Here the second sum is taken over distinct indices (i,j,k,l) such that $i<j$, $k<l$ and $i<k$, and the third sum is taken over distinct indices (i,j,k,l) such that $i<j$ and $k<l$. Note that (7.5) makes sense only when $n \geqq 4$: aside from using four indices, we also need $n-3$ in the denominator! This equation shows that $r_{n-1}(n)=1$. On the other hand, we can show that, for $2 \leqq k \leqq n-2$, the forms f_k are "wilder" than f_{n-1} in that $M_2 \cdot f_k(x_1, \ldots, x_n)$ is still *not* sos; thus, $r_k(n) \geqq 2$ and $r(n) \geqq 2$ for $n \geqq 4$. For $n=4$, an explicit computation shows that $M_2^2 \cdot f_2(x_1, \ldots, x_4)$ *is* sos, and so $r_2(4)=2$, $r(4)=2$. We close with the following:

(7.6) **Question.** Determine $r_k(n)$ and $r(n)$ for $n \geqq 5$ and $2 \leqq k \leqq n-2$.

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Note Added in Proof (June 1987)

Stated in another form, our determination of P_n $(n \geqq 3)$ in (5.2) amounts to the fact that the symmetric cubic $\alpha \sum x_i^3 + \beta \sum x_i^2 x_j + \gamma \sum x_i x_j x_k$ is psd for $x_i \geqq 0$ iff

$$(*) \qquad \left(\alpha - \beta + \frac{\gamma}{3}\right) + k\left(\beta - \frac{\gamma}{2}\right) + k^2 \cdot \frac{\gamma}{6} \geqq 0 \quad \text{for } k = 1, 2, \ldots, n.$$

In the special case when $n=3$, this has been proved earlier by J.F. Rigby (Univ. Beograd. Publ. Elektrotehn. Fak. Sci. Mat. Fiz. No. 412–460 (1973), pp. 217–226). Thus our, complete result on P_n can be viewed as a generalization of Rigby's result from 3 variables to any number of variables. In comparing our results with Rigby's (Theorem 2, *loc. cit.*), note that the first three conditions in $(*)$ are: $\alpha \geqq 0$, $\alpha + \beta \geqq 0$, and $\alpha + 2\beta + \gamma/3 \geqq 0$.