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Titel: Sequence Spaces and Summability Factors.

Autor: DeFranza, J.; Fleming, D.J.

Jahr: 1988

PURL: https://resolver.sub.uni-goettingen.de/purl?266833020_0199|log14

Kontakt/Contact

Digizeitschriften e.V.
SUB Göttingen
Platz der Göttinger Sieben 1
37073 Göttingen

info@digizeitschriften.de

Math. Z. 199, 99–108 (1988)

Mathematische Zeitschrift
© Springer-Verlag 1988

Sequence Spaces and Summability Factors

J. DeFranza and D.J. Fleming
St. Lawrence University, Department of Mathematics, Canton, NY 13617, USA

1. Introduction

Let $A=(a_{nk})$ be an infinite matrix of scalars. Then A defines a sequence to sequence transformation, mapping the sequence $x=(x_k)$ to Ax where

$$(Ax)_n=\Sigma_k a_{nk} x_k, \quad n=1,2,\ldots.$$

A sequence x is summable A if Ax is convergent and x is absolutely summable A or summable $|A|$ if Ax is of bounded variation. A series $\Sigma_n x_n$ is summable A if the corresponding sequence of partial sums is summable A.

If a sequence β has the property that $\Sigma_n \beta_n x_n$ is summable B whenever $\Sigma_n x_n$ is summable A we say β is a summability factor of type (A, B), and we write $\beta\in(A, B)$. Summability factors of type $(|A|, |B|)$ are referred to as absolute summability factors while those of type $(|A|, B)$ are called mixed summability factors.

Problems involving summability factors have drawn the attention of numerous researchers since the turn of the century. With few exceptions, the techniques that have been used to obtain these results have been the techniques of classical analysis. The purpose of this report is to show how some of these results, often in a substantially improved form, and some new results can be obtained using the theory of sequence spaces developed in the past two decades.

2. Notation and Terminology

Let ω denote the linear space of all scalar sequences and φ the subspace of all finitely non-zero sequences. By a sequence space E we shall mean a linear subspace of ω satisfying $\varphi\subseteq E$. For E, F sequence spaces $M(E, F)$ denotes the multipliers from E to F. That is, $M(E, F)=\{x\in\omega: xy\in F\ \forall y\in E\}$ where xy denotes the coordinatewise product. If $E=F$ we write $M(E)$ for $M(E, E)$.

If E is a locally convex Hausdorff sequence space (l.c.s.s.) then $E^f=\{t_f=(f(e_i))_{i=1}^{\infty}: f\in E'\}$ where E' is the topological dual of E and e_i denotes the i-th coordinate sequence. The closure of φ in E is denoted by E°. If $E=E^\circ$

then E is said to have AD. An immediate consequence of the Hahn-Banach Theorem is that $E^f=(E^\circ)^f$.

A l.c.s.s. E on which the coordinate functionals E_i given by $E_i(x)=x_i$ are continuous is called a K-space. A K-space E is called a sum space if $M(E)=E^f$. The concept of a sum space was introduced by Ruckle in [14] and subsequently studied by him in [13, 15, 16]. By an FK-space we shall mean a l.c.s.s. with its (unique) Fréchet topology. An FK-space which is normable is called a BK-space. If E is a BK-space then E^f is a BK-space with $\|t_f\|=\|f\|$. Furthermore E^f is isometrically isomorphic to $(E^\circ)'$ under the correspondence $t_f\to f$. We will assume throughout this report familiarity with the standard sequence spaces and their natural topologies (see e.g. [9, 16] or [19]).

If $x, y\in\omega$ then $\langle x, y\rangle$ will denote the sum $\Sigma_k x_k y_k$ provided the series converges. If E is FK and $x\in\varphi\subseteq E$, then $\forall f\in E'$, $f(x)=\langle t_f, x\rangle$. If, in addition, E has AD and $t_f\in\varphi\subseteq E^f$, then $f(x)=\langle t_f, x\rangle\ \forall x\in E$.

If E is an FK-space and T is an infinite matrix then $E_T=\{x\in\omega\colon Tx\in E\}$ admits a natural FK-topology (see, e.g. [19]). Of course, E_T will contain φ if and only if the columns of T are in E. The matrix T is E-reversible if $T\colon E_T\to E$ is $1-1$ and onto. If T is E-reversible and E is a BK-space, then E_T will be a BK-space isometrically isomorphic to E under the map T.

A continuous linear functional $\mathscr{E}$ on an FK-space E is called a sum if $\mathscr{E}(x)=\Sigma_k x_k\ \forall x\in\varphi$. Clearly E admits a sum if and only if $e\in E^f$ where e denotes the sequence of ones. If E has AD then a sum on E, if such exists, will be unique. It is routine to verify that E admits a sum if and only if $M(E)\subseteq E^f$. If $E\subseteq F$ then clearly $F^f\subseteq E^f$. If, in addition, E has AD then $E\subseteq F$ if and only if $F^f\subseteq E^f$ [17].

3. Sequence Space Theory

In this section we summarize some of the results from sequence space theory that will be used in this sequel and we interpret traditional summability factor questions in the sequence space context. The sequence spaces that arise naturally in summability factor theory are FK-spaces and so we shall state these results in this context despite the fact that many are valid in a more general setting.

Theorem A (Ruckle [15]). *Let E and F be FK-spaces such that E has AD and F admits a sum. Then the following are equivalent:*

a) $F^f\subseteq M(E)$

b) $F^f\subseteq M(E^f)$

c) $M(E, F)=E^f$.

Let $T=(t_{nk})$ be a row-finite matrix and let t^n denote the n-th row of T. For $x\in\omega$ let $H(x)$ denote the net of elements from φ given by $H(x)=\{\Sigma_{i\in F}(t^i-t^{i-1})\ x\colon F\in\mathscr{F}\}$ where $\mathscr{F}$ is the collection of finite subsets of the positive integers directed by containment and $t^\circ=0$. For E a l.c.s.s. define

$$E_{TB}=\{x\in\omega\colon \{t^n x\}_n \text{ is bounded in } E\}$$
$$E_{FTK}=\{x\in\omega\colon \{t^n x\}_n \text{ is weakly Cauchy in } E\}$$
$$E_{STK}=\{x\in E\colon t^n x\to x \text{ weakly in } E\}$$
$$E_{TK}=\{x\in E\colon t^n x\to x \text{ in } E\}.$$

A sequence $x \in E$ is said to have TB (T-sectional boundedness) if $x \in E_{TB}$. If $E \subseteq E_{TB}$, then E is called a TB-space. The remaining T-sectional properties are defined in an analogous way. The unconditional analogs are obtained by replacing the sequence $\{t^n x\}_n$ by the net $H(x)$. For example, $E_{UTB} = \{x \in \omega\colon H(x)$ is bounded in $E\}$. The α_T, β_T and γ_T-duals of a sequence space E are defined by $E^{\alpha_T} = M(E, bv_T)$, $E^{\beta_T} = M(E, c_T)$, $E^{\gamma_T} = M(E, m_T)$. In the case where $T = \Sigma$, the triangle of ones, these yield the familiar α, β and γ-duals of E. A row-finite matrix T is called an sp_a matrix if $\lim_n t_{nk} = a \; \forall k = 1, 2, \ldots$. If, in addition, the columns of T are in bv we say T is an sp_a^*-matrix.

Theorem B (Buntinas [3], Ruckle [13]). *Suppose A and T are c-reversible, row-finite, sp_1-matrices. The following are equivalent:*

a) c_A *is TB*
b) $c_A^f = c_A^{\beta_T}$
c) $\sup_{m,n} \Sigma_j |\Sigma_k t_{mk} a_{nk} a_{kj}^{-1}| < \infty$

where (a_{kj}^{-1}) denotes the matrix A^{-1}.

If, in addition, either $\ell_1 \subseteq c_A$ or c_A has AD, then each of the above is equivalent to

d) c_A *is TK*

Corollary. *Let T be a c-reversible row-finite sp_1-matrix such that $\ell_1 \subseteq c_T$ or c_T has AD. The following are equivalent:*

a) c_T *is a sum space*
b) c_T *is TB*
c) c_T *is TK*

Theorem C (Zeller [20]). *For each $r \geqq 0$, c_T is TK where $T = (C, r)\Sigma$.*

Consequently, convergence domains of the series to sequence Cesàro methods of order $r \geqq 0$ are sum spaces.

Theorem D (Fleming [5]). *Let T be a row-finite, bv-reversible, sp_1^*-matrix. The following are equivalent:*

a) bv_T *is a sum space*
b) bv_T *is a sum space with AD*
c) bv_T *is UTB*
d) bv_T *is UTK*
e) bv_T *is UFTK*
f) bv_t *is USTK*
g) $\sup_F \sup_n \Sigma_m \left| \sum_{j=1}^{n} \langle (t^m - t^{m-1}) t_F, s_j \rangle \right| < \infty$

where $t_F = \Sigma_{i \in F}(t^i - t^{i-1})$ and s_j denotes the j-th column of T^{-1}.

If $A = (a_{nk})$ is a normal sequence to sequence method then $A\Sigma = (\bar{a}_{nk})$ is the corresponding series to sequence method where Σ is the triangle of ones and

$\bar{a}_{nk}=\sum_{j=k}^{n} a_{nj}$ if $k\leqq n$ and $\bar{a}_{nk}=0$ if $k>n$. An ordered pair of normal methods $[A, B]$ is said to belong to the class $\mathscr{S}$ if the conditions

$$\beta_n=d+\sum_{j=n}^{\infty} \bar{a}_{jn}\gamma_j, \tag{3.1.1}$$

$$\beta_n=O(a_{nn}/b_{nn}) \tag{3.1.2}$$

where d is a constant and $\gamma=(\gamma_n)\in\ell_1$ are necessary and sufficient for $\beta\in(A, B)$ (d, γ depend upon β).

In the case where $A=(C, r)$, $B=(C, s)$, $r\geqq 0$, $s\geqq 0$, Peyerimhoff [12] showed that the conditions (3.1.1) and (3.1.2) are equivalent to:

$$\Sigma_n A_n^r|\Delta^{r+1}\beta_n|<\infty, \tag{3.1.3}$$

$$\beta_n=O(A_n^s/A_n^r). \tag{3.14}$$

It follows from this observation and the work of L.S. Bosanquet [1], Jurkat and Peyerimhoff [8] and G.E. Peterson [11] that $[(C, r), (C, s)]\in\mathscr{S}$ $\forall r\geqq 0$, $s\geqq 0$. A considerable effort in summability factor theory has been devoted to identifying pairs of methods in the class $\mathscr{S}$.

If A is normal and regular then clearly $\beta\in(A, A)$ implies $\beta\in m$ and hence $[A, A]\in\mathscr{S}$ if and only if condition (3.1.1) is necessary and sufficient for $\beta\in(A, A)$. It is easy to see that

$$(A, B)=M(c_{A\Sigma}, c_{B\Sigma}).$$

Furthermore if A is normal then it is c-reversible. In this case standard arguments (see e.g. [19]) on representing the dual of a convergence domain yield

$$c_{A\Sigma}^{f}=\left\{\beta\in\omega:\ \beta_n=d+\sum_{j=n}^{\infty}\bar{a}_{jn}\gamma_j,\ d\in\mathbb{C},\ \gamma\in\ell_1\right\}.$$

Thus we have:

Proposition 3.1. *Let A be normal and regular. Then $[A, A]\in\mathscr{S}$ if and only if $c_{A\Sigma}$ is a sum space.*

The following shows that the classic type M condition affords a necessary condition for a regular, normal method A to satisfy $[A, A]\in\mathscr{S}$.

Proposition 3.2. *Let A be regular and normal. If $c_{A\Sigma}$ is a sum space then A is of type M.*

Proof. Since A is regular and normal it is of type M if and only if it is perfect and this in turn is equivalent to $c_{A\Sigma}$ having AD. Now $A\Sigma$ is an sp_1-matrix and $\ell_1\subseteq c_{A\Sigma}$. It follows from the corollary to Theorem B that $c_{A\Sigma}$ will be a sum space if and only if it is a sum space with AD. □

If A is regular, normal and of type M then, since $c_{A\Sigma}$ has AD, $c_{A\Sigma}^{f}$ yields the sequence space representation of the dual of $c_{A\Sigma}$. Thus the normal, regular

methods for which $[A, A] \in \mathscr{S}$ are precisely those for which the summability factors afford a representation of the dual of $c_{A\Sigma}$. For any regular method A, $M(c_{A\Sigma}) \subseteq c_{A\Sigma}^f$ as $\lim\limits_{A\Sigma}$ is a sum on $c_{A\Sigma}$ and thus it is immediate that condition (3.1.1) is necessary for $\beta \in (A, A)$. More generally, if A and B are regular, then the existence of a sum on $c_{B\Sigma}$ implies $M(c_{A\Sigma}, c_{B\Sigma}) \subseteq c_{A\Sigma}^f$ [15]. Furthermore if A, B are normal and if $\beta \in M(c_{A\Sigma}, c_{B\Sigma})$ then condition (3.1.2) is satisfied as the matrix $B\Sigma\, D_\beta (A\Sigma)^{-1}$ will be conservative where D_β is the diagonal matrix with diagonal β.

Proposition 3.3. *Let E be a barrelled K-space and T an sp_a-matrix with $a \neq 0$. If E has AD, then $E^{\beta_T} = E^{\gamma_T}$.*

Proof. Clearly $E^{\beta_T} \subseteq E^{\gamma_T}$. Let $u \in E^{\gamma_T}$. Then $ux \in m_T \; \forall x \in E$. Define $f_n : E \to \mathbb{C}$ by $f_n(x) = \Sigma_k t_{nk} u_k x_k$. Then $\sup\limits_n |f_n(x)| < \infty \; \forall x \in E$ and so (f_n) is equicontinuous. Hence $S = \{x \in E : \lim\limits_n f_n(x) \text{ exists}\}$ is a closed linear subspace of E. For each $i \in N$, $f_n(e_i) = u_i t_{ni}$. Thus $\lim\limits_n f_n(e_i) = u_i a$ as T is an sp_a-matrix and hence $\varphi \subseteq S$. But E has AD so $\lim\limits_n \Sigma_k t_{nk} u_k x_k$ exists $\forall x \in E$. Thus $ux \in c_T \; \forall x \in E$ so $u \in M(E, c_T) = E^{\beta_T}$. □

Corollary 3.4. *Let A and B be normal, regular methods with A of type M. Then $\beta \in (A, B)$ if and only if the B transform of $\Sigma_n \beta_n x_n$ is bounded whenever $\Sigma_n x_n$ is summable A.*

Remark. In [3] Buntinas shows that if E is BK and T is an sp_a-matrix with $a \neq 0$, then

$$(E^f)_{TB} = (E^\circ)^{\gamma_T} \subseteq E^f.$$

Suppose now that A and B are regular and normal with A of type M and let $E = c_{A\Sigma}$, $T = B\Sigma$. Then $E = E^\circ$ and hence by Proposition 3.3 $E^{\gamma_T} = E^{\beta_T}$. Thus the elements of (A, B) are precisely those elements of $c_{A\Sigma}^f$ (i.e. satisfying (3.1.1)) which have T-sectional boundedness in the BK-space $c_{A\Sigma}^f$. This observation strongly suggests that condition (3.1.2) may not be a "natural" candidate for one of the sufficient conditions for a sequence β to be a summability factor from A to B but rather one which evolved through experience with the Cesàro methods. This is reinforced in [4] where an example is given of two regular, normal methods A and B with $[A, A]$ and $[B, B]$ in $\mathscr{S}$ but for which (3.1.1) and (3.1.2) are not sufficient for $\beta \in (A, B)$. More specifically in that instance, condition (3.1.2) does not serve to identify those elements of $c_{A\Sigma}^f$ which have $B\Sigma$-sectional boundedness.

The following representation theorem for the dual of bv is easily verified.

Theorem E. *Let $f \in bv'$ and let t be defined by*

$$t_k = \begin{cases} f(e) & \text{if } k = 1 \\ -f(e_{k-1}) & \text{if } k \geqq 2. \end{cases}$$

Then $t\in bs$, $\|f\|=\|t\|_{bs}$ *and for each* $x\in bv$

$$f(x)=t_1 \lim x+\sum_{k=1}^{\infty}\left(\sum_{j=1}^{k} t_{j+1}\right)(x_{k+1}-x_k)$$

$$=t_1 x_1+\sum_{k=1}^{\infty}\left(\sum_{j=1}^{k+1} t_j\right)(x_{k+1}-x_k)$$

$$=\sum_{k=1}^{\infty}\left(\sum_{j=1}^{k} t_k\right)(x_k-x_{k-1}) \quad \textit{where } x_0=0.$$

Conversely, if $t\in bs$ *then* $f_t\in bv'$ *where, for* $x\in bv$,

$$f_t(x)=\sum_{k=1}^{\infty}\left(\sum_{j=1}^{k} t_k\right)(x_k-x_{k-1}) \quad \textit{and} \quad \|f_t\|=\|t\|_{bs}.$$

As a consequence we have the following:

Theorem 3.5. *Let* $A=(a_{nk})$ *be normal and absolutely regular. Then* $(|A|, |A|)=M(bv_{A\Sigma})\subseteq bv^f_{A\Sigma}\subseteq m$.

Moreover, $\beta\in bv^f_{A\Sigma}$ *if and only if there exists* $d\in\mathbb{C}$, $\lambda=(\lambda_k)\in m$ *such that*

$$\beta_j=d+\sum_{k=j-1}^{\infty}\lambda_k(\bar{a}_{k+1\,j}-\bar{a}_{kj}). \tag{3.5.1}$$

Equivalently, $\beta\in bv^f_{A\Sigma}$ *if and only if there exists* $\lambda\in m$ *such that*

$$\beta_i=\sum_{k=i}^{\infty}\lambda_k(\bar{a}_{ki}-\bar{a}_{k-1,i}). \tag{3.5.2}$$

Proof. Clearly $(|A|, |A|)=M(bv_{A\Sigma})$ and since $\lim_{A\Sigma}$ is a sum on $bv_{A\Sigma}$ we have $M(bv_{A\Sigma})\subseteq bv^f_{A\Sigma}$. Also since A is absolutely regular, $bv\subseteq bv_A$ and hence $\ell_1\subseteq bv_{A\Sigma}$. It follows that $bv^f_{A\Sigma}\subseteq\ell_1^f=m$. The remainder of the theorem follows routinely from Theorem E and the fact that $f\in bv'_{A\Sigma}$ if and only if there exists $g\in bv'$ such that $f=g\circ A\Sigma$. □

Analogous to the class $\mathcal{S}$ for summability factors of type (A, B) we have:

Definition 3.6. A pair of absolutely regular, normal methods $[A, B]$ are said to belong to the class $\mathcal{S}^*$ if the following conditions together with (3.5.1) (or, equivalently (3.5.2)) are necessary and sufficient for $\beta\in(|A|, |B|)$

$$\beta_j=O(a_{jj}/b_{jj}) \tag{3.5.3}$$

$$\beta_j=O(1). \tag{3.5.4}$$

Remark. Since $\lim_{B\Sigma}$ is a sum on $bv_{B\Sigma}$ it follows that

$$(|A|, |B|) = M(bv_{A\Sigma}, bv_{B\Sigma}) \subseteq bv^f_{A\Sigma} \subseteq m.$$

Moreover, if $\beta \in (|A|, |B|)$ then $B\Sigma\, D_\beta (A\Sigma)^{-1}$ is a triangle mapping $bv \to bv$ and so must have bounded diagonal. Thus conditions (3.5.1), (3.5.3) and (3.5.4) are clearly necessary for $\beta \in (|A|, |B|)$.

In the case $A = B$ the above observations together with Theorem D yields

Theorem 3.7. *Let A be normal and absolutely regular. Then $[A, A] \in \mathscr{S}^*$ if and only if $bv_{A\Sigma}$ is a sum space with AD.*

Let $A = (C, r)$, $r \geqq 0$. Then condition (3.5.2) *becomes*

$$\beta_i = \sum_{k=i}^{\infty} \lambda_k \left(\frac{A^r_{k-i}}{A^r_k} - \frac{A^r_{k-i-1}}{A^r_{k-1}} \right)$$

$$= \sum_{k=i}^{\infty} \lambda_k \left(\frac{i}{k} \frac{A^{r-1}_{k-i}}{A^r_k} \right) \quad \text{where } \lambda \in m.$$

Consequently (see [2] and the references therein), $[(C, r), (C, s)] \in \mathscr{S}^*$ for all $r \geqq 0$, $s \geqq 0$. In particular, $bv_{(C,r)\Sigma}$ is a sum space for all $r \geqq 0$.

In [5] it is shown that if E is a BK-space and T is an sp^*_a-matrix with $a \neq 0$ then

$$(E^f)_{UTB} = (E^\circ)^{\alpha T} \subseteq E^f.$$

Let $E = bv_{(C,r)\Sigma}$, $T = (C, s)\Sigma$. Then $(bv^f_{(C,r)\Sigma})_{UTB} = bv^{\alpha T}_{(C,r)\Sigma} = (|(C, r)|, |(C, s)| \subseteq bv^f_{(C,r)\Sigma}$. Thus, since $[(C, r), (C, s)] \in \mathscr{S}^*$ the condition $\beta_n = O(A^s_n / A^r_n)$ identifies those elements $\beta \in bv^f_{(C,r)\Sigma}$ which have unconditional $(C, r)\Sigma$-bounded sections.

Several theorems in [2] allow for the identification of other pairs in $\mathscr{S}^*$. For example, $[(C, r), N_p] \in \mathscr{S}^*$ for all $r > 0$ where N_p is the Riez Harmonic Mean. As in the case of ordinary summability factors such pairs $[A, B]$ are Cesàro-like in that condition (3.5.3) is a necessary and sufficient condition for an element $\beta \in bv^f_{A\Sigma}$ to have unconditional $B\Sigma$-bounded sections.

Also in [2, Theorem D] it is argued that

$$(|(C, r)|, |(C, s)|) = (|(C, r)|), (C, s))$$

for all $r \geqq 0$, $s \geqq 0$. In fact this is always the case for absolutely regular pairs in $\mathscr{S}^*$.

Theorem 3.8. *Let A, B be absolutely regular, normal methods satisfying $[A, B] \in \mathscr{S}^*$. Then*

$$(|A|, |B|) = (|A|, B).$$

Proof. Clearly $(|A|, |B|) \subseteq (|A|, B)$. Let $\beta \in (|A|, B) = bv^{\beta T}_{A\Sigma} \subseteq bv^{\gamma T}_{A\Sigma} \subseteq bv^f_{A\Sigma}$ [3, Theorem 3]. Since A is absolutely regular $bv \subseteq bv_A$ and hence $\ell_1 = bv_\Sigma \subseteq bv_{A\Sigma}$. Thus $bv^f_{A\Sigma} \subseteq \ell^f_1 = m$ and so β satisfies (3.5.1) and (3.5.4). Furthermore $\beta \in (|A|, B)$ implies $B\Sigma\, D_\beta (A\Sigma)^{-1}$ maps bv into c. The conditions for a matrix to map

bv in c (see e.g. [19] Chapter 8) clearly imply that the diagonal of the matrix must be bounded and hence β satisfies (3.5.3). Thus $\beta \in (|A|, |B|)$ as $[A, B] \in \mathcal{S}^*$. □

In particular, if A is normal, absolutely regular such that $bv_{A\Sigma}$ is a sum space then $(|A|, |A|) = (|A|, A)$.

4. Inclusion Theorems

The following theorem was established by Jurkat and Peyerimhoff [8] under the assumption that B be triangular and in the more general setting by Kuttner [10].

Theorem F. *Let $r \geqq 0$ and let B be regular and stronger than (C, r). Then*

$$((C, r), B) = ((C, r), (C, r)).$$

In [10] Kuttner posed the question whether the result of Theorem F is a special property of Cesàro methods or whether it applies more generally. The following result supplies an answer to that question.

Theorem 4.1. *Let A be regular, row finite and c-reversible such that $c_{A\Sigma}$ is a sum space and let B be regular. Then*

$$(A, B) = (A, A)$$

if and only if B is stronger than A.

Proof. Since A is regular $\ell_1 \subseteq c_{A\Sigma}$ and hence by Theorem B, $c_{A\Sigma}$ is a sum space with AD. Furthermore the regularity of B implies $\lim_{B\Sigma}$ is a sum on the space $c_{B\Sigma}$. Thus by Theorem A,

$$\begin{aligned}(A, B) = (A, A) &\Leftrightarrow M(c_{A\Sigma}, c_{B\Sigma}) = M(c_{A\Sigma}) \Leftrightarrow M(c_{A\Sigma}, c_{B\Sigma}) \\ &= c^f_{A\Sigma} \Leftrightarrow c^f_{B\Sigma} \subseteq M(c_{A\Sigma}) \Leftrightarrow c^f_{B\Sigma} \subseteq c^f_{A\Sigma} \Leftrightarrow c_{A\Sigma} \subseteq c_{B\Sigma} \Leftrightarrow c_A \subseteq c_B. \quad \square\end{aligned}$$

The absolute analog is obtained by a similar argument.

Theorem 4.2. *Let A be absolutely regular, row finite and bv-reversible such that $bv_{A\Sigma}$ is a sum space and let B be absolutely regular. Then*

$$(|A|, |B|) = (|A|, |A|)$$

if and only if B is absolutely stronger than A.

In particular if B is absolutely regular then $(|(C, r)|, |B|) = (|(C, r)|, |(C, r)|)$ if and only if B is absolutely stronger than (C, r).

In [7] Jurkat and Peyerimhoff showed that if A is regular, normal and satisfies the mean value property then condition (3.1.1) is necessary and sufficient for $\beta\in(A, A)$. That is, for such methods A, $c_{A\Sigma}$ is a sum space. Using this result Thorpe [18] established the following:

Theorem G. *Let A and B be regular, normal methods satisfying the mean value property. Then $(A, A)\subseteq(B, B)$ if and only if A is stronger than B.*

Arguing as above one readily obtains the following generalization and the absolute analog.

Theorem 4.3. *Let A, B be regular, normal methods such that $c_{A\Sigma}$ and $c_{B\Sigma}$ are sum spaces. Then $(A, A)\subseteq(B, B)$ if and only if A is stronger than B.*

Theorem 4.4. *Let A, B be absolutely regular normal methods such that $bv_{A\Sigma}$ and $bv_{B\Sigma}$ are sum spaces. Then $(|A|, |A|)\subseteq(|B|, |B|)$ if and only if A is absolutely stronger than B.*

Question. In [10] Kuttner gave an example of methods A and B with B stronger than A such that $(A, A)\subseteq(A, B)$. Specifically his A was the binary method and B was the square of the binary method. Of course, as noted in [4], in this case $c_{A\Sigma}$ is not a sum space. It would be interesting to know if Theorem 4.1 characterizes those regular c-reversible methods for which $c_{A\Sigma}$ is a sum space. That is, let A be regular, row-finite, Type M and c-reversible. Does $(A, B)=(A, A)$ for all regular B stronger than A imply $c_{A\Sigma}$ is a sum space?

In [2] Bosanquet and Das show that if $\alpha>0$, $0<\delta<1$ then $|(C, \alpha)|\Rightarrow|(R, \exp(n^{\delta}), 1)|$ is false. They note that they actually obtain "rather more than is required" in that they show $|(C, \alpha)|\Rightarrow(R, \exp(n^{\delta}), 1)$ is false. In fact these statements are equivalent.

Theorem 4.5. *Let A, B be absolutely regular normal methods such that $[A, B]\in\mathscr{S}^*$ and $bv_{A\Sigma}$ is a sum space. Then $|A|\Rightarrow|B|$ if and only if $|A|\Rightarrow B$.*

Proof. $|A|\Rightarrow B\Leftrightarrow bv_{A\Sigma}\subseteq c_{B\Sigma}\Leftrightarrow c^f_{B\Sigma}\subseteq bv^f_{A\Sigma}\Leftrightarrow c^f_{B\Sigma}\subseteq M(bv_{A\Sigma})\Leftrightarrow$ (Theorem A) $M(bv_{A\Sigma}, c_{B\Sigma})=bv^f_{A\Sigma}\Leftrightarrow(|A|, B)=(|A|, |A|)\Leftrightarrow(|A|, |B|)=(|A|, |A|)$ (by 3.8) $\Leftrightarrow|A|\Rightarrow|B|$ (by 4.2). □

We now show how this allows for a much simplified argument of the falsity of the implication

$$|(C, \alpha)|\Rightarrow|(R, \exp(n^{\delta}), 1)|$$

$\alpha>0$, $0<\delta<1$.

Let $A=(C, \alpha)$, $B=(R, \exp(n^{\delta}), 1)$. As noted in [2] if $\alpha\geqq 1$, $0<\delta<1$ then

$$(|A|, |B|)=(|A|, |(C, 1-\delta)|)$$

and if $0<\delta<1$ then

$$|B|\Rightarrow|(C, 1-\delta)|.$$

Thus if $\alpha\geqq 1$, $0<\delta<1$ then $|A|\Rightarrow|B|$ is false for otherwise (Theorem 4.2) we would have $(|A|, |B|)=(|A|, |A|)$ which is not the case. On the other hand if $0<\alpha<1$, $0<\delta<1$ then $|A|\Rightarrow|B|$ is false for if not then we would have $|A|\Rightarrow|(C, 1-\delta)|\ \forall 0<\delta<1$ which is clearly not the case.

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Received July 1, 1987