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Math. Z. 200, 583–589 (1989)

Mathematische Zeitschrift

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An Infinite Family of Minor-Minimal Nonrealizable 3-Chirotopes

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1. Introduction

The language of oriented matroids [1, 8, 9], or chirotopes [3, 4], has turned out to be a very helpful tool for a certain class of geometric realization problems. For example it can be shown that the question to decide the polytopality of simplicial spheres reduces to the realizability problem for simplicial chirotopes, compare [3]. On the other hand, the realizability problem for (oriented) matroids is a fundamental question in matroid theory, regardless of its applications in convexity, see e.g. Welsh [15].

Given a field K, the K-realizable (oriented) matroids form a hereditary class, i.e. it is closed under forming minors, and therefore it is very natural to ask whether the K-realizable (oriented) matroids can be characterized by the exclusion of a *finite* set of minors. It is well-known that in the case $K = F_2$, the two-element field, such a local characterization is possible: a matroid M is F_2-realizable (binary) if and only if no minor of M is isomorphic to $U_{2,4}$, the uniform matroid of rank 2 on 4 elements, see Welsh [15, Sect. 10.2].

The situation is entirely different for the real numbers and the rational numbers. It follows from Vamos' result "the missing axiom of matroid theory is lost forever" [14] that there is no local axiom to characterize the R-realizable or the Q-realizable matroids. The proof deals with products of matroids with one-element characteristic sets. So, Vamos' construction relies on specific properties of finite fields, and it is impossible to carry over his technique to oriented matroids, immediately.

In 1984 A. Dress [private communication] gave a construction of an infinite family of minor-minimal non-realizable chirotopes in the plane which uses the underlying matroid structure. He suggested to glue together Pappus-configurations in such a manner that only in the last step a collinearity is caused by the entire configuration. Unbending this collinearity one obtains an oriented "multi-non-Pappus-matroid" with the desired properties. In his PhD-thesis A. Mandel [10, p. 292] described a family $\{T_n\}$ of minorminimal nonrealizable oriented matroids with increasing rank.

In the present paper we prove the same result for *simplicial* chirotopes or uniform oriented matroids of *fixed* rank 3. While Dress' construction depends on matroid dependencies rather than orientations, the nonrealizability properties of our chirotope family is caused only by the orientations since uniform matroids are trivially realizable.

Some work has been done to find local characterizations for combinatorial spheres which guarantee isomorphisms to boundary complexes of convex polytopes, see [5]. Our infinite class of nonrealizable minor-minimal simplicial chirotopes in Theorem 1 indicates that a local Steinitz-theorem in higher dimensions could be lost forever, too. In fact, in the meantime the second author was able to apply Theorem 1 to obtain such a result [12] for the general (non-simplicial) case, while for the boundary complexes of simplicial polytopes this problem is still open.

In Theorem 2 we show that certain problems in higher dimensions are equivalent to the rank 3 case. This idea was helpful to decide for combinatorial 5-spheres their realizability as boundary complexes of convex 6-polytopes, compare [2, 13].

2. Chirotopes and the Grassmann Manifold

There is a close connection between Grassmann algebra and chirotope theory. By the well known Grassmann-Plücker-relations valid for a given $n \times d$-matrix M, it is seen that the values of the $d \times d$-subdeterminants of M can not be arbitrarily prescribed in order to guarantee the existence of M (representing a d-dimensional subspace of R^n).

The definition of a *chirotope* reflects the same situation when only the signs of the determinants are considered to be important. The signs can not be prescribed arbitrarily in order to guarantee the existence of a geometric structure. The conditions the signs have to fulfill are derived from the Grassmann-Plücker relations.

The set of ordered d-tupels of n elements will be denoted by

$$\Lambda(n, d) := \{(\lambda_1, \ldots, \lambda_d) \mid 1 \leqq \lambda_1 < \lambda_2 < \ldots < \lambda_d \leqq n\}.$$

A mapping $\Xi\colon \Lambda(n, d) \to R$ is called *simple* or *decomposable* if there exists a real-valued $n \times d$-matrix $M = (m_{ij})$ such that

$$\Xi(\lambda) := \det \begin{pmatrix} m_{\lambda_1 1} & m_{\lambda_1 2} & \ldots & m_{\lambda_1 d} \\ m_{\lambda_2 1} & m_{\lambda_2 2} & \ldots & m_{\lambda_2 d} \\ \vdots & \vdots & & \vdots \\ m_{\lambda_d 1} & m_{\lambda_d 2} & \ldots & m_{\lambda_d d} \end{pmatrix}$$

for all $\lambda \in \Lambda(n, d)$. The simple $\Xi \in R^{\Lambda(n,d)}$ are exactly the Plücker coordinate vectors of decomposable d-vectors in the $\binom{n}{d}$-dimensional real vector space $\bigwedge_d R^n$.

Given $\sigma \in \Lambda(n, d-2)$ and $\tau \in \Lambda(n, 4)$ we abbreviate

$$\begin{aligned}\{\sigma|\tau\}_\Xi := &\Xi(\sigma_1, \ldots, \sigma_{d-2}, \tau_1, \tau_2) \cdot \Xi(\sigma_1, \ldots, \sigma_{d-2}, \tau_3, \tau_4) \\ &- \Xi(\sigma_1, \ldots, \sigma_{d-2}, \tau_1, \tau_3) \cdot \Xi(\sigma_1, \ldots, \sigma_{d-2}, \tau_2, \tau_4) \\ &+ \Xi(\sigma_1, \ldots, \sigma_{d-2}, \tau_1, \tau_4) \cdot \Xi(\sigma_1, \ldots, \sigma_{d-2}, \tau_2, \tau_3).\end{aligned}$$

Proposition. *A mapping Ξ: $\Lambda(n, d) \to R \setminus \{0\}$ is simple if and only if for all $\sigma \in \Lambda(n, d-2)$ and all $\tau \in \Lambda(n, 4)$ the Grassmann Plücker relation $\{\sigma|\tau\}_\Xi = 0$ is fulfilled.*

For a new proof of this old result and many references to Grassmann algebra see Havel and Sturmfels [7].

Definition. A mapping χ: $\Lambda(n, d) \to \{-1, 0, +1\}$ or its unique alternating extension χ: $\{1, \ldots, n\}^d \to \{-1, 0, +1\}$ is a *chirotope* if and only if the set

$$\begin{aligned}\{&\chi(\sigma_1, \ldots, \sigma_{d-2}, \tau_1, \tau_2) \cdot \chi(\sigma_1, \ldots, \sigma_{d-2}, \tau_3, \tau_4), \\ &-\chi(\sigma_1, \ldots, \sigma_{d-2}, \tau_1, \tau_3) \cdot \chi(\sigma_1, \ldots, \sigma_{d-2}, \tau_2, \tau_4), \\ &\chi(\sigma_1, \ldots, \sigma_{d-2}, \tau_1, \tau_4) \cdot \chi(\sigma_1, \ldots, \sigma_{d-2}, \tau_2, \tau_3)\}\end{aligned}$$

either contains $\{-1, +1\}$ or equals $\{0\}$.

Chirotopes appeared in the literature for the first time in 1965 in a paper by Novoa [11] under the name of "n-ordered sets". The term *chirotope* was suggested by Dreiding when he and Dress studied certain structures in organic chemistry (chirality). The first author again independently of the others used this concept in connection with the search for an algorithmic solution of the Steinitz-problem, [2, 3].

By the above Grassmann-Plücker relations the definition of a chirotope is an intrinsic necessary condition for a mapping χ: $\Lambda(n, d) \to \{-1, 0, +1\}$ to be obtained as the signs of the $d \times d$-minors of an $n \times d$-matrix, in which case χ is *realizable*. In other words: χ is realizable if and only if $\chi = \operatorname{sign} \Xi$ for some simple $\Xi \in R^{\Lambda(n,d)}$. Given a set of vectors $X := \{x_1, \ldots, x_n\} \in R^d$, the linear chirotope of X is the d-chirotope with n vertices obtained from the matrix with row set X, and, as before, for a set Y of points in affine $(d-1)$-space, we obtain the affine d-chirotope of X as linear chirotope of its homogenous coordinates.

3. The Missing Axiom for Chirotopes is Lost Forever

It is well known in matroid theory and in the theory of oriented matroids that a sufficient realizability criterion by excluding a finite number of minors cannot exist, see Vamos [14], Mandel [10]. By constructing an infinite family of minor-minimal non-realizable chirotopes, we prove that this result holds already in the case of simplicial affine 3-chirotopes, i.e. for non-degenerate arrangements of pseudo-lines in the plane.

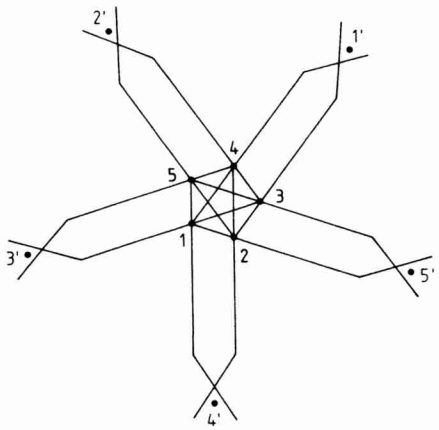

Fig. 1

Definition of χ_l. For $n=2\cdot l$, $l\geqq 5$ we define a 3-chirotope χ_l according to the following plane configuration ($l=5$).

Given a regular l-gon with counterclockwise oriented vertices $1,\ldots,l$ in the plane, consider the parallel halflines $\overrightarrow{i, i+3}$ and $\overrightarrow{i+1, i+2}$ where $i=1,\ldots,l$. (Here and in the following all numerations are understood modulo l.)

For every $i=1,\ldots,l$ insert a new point i' with the following properties.

(i) i' lies to the left of $\overrightarrow{i, i+3}$: $\chi_l(i, i+3, i')=+1$.

(ii) i' lies to the right of $\overrightarrow{i+1, i+2}$: $\chi_l(i+1, i+2, i')=-1$.

(iii) In all other triangle orientations i' is identified with the intersection point (at infinity) of $\overrightarrow{i, i+3}$ and $\overrightarrow{i+1, i+2}$.

Lemma 1. *χ_l is a non-relizable chirotope, but all proper minors of χ_l are realizable.*

Proof. (i) If χ_l is realizable, then there exists an affine realization

$$\{x_1,\ldots,x_l,x_1',\ldots,x_l'\}\subset R^3$$

of χ_l with $x_{i,3}=x_{i',3}=0$ for all i. Let $e_3=(0,0,1)^t$. Then we have in view of the configuration of x_{i+3}, x_{i+2}, x_{i+1}, x_i and x_i'

$$\langle (x_{i+3}-x_i)\wedge(x_{i+2}-x_{i+1}), e_3\rangle>0 \quad \text{for } i=1,\ldots,l,$$

where $\langle\ \rangle$ denotes the scalar product, and $\wedge$ denotes the cross product. The sum over i of all these terms vanishes and hence $\langle 0, e_3\rangle>0$, a contradiction. Therefore χ_l is non-realizable. For this method of proof compare Goodman and Pollack [6].

(ii) The minors by contraction are of rank $\leqq 2$ and hence realizable. But also deletion of a point leads to a realizable configuration. Because of symmetry it suffices to show this for $\chi\backslash\{1\}$ and $\chi\backslash\{1'\}$.

Realization of $\chi_l\backslash\{1\}$. Choose points $x_2, \ldots, x_l \in R^2 \subset R^3$ in counterclockwise order on the unit circle such that $\omega_{l-1} < \ldots < \omega_3 < \omega_2 < \frac{2\pi}{l}$, where ω_i denotes the angle $(x_i, 0, x_{i+1})$. Then points $x'_2, \ldots, x'_{l-3}$ can be placed in the plane R^2 according to the orientations prescribed by χ_l because

$$\langle (x_{i+3} - x_i) \wedge (x_{i+2} - x_{i+1}), e_3 \rangle > 0 \quad \text{for } i = 2, \ldots, l-3.$$

But now also the points x'_{l-2}, x'_{l-1}, x'_l and x'_1 can be chosen because one of their restricting halflines vanished with the deletion of $\{1\}$.

Realization of $\chi_l\backslash\{1'\}$. Starting with a regular counterclockwise oriented l-gon $x_1, \ldots, x_l \in R^2 \subset R^3$, the following procedure leads to a realization of $\chi_l\backslash\{1'\}$:

For $i = 3$ *to* l *let* $x_i := x_i + \varepsilon_i(x_i - x_{i+3})$

where $\varepsilon_i > 0$ *is sufficiently small.*

We have to show that after this process

$$r_i := \langle (x_{i+3} - x_i) \wedge (x_{i+2} - x_{i+1}), e_3 \rangle > 0 \quad \text{for } i = 2, \ldots, l. \qquad (*)$$

Observing that after the first step ($i = 3$) $r_l > 0$, $r_1 < 0$, $r_2 > 0$ and $r_s = 0$, we can prove $(*)$ by induction on i. Assume that $r_k > 0$ for $k = 2, \ldots, i-1$ and $k = l$ and $r_k = 0$ for $k = i, \ldots, l-1$. Replacing x_i by $x_i + \varepsilon_i \cdot (x_i - x_{i+3})$ we can consider the r_k as differentiable function $r_k(\varepsilon_i)$. Obviously $\frac{dr_{i-2}}{d\varepsilon_i} > 0$, $\frac{dr_{i-1}}{d\varepsilon_i} < 0$, $\frac{dr_i i}{d\varepsilon_i} > 0$ and $\frac{dr_k}{d\varepsilon_i} = 0$ for all other k. By assumption, $r_{i-2}(0) > 0$, $r_{i-1}(0) > 0$ and $r_i(0) = 0$. Hence for sufficiently small $\varepsilon_i > 0$ both r_{i-2} and r_{i-1} are positive which proves the induction hypothesis $(*)$ for $i+1$.

(iii) It follows from the definition of a signed d-vector that χ: $\Lambda(n, d) \to \{-1, 0, +1\}$ with $n > d+2$ is a chirotope if every proper minor is realizable. Hence the χ_l are chirotopes. This completes the proof of Lemma 1.

Theorem 1. *For fixed rank* $d \geqq 3$ *and all* $n \in N$ *there exists a nonrealizable simplicial* d*-chirotope* χ *with* $n' > n$ *vertices such that all minors with* n *vertices are realizable.*

Proof. The minor-minimal non-realizable 3-chirotopes χ_l have this property by the above Lemma 1. For $d > 3$ we can construct chirotopes with the desired property as follows. For a simplicial chirotope Π, let Π^i denote any simplicial principle extension, see [10], of Π by i points, and let Π^* denote the dual chirotope of Π.

Given $d > 3$ and $n \in N$, consider the d-chirotope $((\chi_n^*)^{d-3})^*$ with $n' = 2n + d - 3 > n$ vertices. From the properties of the χ_l and the fact that principal extensions preserve realizability, it follows that all minors with n vertices of this chirotope are realizable.

Remark. This disproves a conjecture of Grünbaum [17, p. 408] by the well-known Folkmann and Lawrence equivalence between 3-chirotopes and arrangements of pseudolines.

4. d-Polytopes with $d+4$ Vertices

Using standard oriented matroid techniques we show that the problem to decide the polytopality of a given $(d-1)$-sphere with $d+4$ vertices can be reduced to a plane problem. This idea was helpful in a joint work of I. Shemer and the first author [2] to classify all neighborly 6-polytopes with 10 vertices. In a recent paper of the second author on *affine Gale-diagrams* [13] a unified framework for this transformation technique is suggested.

Theorem 5. *Given a combinatorial $(d-1)$-sphere S with $d+4$ vertices, there is a finite set $\{\chi_1, \ldots, \chi_k\}$ of affine 3-chirotopes with $d+4$ vertices such that S is polytopal if and only if at least one χ_i is realizable.*

Proof. Given S, let $\{\Pi_1, \ldots, \Pi_k\}$ be the set of all $(d+1)$-chirotopes with $d+4$ vertices and face-lattice isomorphic to S. Clearly, S is polytopal if and only if at least one Π_i is realizable. Consider the 3-chirotopes $\{\Pi_1^*, \ldots, \Pi_k^*\}$. By a result of Las Vergnas [8] there exist subsets $M_i \subset \{1, \ldots, d+4\}$ such that the chirotopes $\chi_i := {}_{\bar{M}_i}(\Pi_i^*)$ by sign reversal on M_i are affine or totally acyclic. The theorem follows from the fact that Π_i is realizable if and only if χ_i is realizable.

While in [8] M. Las Vergnas uses the Tutte polynomial to show that for any d-chirotope with n vertices at least one reorientation ${}_{\bar{A}}\chi$ is affine, such a subset can be constructed very easily in the simplicial case.

Remark. Let $\chi\colon \lambda(n,d) \to \{-1, +1\}$ be a chirotope. For any $\mu \in \Lambda(n, d-1)$ define $M_\mu := \{i \in \{1, \ldots, n\} \mid \chi(\mu, i) = -1\}$. Then ${}_{\bar{M}_\mu}\chi$ is an affine chirotope.

In coordinates this construction can be described as follows. Assuming that a $(d-1)$-sphere is isomorphic to the boundary complex of a d-polytope P with $d+4$ vertices, we can write homogenous coordinates for the vertices of P as rows of a $(d+4) \times (d+1)$-matrix M. Without loss of generality, and if necessary, after relabeling the vertices of P, we can further assume that $M = \begin{pmatrix} \mathbf{1} \\ A \end{pmatrix}$, where $\mathbf{1}$ denotes the $(d+1) \times (d+1)$-unit matrix, and A is a $3 \times (d+1)$-matrix. The isomorphism from S to the boundary complex of P imposes certain sign conditions on the subdeterminants of A. In fact, if P is evendimensional and neighborly, compare [3], then for *all* subdeterminants of A the sign is prescribed due to a result of Shemer [18].

If χ is the chirotope associated with P, then the dual χ^* corresponds to a Gale transform of P, the coordinates of which are given by the rows of the $(d+4) \times 3$-matrix $\begin{pmatrix} -A^T \\ 1 \end{pmatrix}$. Here $\mathbf{1}$ denotes the 3×3-unit matrix. After replacing some of the row vectors of this matrix by their negative, we can interprete this matrix as homogenous coordinate matrix of a configuration of $d+4$ points

in the affine plane. The sphere S is polytopal if and only if a configuration can be found which fulfils the above conditions on the minors of $-A^T$ which are (up to some sign reversals) the minors of A.

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Received November 10, 1986