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On Strongly WCG Banach Spaces

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Let X be a Banach space with dual space X^*, and let B and B^* denote the closed unit balls of X and X^*, respectively. In this paper we investigate a strengthened version of the familiar weakly compactly generated (WCG) property for X.

The *strong WCG* (SWCG) property for X is the requirement that there exist a weakly compact subset K of X such that for every weakly compact subset L of X and $\varepsilon > 0$, there is a positive integer n with $L \subset nK + \varepsilon B$. By comparison, the usual WCG property is equivalent to the corresponding assertion for norm compact L. The two notions are distinct: any SWCG space is both WCG and weakly sequentially complete, so the space c_0 is not SWCG.

The space X is SWCG if and only if the topological space (B^*, τ) is (completely) metrizable, where $\tau = \tau(X^*, X)$ denotes the dual Mackey topology on X^*. It is then easy to see (directly or dually) that the SWCG property is enjoyed by reflexive spaces, separable Schur spaces, the space of operators of trace class on a separable Hilbert space, and $L_1(\mu)$, for μ a σ-finite measure. The SWCG property is preserved by countable l_1-sums, and by quotient maps which are "weakly-compact covering"; thus, in particular, the space L_1/H_1 is SWCG. It is not, however, preserved by arbitrary quotients, nor by closed subspaces.

Using fundamental examples due to Batt and Hiermeyer [BH2] and Pisier [Pi], we show that a separable, weakly sequentially complete space need not be SWCG, and that the injective or projective tensor product of two SWCG spaces need not be SWCG. We also present partial results on the important open question of whether X SWCG implies that $L_1(\mu, X)$, the space of Bochner integrable functions from a finite measure space to X, is also SWCG. The work of [BH2] and its subsequent extensions [BS, Schl] have focused attention on the topology $\sigma' = \sigma(L_1(\mu, X), L_\infty(\mu, X^*))$; it is strictly coarser than the usual weak topology of $L_1(\mu, X)$ unless X^* has the Radon-Nikodym property. It is shown here that $L_1(\mu, X)$ is strongly σ'-compactly generated if and only if one of two alternatives holds: either X is reflexive, or X is SWCG and the measure space is purely atomic.

The paper concludes with a list of open questions.

1. Notation and Background

We use [Scha] as a basic reference for the theory of locally convex topological vector spaces. Throughout, X is a Banach space over the real field, with closed unit ball B (or $B(X)$, if confusion might arise). The dual space X^* is the space of continuous linear functionals on X, and B^* (or $B(X^*)$) is its closed unit ball. The notation (X, weak) refers to X with the weak topology $\sigma(X, X^*)$; (B, weak) is the closed unit ball with the relative weak topology. Similarly, (X^*, w^*) denotes X^* with the weak* topology $\sigma(X^*, X)$. The Mackey topology $\tau(X^*, X)$ is the finest locally convex topology on X^* whose dual space is X. The Mackey-Arens Theorem [Scha, p. 131] characterizes τ as the topology on X^* of uniform convergence on weakly compact, absolutely convex subsets of X. Since the closed absolutely convex hull of a weakly compact subset of a Banach space is still weakly compact [Scha, p. 189], we will usually omit the "absolutely convex" stipulation in our discussion.

We recall some well-known facts about the Mackey topology and weakly compact sets, with brief indication of proof.

1.1. Proposition. *(X^*, τ) is a complete locally convex space.*

Proof. Applying Grothendieck's Completeness Theorem [Scha, p. 149], let f be a linear functional on X whose restriction to each weakly compact set is weakly continuous. Then f is continuous on norm-null sequences, so $f \in X^*$. □

1.2. Proposition. *Let $(X_\alpha)_{\alpha \in A}$ be a family of Banach spaces. Let $X = \{x \in \prod X_\alpha : \sum \|x_\alpha\| < \infty\}$ be the l_1-sum of the spaces X_α. Then $(B(X^*), \tau(X^*, X))$ can be identified with the topological product of the spaces $(B(X_\alpha^*), \tau(X_\alpha^*, X_\alpha))$.*

Proof. Since X^* is the l_∞-sum of the spaces X_α^*, the identification of the sets $B(X^*)$ and $\prod B(X_\alpha^*)$ is clear. Also, $\tau(X^*, X)$ is evidently finer than $\prod \tau(X_\alpha^*, X_\alpha)$ on $B(X^*)$. The converse can be established using the following: if H is a weakly compact subset of X, and $\varepsilon > 0$, then there is a finite subset D of A such that $\sum\{\|x_\alpha\| : \alpha \in A - D\} < \varepsilon$ for all $x \in H$. The argument is like the usual "gliding hump" proof of Schur's Lemma for l_1.

1.3. Grothendieck's Criterion for Weak Compactness [D2, p. 227]. *Let H be a subset of a Banach space X such that for every $\varepsilon > 0$ there exists a weakly compact subset K of X with $H \subset K + \varepsilon B$. Then H is relatively weakly compact in X.*

2. Strongly Weakly Compactly Generated Spaces

The space (X^*, τ) is not metrizable, except in the trivial case where X is reflexive. However, the situation for (B^*, τ) is much different. Since B^* is absolutely convex, it is a standard result that the topology τ is metrizable if and only if the associated uniformity on B^* is metrizable. In view of 1.1, this means that (B^*, τ) is metrizable if and only if it is completely metrizable.

2.1. Theorem. *The following conditions on X are equivalent:* (a) (B^*, τ) *is (completely) metrizable;* (b) *there is a sequence (K_n) of weakly compact (absolutely* *convex) subsets of* X *such that for every weakly compact subset* L *of* X *and every* $\varepsilon>0$, *there is a positive integer* n *such that* $L\subset K_n+\varepsilon B(X)=\{x\in X: \exists\, y\in K_n, \|x-y\|\leqq\varepsilon\}$*;* (c) *there is a weakly compact (absolutely convex) subset* K *of* X *such that for every weakly compact subset* L *of* X *and every* $\varepsilon>0$, *there is a positive integer* n *such that* $L\subset nK+\varepsilon B(X)$.

Proof. (a) $\Rightarrow$ (b): Choose a sequence (K_n) of weakly compact, absolutely convex subsets of X such that $(K_n^0\cap B^*)$ is a neighborhood base at 0 for (B^*,τ). Given a weakly compact set L and $0<\varepsilon<1$, let $c=\frac{1}{\varepsilon}$, and choose n such that $(cL)^0\cap B^*\supset K_n^0\cap B^*$. Then $((cL)^0\cap B^*)^0\subset(K_n^0\cap B^*)^0$, so $cL\subset((cL)^{00}\cup B)^{00}\subset(K_n^{00}\cup B)^{00}\subset K_n+B$, since the last of these sets is closed and absolutely convex. It follows that $L\subset K_n+\varepsilon\cdot B(X)$.

(b) $\Rightarrow$ (c): Given (K_n) as in (b), let $p_n=\sup\{\|x\|: x\in K_n\}$, and let $K=\left\{\sum_{n=1}^{\infty}\frac{1}{2^n\cdot p_n}\cdot x_n: x_n\in K_n\right\}$. Now K is weakly closed, by a straightforward argument, and an application of 1.3 shows it to be weakly compact. Since $K_n\subset 2^n\cdot p_n\cdot K$, (c) now follows.

(c) $\Rightarrow$ (a): $d(x^*, y^*)=\max\{|x^*(x)-y^*(x)|: x\in K\}$ is a metric on X^*, and the metric topology τ' is coarser than τ. Let $f_\alpha\to f$ in (B^*,τ'). Given weakly compact L and $\varepsilon>0$, choose n so that $L\subset nK+\left(\frac{\varepsilon}{4}\right)\cdot B$. Choose α_0 such that if $\alpha\geqq\alpha_0$, then $|f_\alpha(y)-f(y)|<\varepsilon/2n\ \forall\, y\in K$. Then if $x\in L$ and $y\in K$ with $\|x-ny\|\leqq\frac{\varepsilon}{4}$, we have $|f_\alpha(x)-f(x)|<\varepsilon$ for all $\alpha\geqq\alpha_0$. Thus (f_α) is τ-convergent to f. □

We refer to any space X satisfying the conditions of 2.1 as *strongly weakly compactly generated* (SWCG). The sets (K_n) in (b) (resp. K in (c)) are called strongly generating sets for X.

2.2. Remark. In 2.1 (b) and (c), it suffices to verify the condition for L a weakly null sequence. In (c), for example suppose that L is a weakly compact set such that $x_n\in L\backslash(nK+\varepsilon B)$ for each n. We may assume that the sequence (x_n) converges weakly to $x_0\in L$. Then (x_n-x_0) converges weakly to 0, so $x_n-x_0\in mK+\left(\frac{\varepsilon}{2}\right)B$ for some m and all n. However, $x_0\in pK+\left(\frac{\varepsilon}{2}\right)B$ for some p, and so $x_n\in(m+p)K+\varepsilon B$ for all n, a contradiction.

2.3. Examples of SWCG Spaces. (a) reflexive spaces; (b) separable Schur spaces. Since weakly compact subsets of X are norm compact, the topologies τ, uniform convergence on norm compact sets, and $\sigma(X^*, X)$ coincide on B^*. The latter is compact metrizable for separable X. (c) the space X of operators of trace class on a separable Hilbert space H. It is well-known that X^* can be represented as the space of all bounded operators on H. Akemann [Ak, Th. II.7] showed that $\tau(X^*, X)$ coincides on B^* with the strong* topology, generated by the semi-norms $\|a\|_f = [f(a^* a + a a^*)]^{1/2}$, f a positive operator of trace class. But if H is separable, then the strong* topology on B^* is metrizable [Tak, p. 71]. (d) $L^1(\mu)$, μ a finite measure. Grothendieck [Gro, Sect. 1.3] proved that the topology $\tau(L_\infty, L_1)$ coincides with the topology of convergence in μ-measure (or the $\| \ \|_1$-topology) on B^*. Using 1.2, this extends immediately to all σ-finite measures. □

The reader may wish to locate strongly generating sets for each of the SWCG spaces mentioned above. In (d), a useful choice is K = closed unit ball of $L_\infty(\mu)$, considered as a subset of $L_1(\mu)$. The proof of the next result is also left to the reader.

2.4. Proposition. *The following conditions on X are equivalent:* (a) *X is WCG;* (b) *(B^*, τ) is submetrizable (i.e., admits a coarser metric topology);* (c) *there is a weakly compact (absolutely convex) subset K of X such that for every norm compact subset L of X and every $\varepsilon > 0$, there is a positive integer n such that $L \subset nK + \varepsilon B$.*

In the other direction, we could ask that X satisfy a stronger condition: there is a weakly compact subset K of X and a sequence (x_n) in X such that for every weakly compact L and $\varepsilon > 0$, $L \subset \{x_1, \ldots, x_n\} + K + \varepsilon B$ for some n. Every separable Schur space satisfies this condition (let (x_n) be a dense sequence in X, and let $K = \{0\}$). But there are no other spaces X: According to the definitions there is for a given $\varepsilon > 0$ and for each $m \in \mathbb{N}$ an $n(m) \in \mathbb{N}$ such that $mK \subset \{x_1, \ldots, x_{n(m)}\} + K + \frac{\varepsilon}{2} B$. Select an $m \geqq 2$ such that $\frac{1}{m} K \subset \frac{\varepsilon}{2} B$. Then $K \subset \frac{1}{m} \{x_1, \ldots, x_{n(m)}\} + \varepsilon B$, and so K is norm compact. Hence X is a separable Schur space.

The next result shows that the WCG and SWCG properties are not equivalent.

2.5. Theorem. *If X is SWCG, then X is WCG and weakly sequentially complete.*

Proof. Let (x_n) be a weak Cauchy sequence in X, and let K be a strongly generating (weakly compact) subset of X. We use 1.3 to show that (x_n) is relatively weakly compact in X.

Suppose there is an $\varepsilon > 0$ such that: (1) for all $m \in \mathbb{N}$, there is an $n_m \in \mathbb{N}$ with $x_{n_m} \notin mK + \varepsilon B$. For $n \in \mathbb{N}$ and $i = 1, 2$, define $m_i(n) = \min\{m \in \mathbb{N}: x_n \in mK + (\varepsilon/i) B\}$. Clearly $m_2(n) \geqq m_1(n)$ for all n. Using (1), select a subsequence (x_{n_k}) of (x_n) such that for all k: (2) $m_1(n_{k+1}) > (m_2(n_k))^2$. Then (x_{n_k}) is again a weak-Cauchy sequence, and so there is an m_0 such that $x_{n_{k+1}} - x_{n_k}$ belongs to $m_0 K + (\varepsilon/2) B$ for every k. Then $(\frac{1}{2}) x_{n_k} \in (\frac{1}{2}) m_2(n_k) K + \left(\frac{\varepsilon}{4}\right) B$ for all k, by definition of m_2, so that

$$(\tfrac{1}{2}) x_{n_k} + (\tfrac{1}{2})(x_{n_{k+1}} - x_{n_k}) \in (\tfrac{1}{2})(m_2(n_k) + m_0) K + \left(\frac{\varepsilon}{2}\right) B. \tag{3}$$

Thus

$$x_{n_{k+1}} \in (m_2(n_k) + m_0) K + \varepsilon B \qquad \text{for all } k. \tag{4}$$

Now $(m_2(n_k))$ forms by construction a strictly monotone increasing sequence. Hence there is a k_0 such that $x_{n_{k+1}} \in 2m_2(n_k)K + \varepsilon B$ for all $k \geqq k_0$. If $k > 2$, then also $m_2(n_k) > 2$, and so $m_1(n_{k+1}) \leqq (m_2(n_k))^2$, contradicting (2). Thus (1) must fail, and now the relative weak compactness of (x_n) follows from 1.3. □

A weakly sequentially complete, WCG (even separable) space need not be SWCG, as the following example of Batt and Hiermeyer [BH2, pp. 417–419] shows.

2.6. Example. A weakly sequentially complete and separable Banach space need not be SWCG. Define $C = \cup\{0,1\}^n$, the usual binary tree. For ξ, $\xi' \in C$, say $\xi < \xi'$ if ξ is an initial portion of ξ'. A totally ordered finite subset Q of C is called a segment.

Let

$$X_0 = \{z = (z_\xi)_{\xi \in C};\ \|z\| = \sup_F(\sum_{Q \in F} (\sum_{\xi \in Q} |z_\xi|)^2)^{1/2} < \infty\}$$

where the supremum is taken over all finite sets F of disjoint segments. Now the desired space X is the closed linear space of the unit vectors $\{e_\xi\colon \xi \in C\}$ in X_0, where $e_\xi = (\delta_{\xi\xi'})_{\xi' \in C}$. It is shown in [BH2] that X has a monotone boundedly complete unconditional basis. Consequently, X is a weakly sequentially complete, separable dual space, and has the Radon-Nikodym property.

There exist uncountably many copies of the l^2-unit vector basis in X. To see this let $\theta(0) = 1$, $\theta(1) = 0$ and define for $x = (x_n) \in \{0,1\}^{\mathbb{N}}$, $\varepsilon_1(x) = (x_1)$, $\varepsilon_2(x) = (\theta x_1, x_2)$, $\ldots$, $\varepsilon_n(x) = (\theta x_1, \ldots, \theta x_{n-1}, x_n)$, $\ldots$. Then $(\varepsilon_n(x))_{n \in \mathbb{N}}$ is a sequence of pairwise incomparable elements in C. The map $x \mapsto (\varepsilon_n(x)) \in C^{\mathbb{N}}$ is injective and $\varepsilon_i(x)$ and $\varepsilon_j(x)$ are comparable iff $i = j$. Thus for $x \in \{0,1\}^{\mathbb{N}}$ and (a_n) a sequence of scalars

$$\left\|\sum_{n=1}^{m} a_n e_{\varepsilon_n(x)}\right\| = \sup_F \left(\sum_{Q \in F}\left(\sum_{\xi \in Q}\left|\sum_{n=1}^{m} a_n \cdot \delta_{\varepsilon_n(x),\xi}\right|\right)^2\right)^{1/2}$$

$$= \left(\sum_{i=1}^{m}\left|\sum_{n=1}^{m} a_n\, \delta_{\varepsilon_n(x),\varepsilon_i(x)}\right|^2\right)^{1/2}$$

$$= \left(\sum_{n=1}^{m} a_n^2\right)^{1/2}.$$

Thus for all $x \in \{0,1\}^{\mathbb{N}}$, $(e_{\varepsilon_n(x)})_{n \in \mathbb{N}}$ is relatively weakly compact in X.

Now suppose X is SWCG, and choose a weakly compact absolutely convex set K such that for all $x \in \{0,1\}^{\mathbb{N}}$, $\{e_{\varepsilon_n(x)};\ n \in \mathbb{N}\} \subset m(x)K + \frac{1}{2}B$ for a suitable positive integer $m(x)$. Then there is an $m \in \mathbb{N}$ and an uncountable subset $D \subset \{0,1\}^{\mathbb{N}}$ such that

$$\bigcup_{x \in D} \{e_{\varepsilon_n(x)};\ n \in \mathbb{N}\} \subset mK + \tfrac{1}{2}B.$$

By induction, select a totally ordered sequence $\{\tilde{\varepsilon}_k;\ k \in \mathbb{N}\} \subset \bigcup_{x \in D} \{\varepsilon_n(x);\ n \in \mathbb{N}\}$. It is easy to see that $(e_{\tilde{\varepsilon}_k})_{k \in \mathbb{N}}$ is 1-equivalent to the l_1-unit vector basis. For $k \in \mathbb{N}$

choose $y_k \in \frac{1}{2}B$ so that $e_{\tilde{\varepsilon}_k} - y_k \in mK$. We show that $(e_{\tilde{\varepsilon}_k} - y_k)_{k\in\mathbb{N}}$ is equivalent to the l_1-vector basis, yielding a contradiction. Indeed for a sequence $(a_k)_{k\in\mathbb{N}}$ of scalars we have

$$\begin{aligned}
\left\|\sum_{k=1}^{n} a_k(e_{\tilde{\varepsilon}_k} - y_k)\right\| &\geqq \sum_{i=1}^{n}\left|\sum_{k=1}^{n} a_k(e_{\tilde{\varepsilon}_k} - y_k)(\tilde{\varepsilon}_i)\right| \\
&= \sum_{i=1}^{n}\left|\left(\sum_{k=1}^{n} a_k\,\delta_{\tilde{\varepsilon}_k,\tilde{\varepsilon}_i}\right) - \left(\sum_{k=1}^{n} a_k\, y_k(\tilde{\varepsilon}_i)\right)\right| \\
&\geqq \sum_{i=1}^{n}\left(\left|\sum_{k=1}^{n} a_k\,\delta_{\tilde{\varepsilon}_k,\tilde{\varepsilon}_i}\right| - \left|\sum_{k=1}^{n} a_k\, y_k(\tilde{\varepsilon}_i)\right|\right) \\
&\geqq \sum_{k=1}^{n} |a_k| - \sum_{k=1}^{n}\sum_{i=1}^{n} |a_k\, y_k(\tilde{\varepsilon}_i)| \\
&\geqq \sum_{k=1}^{n} |a_k| - \sum_{k=1}^{n} |a_k|\,\|y_k\| \geqq \tfrac{1}{2}\sum_{k=1}^{n} |a_k|. \qquad \square
\end{aligned}$$

We see from 2.5 that the SWCG property need not be preserved by quotient maps (c_0 is a quotient of l_1). However, if a quotient map q: $X \to X/Y$ has the additional property that every weakly compact subset of X/Y is the image under q of a weakly compact subset of X, then it is easy to see (directly or dually) that the SWCG property passes from X to X/Y. For example, the space L_1/H_1 is such that a quotient image of L_1 [Pe, Th. 7.1]. This extends the well-known result that L_1/H_1 is weakly sequentially complete.

2.7. Theorem. *Let Y be a reflexive subspace of X. Then X is SWCG if and only if X/Y is SWCG.*

Proof. We first observe that q: $X \to X/Y$ is "weakly-compact covering" in the sense of the discussion above. Let $L \subset X/Y$ be weakly compact. For each $z \in L$, choose $x \in q^{-1}(z)$ with $\|x\| \leqq \|z\| + 1$, and let T denote the set of these x's.

We prove that T is relatively weakly compact in X. Let (x_n) be a sequence in T. We may assume that $(q(x_n))$ converges weakly to $q(x_0)$, for some $x_0 \in T$. It now suffices to prove that the weak*-closure of $\{x_n\}$ in X^{**} is a subset of X. If F is a member of this weak*-closure, then $q^{**}(F) = q(x_0)$, so $F - x_0 \in \ker(q^{**}) = Y$ (since Y is reflexive). Thus $F \in x_0 + Y \subset X$.

It follows easily from this that X SWCG implies X/Y SWCG. Conversely, let K be a weakly compact absolutely convex subset of X/Y which is strongly generating, and choose a weakly compact absolutely convex subset Q of X with $q(Q) = K$, using the construction above. Then $P = Q + B(Y)$ is still weakly compact and absolutely convex, with $q(P) = K$. Let $L \subset X$ be weakly compact, and let $\varepsilon > 0$. Choose a positive integer n such that $q(L) \subset nK + \left(\frac{\varepsilon}{2}\right) B(X/Y)$. Choose m such that $m \geqq \sup_{x\in L} \|x\| + n \sup_{x\in P} \|x\| + \varepsilon$. Note that $m \geqq n$, since $B(Y) \subset P$.

Fix $x \in L$, and choose $w \in Q$, $z \in \varepsilon \cdot B(X)$ with $q(x) = n q(w) + q(z)$. Then $x - n w - z \in Y$, and $\|x - n w - z\| \leqq m$. Hence $x \in nQ + mB(Y) + \varepsilon B(X) \subset mP + \varepsilon B(X)$. This proves that P is strongly generating for X. □

2.8. Remark. It is well-known that the analogue of 2.7 is true for WCG spaces. It is also true that if X/Y is separable, then X is WCG if Y is WCG [Li, p. 24]. This is far from true for SWCG spaces ($X = c_0$, $Y = \{0\}$). The SWCG property is also not preserved by closed subspaces, since there is a non-WCG subspace of an $L_1(\mu)$ space [R 2].

2.9. Proposition. *Let X be the l_1-sum of a sequence of spaces (X_n). Then X is SWCG if and only if each X_n is SWCG.*

Proof. Since a countable product of topological spaces is metrizable if and only if each factor is metrizable, this follows immediately from 1.2.

2.10. Proposition. *If X contains no isomorphic copy of l_1, then X is SWCG if and only if X is reflexive.*

Proof. This is an immediate application of 2.5 and the Rosenthal l_1 Theorem [R 1]. □

There is a superficial resemblance between the class of SWCG spaces and the class of spaces having Pelczynski's property (V^*) (see [Sa]). Indeed every $L^1(\mu)$ space has (V^*), and every space with (V^*) is weakly sequentially complete. However, the two notions are unrelated. Indeed the Batt-Hiermeyer space is a Banach lattice not containing c_0, so it has property (V^*) [Sa, p. 208] without being SWCG. Conversely, Bourgain and Delbaen [BD] gave an example of a separable Schur space (hence SWCG) which has been shown to fail (V^*) [Sa, pp. 209–210].

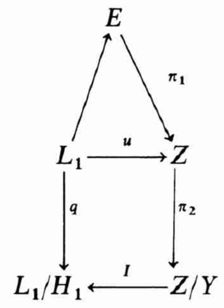
It can be shown that the Banach space X_0 constructed by Azimi and Hagler [AH], a separable non-Schur space which contains l_1 hereditarily, is SWCG. We omit the details.

2.11. Example. The tensor product of two SWCG spaces need not be SWCG. Pisier [Pi, Sect. 4] described a separable weakly sequentially complete Banach space Z such that the completed tensor product of L_1/H_1 and Z, for any tensor norm between the injective and projective norms, contains an isomorphic copy of c_0, and hence is not weakly sequentially complete.

We observed above that, L_1/H_1 is SWCG; now we show the same for Z. Here is a construction for Z: let $E = L_1 \oplus l_2$, with $\|(f, x)\| = \|f\|_1 + \|x\|_2$, and consider the bounded linear operator $P\colon H_1 \to l_2$ given by $P(h) = (\hat{h}(3^n))$, where $\hat{h}(i)$ is the ith Fourier coefficient of $h \in H_1$. Let $N = \{(h, -P(h))\colon h \in H_1\}$, a closed subspace of E, and let $Z = E/N$. Denote the quotient map from E onto E/N by π_1.

Define $u\colon L_1 \to Z$ by $u(f) = \pi_1(f, 0)$ and $j\colon l_2 \to Z$ by $j(x) = \pi_1(0, x)$. According to [Pi, Th. 2.2], j is an isometry of l_2 onto $j(l_2) = Y \subset Z$. Let $q\colon L_1 \to L_1/H_1$ and $\pi_2\colon Z \to Z/Y$ be quotient maps. Define $I\colon Z/Y \to L_1/H_1$ by $I((\pi_2 \circ \pi_1)(f, x)) = q(f)$. This is well-defined, for if (f, x), $(g, t) \in E$ and have the same image under $\pi_2 \circ \pi_1$, then $\pi_1(f - g, x - t) = \pi_1(0, s)$ for some $s \in l_2$. Thus $f - g \in H_1$, so $q(f) = q(g)$.

The diagram



commutes, since any $f \in L_1$ satisfies $I \circ \pi_2 \circ u(f) = I \circ (\pi_2 \circ \pi_1)(f, 0) = q(f)$. Thus I is surjective, and it is also an isometry: if $f \in L_1$, $x \in l_2$, then

$$\begin{aligned} \|(\pi_2 \circ \pi_1)(f, x)\| &= \inf\{\|\pi_1(f, x) + \pi_1(0, t)\| : t \in l_2\} \\ &= \inf\{\|\pi_1(f, s)\| : s \in l_2\} \\ &= \inf\{\|f + h\|_1 + \|s - P(h)\|_2 : s \in l_2, h \in H_1\} \\ &= \inf\{\|f + h\|_1 : h \in H_1\} \quad \text{(since } P \text{ has dense range)} \\ &= \|q(f)\|. \end{aligned}$$

This implies that Z/Y is SWCG; and since Y is reflexive, Z is SWCG by 2.7.

3. The Space $L_1(\mu, X)$

Let (S, Σ, μ) be a finite measure space, and let $L_1(\mu, X)$ denote the Banach space of (equivalence classes of) Bochner integrable functions from S into X. Diestel [D 1] and Talagrand [Tal] have shown that if X is WCG (resp., weakly sequentially complete), then $L_1(\mu, X)$ enjoys the same property. Thus it is natural to inquire if the SWCG property passes from X to $L_1(\mu, X)$. This does hold if X is reflexive, since $B(L_\infty(\mu, X))$ is weakly compact [DU, p. 101] and strongly generates $L_1(\mu, X)$. It also holds if $X = L_1(\nu)$, ν a finite measure, since then $L_1(\mu, X)$ is isometrically isomorphic to $L_1(\mu \times \nu)$, and 2.3(d) applies. The general case remains open. The most obvious way to prove it would be to establish the converse of 2.5, but this is barred by the Example 2.6.

It is convenient here to introduce a generalization of the SWCG property.

3.1. Definition. Let X be a Banach space, and let $Y \subset X^*$ be a closed norming subspace (i.e., $\|x\| = \sup\{|f(x)|: f \in B(Y)\}$ for all $x \in X$). Then X is *strongly* $\sigma(X, Y)$*-compactly generated* if there is an absolutely convex $\sigma(X, Y)$-compact subset K of X such that for every $\sigma(X, Y)$-compact subset L of X and $\varepsilon > 0$, $L \subset nK + \varepsilon B(X)$ for some n. □

The analogue of 1.3 for $\sigma(X, Y)$-compactness is valid: embed X in Y^* in the canonical way, and proceed as in [D2, p. 227]. Also, if Y has the additional property that the absolutely convex hull of a $\sigma(X, Y)$-compact set is relatively

$\sigma(X, Y)$-compact, then the proof of 2.1 goes through to characterize $\tau(Y, X)$-metrizability of $B(Y)$. Finally, the proof of 2.5 shows that if Y is a norming subspace of X^*, and X is strongly $\sigma(X, Y)$-compactly generated, then X is $\sigma(X, Y)$-sequentially complete.

We now consider the norming subspace of $L_1(\mu, X)^*$ given by $L_\infty(\mu, X^*)$, the Banach space of (equivalence classes of) essentially bounded Bochner measurable functions from S into X^*. Note that $L_\infty(\mu, X^*)$ is the entire dual space of $L_1(\mu, X)$ if and only if X^* has the Radon-Nikodym property with respect to (S, Σ, μ) [Ba]. Let $\tau' = \tau(L_\infty(\mu, X^*), L_1(\mu, X))$ and $\sigma' = \sigma(L_1(\mu, X), L_\infty(\mu, X^*))$. The absolutely convex hull of a σ'-compact set is relatively σ'-compact in $L_1(\mu, X)$ [BH2]. Every σ'-compact set is an Eberlein compact [BS].

3.2. Theorem. *Let X be a Banach space, and let (S, Σ, μ) be a finite measure space. Then the following conditions are equivalent:*

(a) *$L_1(\mu, X)$ is strongly σ'-compactly generated*
(b) *$(B(L_\infty(\mu, X^*), \tau')$ is metrizable*
(c) *Either X is reflexive, or X is SWCG and (S, Σ, μ) is purely atomic.*

Proof. (a) $\Leftrightarrow$ (b): This is proved as in 2.1.

(c) $\Rightarrow$ (a): If (S, Σ, μ) is purely atomic, then $L_1(\mu, X)$ is a countable l_1-sum of the SWCG space X, hence SWCG, by 2.9. Moreover, $L_1(\mu, X)^*$ and $L_\infty(\mu, X^*)$ can both be identified with a countable l_∞-sum of X^*, so the weak and σ' topologies coincide. As mentioned above, $L_1(\mu, X)$ is SWCG for X reflexive, and again the weak and σ' topologies coincide. $\square$

Before proceeding with the proof that (a) $\Rightarrow$ (c), we note the following unpublished result of Batt and Hiermeyer [BH 1]. The proof is included here with their kind permission.

3.3. Lemma [BH 1]. *Let X be a Banach space, and let (S, Σ, μ) be a finite measure space. Then $L_1(\mu, X)$ is σ'-sequentially complete if and only if X is weakly sequentially complete and has the Radon-Nikodym property with respect to (S, Σ, μ).*

Proof. Necessity: Since (X, weak) embeds as a closed subspace of $(L_1(\mu, X), \sigma')$, it is weakly sequentially complete. Suppose X fails the RNP with respect to (S, Σ, μ). According to [BH2, Prop. 2.4], there is a μ-continuous vector measure $m: \Sigma \to X$ of bounded variation, an increasing sequence (π_n) of finite Σ-partitions of S, and a $T \in L(L_1(\mu, X), c_0)$ such that $T(K)$ is not relatively weakly compact in c_0, where $K = \{h_n = \sum_{A \in \pi_n} \mu(A)^{-1} m(A) \chi_A : n \in \mathbb{N}\}$, and $T \in \Lambda_1(c_0, (\pi_n))$. Then [BH2, Th. 2.1] shows that K is not relatively σ'-compact.

By the μ-continuity and bounded variation of m, K is bounded and uniformly integrable. The definition of the sequence (h_n) reveals that for each $A \in \bigcup \pi_m$ and $x^* \in X^*$, $\langle h_n, x^* \otimes \chi_A \rangle$ is a Cauchy sequence of scalars. Let β be the σ-algebra generated by $\bigcup \pi_m$, and let $E_\beta(h_n)$ denote the conditional expectation of h_n with respect to β. The Egoroff-type argument suggested on p. 411 of [BH2] shows that

$$\lim_{m \to \infty} |\langle E_{\pi_m}(h_n) - E_\beta(h_n), g \rangle| = 0$$

uniformly with respect to n, for each $g \in L_\infty(\mu, X^*)$. This and the preceding shows that (h_n) is a σ'-Cauchy sequence, so it is σ'-convergent, by hypothesis. This contradicts the fact that K is not relatively σ'-compact.

For the converse implication suppose that (f_n) is a σ'-Cauchy sequence. By [Din, p. 374, Th. 1], (f_n) is bounded, uniformly integrable, and $\{\int_A f_n \, d\mu;\ n \in \mathbb{N}\}$ is conditionally weakly compact, hence relatively weakly compact by hypothesis for all $A \in \Sigma$. Because X has the RNP with respect to (S, Σ, μ), (f_n) is relatively σ'-compact according to [Din, p. 375, Th. 2]. Thus (f_n) is σ'-convergent.

Proof of 3.2. (a) $\Rightarrow$ (c): It is easy to see that X is a complemented subspace of $L_1(\mu, X)$, under the $\sigma' - \sigma(X, X^*)$ continuous projection $f \to \frac{1}{\mu(S)} \cdot \int_S f \, d\mu$, and that the relative weak and σ'-topologies coincide on X. Hence X must be SWCG.

Suppose X is not reflexive and μ is not purely atomic. Then we can find $S' \in \Sigma$ such that $(S', \Sigma \cap S', \mu | \Sigma \cap S')$ is purely non-atomic, and a sub$-\sigma$-algebra Σ_0 of $\Sigma \cap S'$ such that $(S', \Sigma_0, \mu | \Sigma_0)$ is measure isomorphic to the measure algebra for Lebesgue measure λ on $[0, 1]$. Using a conditional expectation to define a $\sigma' - \sigma'$ continuous projection of $L_1(\mu, X)$ onto $L_1(\lambda, X)$, we obtain that $L_1(\lambda, X)$ is strongly σ'-compactly generated. Hence we may assume that (S, Σ, μ) is the Lebesgue measure algebra on $[0, 1]$.

Let (r_j) denote the sequence of Rademacher functions in $L_1[0, 1]$ [DU, p. 103]. Since X is weakly sequentially complete (by 2.5) but not reflexive, the Rosenthal l_1 Theorem guarantees the existence of $\gamma > 0$ and a sequence (e_n) in $B(X)$ such that $\left\| \sum_{i=1}^{n} a_i e_i \right\| \geqq \gamma \sum_{i=1}^{n} |a_i|$ for every sequence of scalars $\{a_1, \ldots, a_n\}$. Let ω_c be the least ordinal having the cardinality of the continuum, and let $\{N_\alpha : \alpha < \omega_c\}$ be the collection of all infinite subsets of $\mathbb{N}$. Let $N_\alpha = \{n_{\alpha,j}\}$, where $n_{\alpha,j} < n_{\alpha,j+1}$ for all j. Let $K_\alpha = \{r_j \otimes e_{n_{\alpha,j}} : j \in \mathbb{N}\} \cup \{0\}$. Then K_α is a bounded, uniformly integrable subset of $L_1(\mu, X)$. We show it is relatively σ'-compact.

For each Lebesgue measurable set A, define an operator I_A: $L_1(\mu, X) \to X$ by $I_A(f) = \int_A f \, d\mu$. Then I_A is $\sigma' - \sigma(X, X^*)$ continuous. Let $\{A(i, j) : 0 \leqq i \leqq 2^j - 1\}$ be the partition of $[0, 1]$ associated with r_j. A short calculation shows that

$$I_{A(i, j_0)}(r_j \otimes e_{n_{\alpha,j}}) = 0 \quad \text{whenever} \quad j > j_0 \text{ and } 0 \leqq i \leqq 2^{j_0} - 1. \tag{1}$$

Thus $I_A(K_\alpha)$ is weakly compact in X for each α and each A of the form $A(i, j_0)$. Now using 1.3 and the uniform integrability of each K_α, the same holds for every Lebesgue measurable subset A of $[0, 1]$. According to the remarks after 3.1, $L_1(\mu, X)$ is σ'-sequentially complete, and so X has the RNP, by 3.3 (see [DU, V.3.8]). Thus each K_α is relatively σ'-compact, by [BH2, Th. 2.1, Cor. 2.5].

By hypothesis, there is a σ'-compact subset K_0 of $L_1(\mu, X)$ such that for all $\varepsilon > 0$ and $\alpha < \omega_c$, there is a positive integer $m(\alpha, \varepsilon)$ with $K_\alpha \subset m(\alpha, \varepsilon) K_0 + \varepsilon \cdot B(L_1(\mu, X))$. Let $D_m = \{\alpha < \omega_c : K_\alpha \subset m K_0 + (\gamma/4) B\}$. For $j \in \mathbb{N}$, let $P_j = \{t \in [0, 1] : r_j(t) > 0\}$. We claim that there exist $m, j \in \mathbb{N}$ such that $I_{P_j}(\cup \{K_\alpha : \alpha \in D_m\}) = \{\frac{1}{2} e_{n_{\alpha,j}} : \alpha \in D_m\} \cup \{0\}$ is infinite. If not, let $n_1 = \max\{n_{\alpha,1} : \alpha \in D_1\} + 1$ (or 1, if $D_1 = \emptyset$), and for $m>1$ let $n_m=\max\{n_{m-1}, n_{\alpha,j}: 1\leqq j\leqq m, \alpha\in D_m\}+1$. Then $\{n_m\}$ is an infinite subset of $\mathbb{N}$, so it is N_{α_0} for some $\alpha_0<\omega_c$. The construction reveals that α_0 belongs to no D_m. Since $[0, \omega_c)=\bigcup D_m$, this is a contradiction.

Now choose m, j as claimed above, and let $\{\frac{1}{2}e_{n_{\alpha,j}}: \alpha\in D_m\}=\{\frac{1}{2}e_{n_k}\}$, where $n_1<n_2<\dots$. Since $\|I_{p_j}\|\leqq 1$, $\{\frac{1}{2}e_{n_k}\}\subset m\,I_{P_j}(K_0)+(\gamma/4)\,B(X)$. For all k, choose $b_k\in(\gamma/2)\,B(X)$ so that $e_{n_k}-b_k\in 2\,m\,I_{P_j}(K_0)$. The sequence $e_{n_k}-b_k$ is then equivalent to the unit vector basis of l_1, since

$$\left\|\sum_{k=1}^{p} a_k(e_{n_k}-b_k)\right\|\geqq\left|\left\|\sum_{k=1}^{p} a_k e_{n_k}\right\|-\left\|\sum_{k=1}^{p} a_k b_k\right\|\right|$$

$$\geqq\gamma\cdot\sum_{k=1}^{p}|a_k|-\frac{\gamma}{2}\sum_{k=1}^{p}|a_k|=\frac{\gamma}{2}\sum_{k=1}^{p}|a_k|.$$

for every sequence of scalars $\{a_1, \dots, a_p\}$. This contradicts the weak compactness of $I_{P_j}(K_0)$, and the proof is complete.

4. Open Questions

(A) Must a WCG subspace of an SWCG space again be SWCG? Specifically, must every subspace of a separable SWCG space again be SWCG? This is of particular interest for the space $L_1[0, 1]$.
(*B*) Must a separable, weakly sequentially complete space which contains l_1 hereditarily be SWCG?
(C) If X is SWCG, must $L_1(\mu, X)$ be SWCG?

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