 1]$. Using this estimate and writing $R(t) = \int_0^t A^\alpha e^{A(t-s)} \overline{A^{-\alpha} B} e^{Ds}\, ds$ we obtain $\|R(t) \leqq c_0 \int_0^t (t-s)^{-\alpha}\, ds \leqq c_1$ for $t \in [0, 1]$, hence condition (b) in Prop. 3.1. □

Example 4. Take $E = F = L^2(\mathbb{R}^n)$ and

$$\mathscr{A} = \begin{pmatrix} \Delta & b \cdot \operatorname{grad} \\ c \cdot \operatorname{grad} & b \cdot \operatorname{grad} \end{pmatrix}$$

for $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$, $\operatorname{grad} = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right)$ and $b, c \in \mathbb{R}^n$.

For the matrix $\mathscr{A}_0 = \begin{pmatrix} \Delta & b \cdot \operatorname{grad} \\ 0 & b \cdot \operatorname{grad} \end{pmatrix}$ and the canonical domains the assumptions of Corollary 3.4 are satisfied. Since $\mathscr{B} := \begin{pmatrix} 0 & 0 \\ c \cdot \operatorname{grad} & 0 \end{pmatrix}$ is bounded from $D(\mathscr{A}_0)$ into $D(\mathscr{A}_0)$ it follows from [9] that $\mathscr{A} = \mathscr{A}_0 + \mathscr{B}$ is a generator.

4. An Example from Fluid Dynamics

In this section we solve – in the spirit of the above considerations – a quite complicated system of linear evolution equations. This means that we show that a certain operator matrix is the generator of a strongly continuous semigroup on some product space.

As test sample we take a system appearing in [19] as the linearization of equations describing the flow of viscous, compressible and heat conducting fluids in $\mathbb{R}^3$ (see also [15, 16]). In [24], Ströhmer writes this system as an equation of type (1.1) with the matrix

$$\mathscr{A}=\begin{pmatrix} \Delta_n+\text{grad}\cdot\text{div} & \text{grad} & \text{grad} \\ \text{div} & \Delta & 0 \\ \text{div} & 0 & 0 \end{pmatrix}$$

on the Banach space $E\times F\times G$ where we put the various coefficients in [24] equal to one. We take $E=L^p(\Omega;\mathbb{C}^n)\cong L^p(\Omega;\mathbb{C})^n$, $F=L^p(\Omega;\mathbb{C})$, $G=W^{p,1}(\Omega;\mathbb{C})$ and Ω a bounded domain in $\mathbb{R}^n$ having smooth boundary. Based on rather complicated resolvent estimates Ströhmer [24] shows that the operator matrix $\mathscr{A}$ with a suitable domain generates an analytic semigroup on $E\times F\times G$. Subsequently we present an alternate proof of this result using some of the matrix techniques developed in Sect. 3.

For more transparency let us only consider Dirichlet boundary conditions and therefore choose the domain of the Laplacian as

$$D(\Delta):=W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega) \text{ in } F,$$

$$D(\Delta_n):=D(\Delta)\times\ldots\times D(\Delta) \text{ in } E.$$

We now proceed by studying the 3×3-matrix $\mathscr{A}$ in three consecutive steps:

Step 1. The $n\times n$-matrix $-\Delta_n-\text{grad}\cdot\text{div}$ is uniformly strongly elliptic, hence it follows from Amann's theory of parabolic systems (use [2], Thm. 13.4 in combination with [3], Thm. 6.6.i) that the upper left entry of $\mathscr{A}$ with domain $D(\Delta_n)$ is the generator of an analytic semigroup on E.

Step 2. Consider the operator matrix

$$\mathscr{A}_1:=\begin{pmatrix} \Delta_n+\text{grad}\cdot\text{div} & \text{grad} \\ \text{div} & \Delta \end{pmatrix}$$

with domain $D(\Delta_n)\times D(\Delta)$ on $E\times F$. From Ehrling's Lemma (see [1], Thm. 4.14) it follows that the off-diagonal matrix $\begin{pmatrix} 0 & \text{grad} \\ \text{div} & 0 \end{pmatrix}$ is relatively bounded with

respect to the diagonal matrix $\begin{pmatrix} \Delta_n + \text{grad} \cdot \text{div} & 0 \\ 0 & \Delta \end{pmatrix}$ and has relative bound zero. Therefore the usual analytic perturbation theorem (e.g., [21], Thm. 3.2.1) shows that $\mathscr{A}_1$ generates an analytic semigroup on $E \times F$.

Step 3. Finally we write $\mathscr{A} = \begin{pmatrix} \mathscr{A}_1 & \mathscr{B} \\ \mathscr{C} & 0 \end{pmatrix}$ on $(E \times F) \times G$ with $\mathscr{B} = \begin{pmatrix} \text{grad} \\ 0 \end{pmatrix}$ and $\mathscr{C} = (\text{div}, 0)$. Then we observe the following:

- $\mathscr{A}_1$ and 0 generate analytic semigroups on $E \times F$, resp. on G.
- $\mathscr{C}$ is bounded from $D(\mathscr{A}_1)$ into G.
- $\mathscr{B}$ is bounded from G into $E \times F$.

Therefore $\begin{pmatrix} \mathscr{A}_1 & 0 \\ \mathscr{C} & 0 \end{pmatrix}$ generates an analytic semigroup by Cor. 3.3 and again [21], Thm. 3.2.1 yields that the full matrix $\mathscr{A}$ is the generator of an analytic semigroup.

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Received February 12, 1988