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Titel: Pointwise Convergence Theorems in L2 over a von Neumann Algebra.

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Jahr: 1986

PURL: https://resolver.sub.uni-goettingen.de/purl?266833020_0193|log40

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Math. Z. 193, 413–429 (1986)

Mathematische Zeitschrift
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Pointwise Convergence Theorems in L_2 over a von Neumann Algebra

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0. Introduction

There are many important theorems well-known in probability and ergodic theory concerning the almost sure convergence of sequences of functions in L_2. As a rule, these theorems are non-trivial extensions of much easier results concerning the convergence in L_2-norm.

In this paper we prove several theorems of this type in the von Neumann algebra context. Namely, we prove some "pointwise convergence" theorems for sequences in a Hilbert space H, where $H=L_2(M, \Phi)$ is the completion of a von Neumann algebra M under the norm $x \to \Phi(x^*x)^{1/2}$, Φ being a faithful normal state on M. The motivation for this is to be found in the mathematical foundations of quantum statistical mechanics.

The first individual ergodic theorem for automorphism of a von Neumann algebra has been proved by E.C. Lance [10]. There are quite a lot of papers devoted to the extensions of the classical pointwise convergence theorems to their noncommutative versions. Let us mention here the works of Y. Sinai and V. Anshelevich [16], E.C. Lance [11], F. Yeadon [19, 20], B. Kümmerer [9], N. Dang Ngoc [3], M.S. Goldstein [6], S. Goldstein and A. Łuczak [7], D. Petz [14, 15]; see also [8].

We shall now briefly discuss the content of the paper. In Sect. 1 we introduce the basic definitions and collect some auxiliary results. In Sect. 2 we prove an individual ergodic theorem for a unitary operator in $L_2(M,\Phi)$ induced by a Φ-preserving automorphism α of M, and an asymptotic formula for the Césaro means of an arbitrary unitary operator in $L_2(M, \Phi)$. This formula extends the classical Gaposhkin's result [5] for weakly stationary sequences in a Hilbert space. In Sect. 3 we discuss some problems concerning the existence of the ergodic Hilbert transform. All the results in this section seem to be new also in the classical case $M=L_\infty$ (over a probability space). In Sect. 4 we prove the non-commutative version of the Rademacher-Menshov theorem and some of its consequences.

1. Auxiliary Results

Let us begin with some notations and definitions. Throughout the paper, M is a σ-finite von Neumann algebra with a faithful normal state Φ. Proj M will denote the lattice of all selfadjoint projections in M. We assume that M acts, in a standard way, on the Hilbert space $H=L_2(M, \Phi)$ which is the completion of M under the norm $x\to\Phi(x^*x)^{1/2}$ with a cyclic and separating vector ξ_0 such that $\Phi(x)=(x\xi_0, \xi_0)$ for $x\in M$ ([18]). We shall identify M with the subset $M\xi_0=\{x\xi_0: x\in M\}$ of H. The norm in H will be denoted by $\|\ \|$, and the norm in M by $\|\ \|_\infty$.

For a $\xi\in H$ and $p\in$Proj M we set

$$S_{\xi,p}=\left\{(x_k)\subset M: \sum_{k=1}^{\infty} x_k\xi_0=\xi \text{ in } H \text{ and } \sum_{k=1}^{\infty} x_k p \text{ converges in norm in } M\right\}$$

and

$$\|\xi\|_p=\inf\left\{\left\|\sum_{k=1}^{\infty} x_k p\right\|_\infty : (x_k)\in S_{\xi,p}\right\}$$

(with the usual convention $\inf\emptyset=+\infty$).

Obviously, for all $\xi, \eta\in H$, we have

$$\|\xi+\eta\|_p\leqq\|\xi\|_p+\|\eta\|_p$$

and for $x\in M$

$$\|x\xi_0\|_p\leqq\|xp\|_\infty.$$

We adopt the following definition of the almost sure convergence in H.

1.1. Definition. A sequence (ξ_n) in $H=L_2(M, \Phi)$ is said to be *almost surely (a.s.) convergent* to $\xi\in H$ if for every $\varepsilon>0$, there exists a projection p in M such that $\Phi(1-p)<\varepsilon$ and $\|\xi_n-\xi\|_p\to 0$ as $n\to\infty$. In other words, $\xi_n\to 0$ *a.s.* if for every $\varepsilon>0$, there is a $p\in$Proj M with $\Phi(1-p)<\varepsilon$ and a matrix $(x_{n,l})$ with entries in M such that $\sum_{l=1}^{\infty} x_{n,l}\xi_0=\xi_n$ in H and $\left\|\sum_{l=1}^{\infty} x_{n,l}p\right\|_\infty\to 0$ as $n\to\infty$.

It is easily seen that in the classical commutative case of $M=L_\infty$ (over a probability space) the convergence just defined coincides with the usual almost everywhere convergence (via Egorov's theorem). For the elements of the algebra M the following kind of convergence (introduced by E.C. Lance) is mostly used. A sequence $(x_n)\subset M$ is said to be almost uniformly convergent to $x\in M$ if for every $\varepsilon>0$ there exists a projection $p\in$Proj M with $\Phi(1-p)<\varepsilon$ such that $\|(x_n-x)p\|_\infty\to 0$ as $n\to\infty$. Obviously, the almost uniform convergence implies the almost sure convergence but the converse implication seems not to be true.

1.2 (Kronecker's lemma). *Let* $(\xi_n)\subset H$ *and* $\sum_{k=1}^{n}\xi_k\to\eta$ *a.s. Then* $\frac{1}{n}\sum_{k=1}^{n} k\xi_k\to 0$ *a.s. as* $n\to\infty$.

Proof. Let us remark first that, for every sequence $(\eta_n)\subset H$ and every $p\in$Proj M, $\|\eta_n\|_p\to 0$ implies $\left\|\frac{1}{n}\sum_{k=1}^{n}\eta_k\right\|_p\to 0$. Indeed, let $\|\eta_n\|_p\to 0$, then there exists a matrix

$(x_{n,l})$ with entries in M such that $\sum_{l=1}^{\infty} x_{n,l}\,\xi_0=\eta_n$ $(n=1, 2, \ldots)$ and $\left\|\sum_{l=1}^{\infty} x_{n,l}\,p\right\|_\infty \to 0$ as $n\to\infty$. Thus

$$\frac{1}{n}\sum_{k=1}^{n}\eta_k=\frac{1}{n}\sum_{k=1}^{n}\sum_{l=1}^{\infty}x_{k,l}\,\xi_0$$

and

$$\left\|\frac{1}{n}\sum_{k=1}^{n}\sum_{l=1}^{\infty}x_{k,l}\,p\right\|_\infty \leqq \frac{1}{n}\sum_{k=1}^{n}\left\|\sum_{l=1}^{\infty}x_{k,l}\,p\right\|_\infty \to 0$$

which means that $\left\|\frac{1}{n}\sum_{k=1}^{n}\eta_k\right\|_p \to 0$.

Let us put $\eta_n=\sum_{k=1}^{n}\xi_k$. By the assumptions, for every $\varepsilon>0$, there exists $p\in \operatorname{Proj} M$ such that $\Phi(1-p)<\varepsilon$ and $\|\eta_n-\eta\|_p\to 0$. Then $\left\|\frac{1}{n}\sum_{k=1}^{n}(\eta_k-\eta)\right\|_p \to 0$. Consequently,

$$\left\|\frac{1}{n}\sum_{k=1}^{n}k\xi_k\right\|_p=\left\|\eta_n-\frac{1}{n}\sum_{k=1}^{n}\eta_{k-1}\right\|_p$$

$$\leqq \|\eta_n-\eta\|_p+\left\|\frac{1}{n}\sum_{k=1}^{n}(\eta_{k-1}-\eta)\right\|_p \to 0$$

which ends the proof.

1.3. Lemma. *Let $\alpha_1, \ldots, \alpha_N$ be complex numbers, and $x_1, \ldots, x_N\in M$. Then*

$$\left|\sum_{i=1}^{N}\alpha_i x_i\right|^2 \leqq \sum_{i=1}^{N}|\alpha_i|^2\sum_{i=1}^{N}|x_i|^2$$

where

$$|x|=(x^*x)^{1/2} \quad \textit{for } x\in M.$$

Proof. This easily follows by induction from the inequality $x^*y+y^*x\leqq x^*x+y^*y$, $(x, y\in M)$.

The following result of M.S. Goldstein [6] is a basic tool in our considerations. We formulate it for a $*$-automorphism of M.

1.4 (Goldstein's Maximal Theorem). *Let α be a Φ-preserving $*$-automorphism of M* (i.e. $\Phi(\alpha x)=\Phi(x)$ *for all* $x\in M$), *(y_n) be a sequence of positive operators from M and (ε_n) be a sequence of positive numbers. Set $s_n(x)=\frac{1}{n}\sum_{k=0}^{n-1}\alpha^k x$ for $x\in M$, $n=1, 2, \ldots$ Then there exists a $p\in\operatorname{Proj} M$ such that*

$$\Phi(1-p)\leqq 2\sum_{n=1}^{\infty}\varepsilon_n^{-1}\,\Phi(y_n)$$

and

$$\|p\,s_n(y_k)\,p\|_\infty \leqq 2\varepsilon_k \quad \textit{for } n, k=1, 2, \ldots$$

For a Φ-preserving $*$-automorphism α of M the (unique) unitary operator u in H such that $u(x\xi_0)=\alpha(x)\xi_0$ is said to be induced by α. In the sequel (also in Sects. 2 and 3) we use constantly the following notation. For $n=1, 2, \ldots$ and $x\in M$ we put

$$s_n(x)=\frac{1}{n}\sum_{k=0}^{n-1}\alpha^k x$$

and for $\xi\in H$

$$S_n(\xi)=\frac{1}{n}\sum_{k=0}^{n-1}u^k\xi.$$

1.5 (Maximal Ergodic Lemma). *Let u be a unitary operator in H induced by a Φ-preserving $*$-automorphism α of M. Let (D_n) be a sequence of positive operators from M, and (δ_n) be a sequence of positive numbers. Let (ξ_n) be a sequence in H. Then there exists a $p\in\mathrm{Proj}\,M$ such that*

$$\Phi(1-p)\leqq 4\sum_{k=1}^{\infty}\delta_k^{-1}(\Phi(D_k)+\|\xi_k\|^2)$$

and

$$\|ps_n(D_k)p\|_\infty\leqq 2\delta_k, \quad n, k=1, 2, \ldots,$$
$$\|S_n(\xi_k)\|_p<5\delta_k^{1/2}, \quad n, k=1, 2, \ldots.$$

Proof. Let $(x_{k,l})$ be a matrix with entries in M such that $\xi_k=\sum_{l=1}^{\infty}x_{k,l}\xi_0$ and $\|x_{k,l}\xi_0\|\leqq 2^{-l+1}\|\xi_k\|$ for all $k, l=1, 2, \ldots$ Putting $y_{k,l}=|x_{k,l}|^2$, $y_{k,0}=D_k$ and $\varepsilon_{k,l}=\delta_k 2^{-l+1}$, $\varepsilon_{k,0}=\delta_k$ we obtain

$$\sum_{k=1}^{\infty}\sum_{l=0}^{\infty}\varepsilon_{k,l}^{-1}\,\Phi(y_{k,l})=\sum_{k=1}^{\infty}\delta_k^{-1}\,\Phi(D_k)+\sum_{k,l=1}^{\infty}\delta_k^{-1}2^{l-1}\|x_{k,l}\,\xi_0\|^2$$
$$\leqq\sum_{k=1}^{\infty}\delta_k^{-1}\,\Phi(D_k)+2\sum_{k=1}^{\infty}\delta_k^{-1}\|\xi_k\|^2.$$

Now, by 1.4 there exists a $p\in\mathrm{Proj}\,M$ such that

$$\Phi(1-p)\leqq 4\sum_{k=1}^{\infty}\delta_k^{-1}\,(\Phi(D_k)+\|\xi_k\|^2)$$

and

$$\|ps_n(D_k)p\|_\infty\leqq 2\delta_k, \qquad (n, k=1, 2, \ldots),$$
$$\|ps_n(|x_{k,l}|^2)p\|_\infty\leqq 2^{-l+2}\,\delta_k \qquad (n, k, l=1, 2, \ldots).$$

By Lemma 1.3 for $z_1, \ldots, z_N\in M$ we have

$$\left|\sum_{k=1}^{N}z_k\right|^2\leqq N\sum_{k=1}^{N}|z_k|^2.$$

Thus, for every $x \in M$ we get

$$|s_n(x)|^2 = \frac{1}{n^2}\left|\sum_{k=0}^{n-1} \alpha^k x\right|^2 \leqq \frac{1}{n}\sum_{k=0}^{n-1} |\alpha^k x|^2$$
$$= \frac{1}{n}\sum_{k=0}^{n-1} \alpha^k(|x|^2) = s_n(|x|^2).$$

Consequently, we obtain

$$\| s_n(x_{k,l})p\|_\infty^2 = \|p|s_n(x_{k,l})|^2 p\|_\infty$$
$$\leqq \|ps_n(|x_{k,l}|^2)\, p\|_\infty \leqq 2^{-l+2}\,\delta_k \qquad (n, k, l = 1, 2, \ldots).$$

Moreover, we have

$$S_n(\xi_k) = \sum_{l=1}^{\infty} S_n(x_{k,l}\,\xi_0) = \sum_{l=1}^{\infty} s_n(x_{k,l})\xi_0$$

so

$$\| S_n(\xi_k)\|_p \leqq \sum_{l=1}^{\infty} \| s_n(x_{k,l})p\|_\infty \leqq \delta_k^{1/2} \sum_{l=1}^{\infty} 2^{-l/2+1} < 5\delta_k^{1/2}$$

for $n, k = 1, 2, \ldots$ which completes the proof.

Taking $D_k = 0$ and as u the identity operator we obtain

1.6. Corollary. *Let (ξ_k) be a sequence in H such that $\sum_{n=1}^{\infty} \|\xi_n\|^2 < \infty$. Then $\xi_n \to 0$ a.s. (comp. [7]).*

1.7. The proofs in Sects. 2 and 3 depend heavily on the properties of some kernels. For $|\lambda| \leqq \pi$ and $n = 1, 2, \ldots$, let us put

$$K_n(\lambda) = \frac{e^{in\lambda} - 1}{n(e^{i\lambda} - 1)}$$
$$L_n(\lambda) = \sum_{k=1}^{n} \frac{\sin k\lambda}{k} \tag{1}$$
$$\tilde{K}_n(\lambda) = \sum_{k=n+1}^{\infty} \frac{\sin k\lambda}{k}.$$

We have

$$L_n(\lambda) + \tilde{K}_n(\lambda) = -(\lambda/2) + (\pi/2)\,(\operatorname{sgn}\lambda). \tag{2}$$

The following estimates are well known and easy to verify: for all $n = 1, 2, \ldots$

$$|K_n(\lambda) - 1| \leqq Cn|\lambda|$$
$$|\tilde{K}_n(\lambda) - (\pi/2)\operatorname{sgn}\lambda| \leqq Cn|\lambda|, \qquad |\lambda| \leqq \pi, \tag{3}$$

$$|P_n(\lambda)| \leqq Cn^{-1}|\lambda|^{-1}, \qquad 0 < |\lambda| \leqq \pi \tag{4}$$

and, for $m > n$

$$|P_n(\lambda) - P_m(\lambda)| \leqq C(m-n)|\lambda|, \qquad m|\lambda| \leqq \pi, \tag{5}$$

$$|P_n(\lambda)-P_m(\lambda)|\leqq C(m-n)^{-1}n-1, \quad |\lambda|\leqq\pi, \tag{6}$$

$$|P_n(\lambda)-P_m(\lambda)|\leqq Cn^{-1}|\lambda|^{-1} \quad \text{for } 0<|\lambda|\leqq\pi, \tag{7}$$

where P_n denotes K_n or $\tilde{K}_n$. (The constant C is different in different formulae.)

2. Individual Ergodic Theorem and Asymptotic Formula

We start with the following individual ergodic theorem

2.1. Theorem. *Let u be a unitary operator in H induced by a Φ-preserving $*$-automorphism α of M. Then, for every $\xi\in H$*

$$S_n(\xi)=\frac{1}{n}\sum_{k=0}^{n-1}u^k\xi\to\xi'$$

almost surely as $n\to\infty$, where ξ' is given by the mean ergodic theorem for u.

Proof. Let us define the closed linear subspaces of H putting

$$H_1=[\xi\in H: u\xi=\xi]$$

and

$$H_2=[(y-\alpha y)\xi_0 : y\in M]^-.$$

Then, by a rather standard reasoning, we get $H_1\oplus H_2=H$. Let $\xi\in H$. Obviously, $\xi-\xi'$ lies in H_2, thus for every $\varepsilon>0$ there exist $y\in M$ and $\eta\in H$ such that $\|\eta\|<\varepsilon$ and

$$\xi-\xi'=\eta+(y-\alpha y)\xi_0. \tag{8}$$

Let us fix some $\varepsilon>0$ and take $\delta_k>0$ with $\sum_{k=1}^{\infty}\delta_k<\varepsilon/4$. Then there exist $\eta_k\in H$ with $\|\eta_k\|<\delta_k$ and $y_k\in M$ $(k=1, 2, \ldots)$ such that

$$\xi-\xi'=\eta_k+(y_k-\alpha y_k)\xi_0 \qquad (k=1, 2, \ldots).$$

Since we have

$$\sum_{k=1}^{\infty}\delta_k^{-1}\|\eta_k\|^2<\sum_{k=1}^{\infty}\delta_k<\varepsilon/4$$

then we obtain by Lemma 1.5 the existence of an element $p\in\mathrm{Proj}M$ with $\Phi(1-p)<\varepsilon$ such that

$$\|S_n(\eta_k)\|_p<5\delta_k^{1/2} \qquad (n, k=1, 2, \ldots).$$

For given $\delta>0$ let us fix k such that $\|S_n(\eta_k)\|_p<\frac{\delta}{2}$ *for* $n=1, 2, \ldots$. Thus, we have

$$\begin{aligned}\|S_n(\xi-\xi')\|_p&\leqq\|S_n(\eta_k)\|_p+\|S_n(y_k-\alpha y_k)\xi_0\|_p\\&\leqq\delta/2+\|s_n(y_k-\alpha y_k)p\|_\infty\leqq\delta/2\\&\quad+\|s_n(y_k-\alpha y_k)\|_\infty\leqq\delta/2+\frac{2}{n}\|y_k\|<\delta\end{aligned}$$

for n large enough, which means that $S_n(\xi)\to\xi'$ *a.s.* The proof is completed.

It is well known that, in general, the individual ergodic theorem does not hold for an arbitrary unitary operator u in $L_2(\Omega, \mathscr{F}, p)$ where $(\Omega, \mathscr{F}, p)$ is a probability space. It is also well known that the asymptotic behaviour of the Césaro means $\frac{1}{n}\sum_{k=0}^{n-1} u^k$ of a unitary operator u depends only on the local properties of the spectrum of u near the value 1.

In the next theorem we give the asymptotic formula for the Césaro means of an arbitrary unitary operator u acting in H. This formula describes the connection between the ergodic averages of u and the spectral measure of u, in the context of the almost sure convergence.

2.2. Theorem. *Let u be a unitary operator acting in H, and let*

$$u=\int_{-\pi}^{\pi} e^{i\lambda}\, E(d\lambda)$$

be its spectral representation with the spectral measure $E(\cdot)$. Let $S_n=\frac{1}{n}\sum_{k=0}^{n-1} u^k$ $(n=1, 2, \ldots)$. Then, for every $\xi\in H$ we have the following asymptotic formula

$$S_n(\xi)-E(|\lambda|\leqq 2^{-[\log_2 n]})\xi\to 0, \tag{9}$$

almost surely as $n\to\infty$, where $[\log_2 n]$ denotes the largest integer m with $m\leqq\log_2 n$.

Proof. We are going to follow the general idea of Gaposhkin ([5], Th. 1), so we keep some of his notations. Let us fix $\xi\in H$ and put $\sigma_n=S_n(\xi)$ for $n=1, 2, \ldots$, $Z(\cdot)=E(\cdot)\xi$ and $F(\cdot)=\|Z(\cdot)\|^2$. Then Z is an orthogonally scattered measure in H on $[-\pi, \pi]$, F is a positive measure. Moreover,

$$\sigma_n=\int_{-\pi}^{\pi} K_n(\lambda)Z(d\lambda) \qquad n=1, 2, \ldots$$

where K_n is given by (1).

First, we observe that putting for $n=1, 2, \ldots$

$$\delta_n=\sigma_{2^n}-Z(|\lambda|\leqq 2^{-n})$$

we get easily

$$\sum_{n=1}^{\infty}\|\delta_n\|^2<\infty \tag{10}$$

using the estimations (3) and (4). (The details can be found in [5].)

In the sequel, we put for $n=1, 2, \ldots$

$$\delta_{m,n}=\sigma_{2^n+m}-\sigma_{2^n} \qquad (m=1, 2, \ldots, 2^n-1).$$

Writing m in the form $m=\sum_{q=1}^{n}\varepsilon_q 2^{n-q}$ with $\varepsilon_q=1$ or 0 and applying the dyadic expansion method, well known in the theory of orthogonal series ([1]), we obtain the following representation

$$\delta_{n,m}=\sum_{q=1}^{n}\varepsilon_q\,\Delta_q^{j_q} \tag{11}$$

where

$$\begin{aligned} \Delta_q^j &= \int_{-\pi}^{\pi} (K_{2^n+j2^{n-q}}(\lambda) - K_{2^n+(j-1)2^{n-q}}(\lambda)) Z(d\lambda) \\ &= \int_{-\pi}^{\pi} R_{n,q,j}(\lambda) Z(d\lambda) \qquad (j=1, 2, \ldots, 2^q,\ q=1, 2, \ldots, n). \end{aligned} \tag{12}$$

(The indices $j_q \leqq 2^q$ in (11) depend only on m when n is fixed.)

Obviously, we have

$$\|\Delta_q^j\|^2 = \int_{-\pi}^{\pi} |R_{n,q,j}(\lambda)|^2 F(d\lambda). \tag{13}$$

Next, we shall find sufficiently good approximations of these integrals by sums with values in M. We shall do it in two steps.

By the uniform continuity of (a finite set of) functions $R_{n,q,j}$, taking a suitable partition of $[-\pi, \pi]$, we can write

$$\Delta_q^j = \eta_q^j + \sum_{t=1}^{t_n} R_{n,q,j}(\lambda_t^n) \zeta_t^n \tag{14}$$

($j=1, 2, \ldots, 2^q$, $q=1, \ldots, n$) where all vectors $\zeta_t^n \in H$ ($t=1, \ldots, t_n$) are mutually orthogonal,

$$\sum_{t=1}^{t_n} \|\zeta_t^n\|^2 = F([-\pi, \pi]),$$

and

$$\|\eta_q^j\|^2 < 2^{-n} n^{-5} \qquad \text{for } j=1, \ldots, 2^q,\ q=1, \ldots, n. \tag{15}$$

In particular, we have

$$\sum_{q=1}^{n} q^2 \sum_{j=1}^{2^q} \|\eta_q^j\|^2 < \frac{1}{n^2}. \tag{16}$$

Now, we approximate the vectors ζ_t^n in H by the "almost orthogonal" operators $x_{n,t}$ from M. Namely, we choose $x_{n,t} \in M$ and $\xi_t^n \in H$ ($t=1, \ldots, t_n$) such that

$$\zeta_t^n = \xi_t^n + x_{n,t}\, \xi_0 \qquad (t=1, \ldots, t_n) \tag{17}$$

and

$$\begin{aligned} \|\xi_t^n\|^2 &< 2^{-n} n^{-5} t_n^{-3} \\ |\Phi(x_{n,t}^* x_{n,v})| &< 2^{-n} n^{-5} t_n^{-3} \qquad (t, v=1, \ldots, t_n,\ t \neq v). \end{aligned}$$

In particular, we have

$$\sum_{t=1}^{t_n} \|\xi_t^n\|^2 < 2^{-n} n^{-5} t_n^{-2} < 2^{-n} n^{-5} \tag{18}$$

$$\sum_{q=1}^{n} q^2 2^q \sum_{t=1}^{t_n} \|\xi_t^n\|^2 < n^{-2}$$

and

$$\sum_{q=1}^{n} q^2\, 2^q \sum_{\substack{t,v=1 \\ t\neq v}}^{t_n} |\Phi(x^*_{n,t}\, x_{n,v})| < n^{-2}. \tag{19}$$

Finally, we obtain

$$\delta_{n,m} = \eta_{n,m} + \xi_{n,m} + y_{m,n}\, \xi_0$$

where

$$\eta_{n,m} = \sum_{q=1}^{n} \varepsilon_q\, \eta_q^{j_q},$$

$$\xi_{n,m} = \sum_{q=1}^{n} \varepsilon_q \sum_{t=1}^{t_n} R_{n,q,j_q}(\lambda_t^n)\, \xi_t^n,$$

$$y_{n,m} = \sum_{q=1}^{n} \varepsilon_q \sum_{t=1}^{t_n} R_{n,q,j_q}(\lambda_t^n)\, x_{n,t}$$

$(m=1, \ldots, 2^n-1,\ n=1, 2, \ldots)$.

Setting, for $j=1, \ldots 2^q,\ q=1, \ldots, n$

$$d_{n,q,j} = \sum_{t=1}^{t_n} R_{n,q,j}(\lambda_t^n)\, x_{n,t} \tag{20}$$

we get, by Lemma 1.3,

$$\begin{aligned} |y_{n,m}|^2 &= \left| \sum_{q=1}^{n} \varepsilon_q\, d_{n,q,j_q} \right|^2 \\ &\leqq \left(\sum_{q=1}^{n} q^{-2} \right) \left(\sum_{q=1}^{n} q^2\, |d_{n,q,j_q}|^2 \right) \\ &\leqq 2 \sum_{q=1}^{n} q^2\, |d_{n,q,j_q}|^2. \end{aligned}$$

Putting

$$D_n = 2 \sum_{q=1}^{n} q^2 \sum_{j=1}^{2^q} |d_{n,q,j}|^2$$

we obtain $D_n \in M$ and

$$|y_{n,m}|^2 \leqq D_n \qquad (m=1, 2, \ldots, 2^{n-1}). \tag{21}$$

We shall prove that

$$\sum_{n=1}^{\infty} \Phi(D_n) = 2 \sum_{n=1}^{\infty} \sum_{q=1}^{n} q^2 \sum_{j=1}^{2^q} \Phi(|d_{n,q,j}|^2) < \infty. \tag{22}$$

By (20) we have

$$\Phi(|d_{n,q,j}|^2)=\sum_{t=1}^{t_n}|R_{n,q,j}(\lambda_t^n)|^2\Phi(|x_{n,t}|^2)$$
$$+\sum_{\substack{t,v=1\\ t\neq v}}^{t_n}\overline{R_{n,q,j}(\lambda_t^n)}R_{n,q,j}(\lambda_v^n)\Phi(x_{n,t}^*\,x_{n,v})$$
$$=A_{n,q,j}+B_{n,q,j}.$$

Then (17) leads us to

$$\Phi(|x_{n,t}|^2)=\|x_{n,t}\,\xi_0\|^2\leqq 2\|\zeta_t^n\|^2+2\|\xi_t^n\|^2$$

so, by (14) and (18) with $|R_{n,q,j}(\lambda)|\leqq 2$, using the orthogonality of the vectors ζ_q^i, we get

$$A_{n,q,j}\leqq 2\sum_{t=1}^{t_n}|R_{n,q,j}(\lambda_t^n)|^2\,(\|\zeta_t^n\|^2+\|\xi_t^n\|^2)$$
$$\leqq 2\|\Delta_q^j-\eta_q^j\|^2+8\cdot 2^{-n}n^{-5}$$
$$\leqq 4\|\Delta_q^j\|^2+4\|\eta_q^j\|^2+8\cdot 2^{-n}n^{-5}.$$

Thus, by (16) and (13)

$$\sum_{n=1}^{\infty}\sum_{q=1}^{n}q^2\sum_{j=1}^{2q}A_{n,q,j}\leqq 4\sum_{n=1}^{\infty}\sum_{q=1}^{n}q^2\sum_{j=1}^{2q}(\|\Delta_q^j\|^2+\|\eta_q^j\|^2)+8\sum_{n=1}^{\infty}\sum_{q=1}^{n}q^2\,n^{-5}$$
$$\leqq 4\sum_{n=1}^{\infty}\sum_{q=1}^{n}q^2\sum_{j=1}^{2q}\int_{-\pi}^{\pi}|R_{n,q,j}(\lambda)|^2F(d\lambda)+12\sum_{n=1}^{\infty}n^{-2}.$$

We have the following estimations

$$|R_{n,q,j}(\lambda)|\leqq\begin{cases}C\,2^{n-q}|\lambda| & 0\leqq|\lambda|\leqq 2^{-n}\\ C\,2^{-q} & 2^{-n}<|\lambda|<2^{-(n-q)}\\ C\,2^{-n}|\lambda|^{-1} & 2^{-(n-q)}<|\lambda|\leqq\pi\end{cases}\tag{23}$$

which can be obtained easily from (5), (6) and (7). This implies the convergence of the series

$$\sum_{n=1}^{\infty}\sum_{q=1}^{n}\sum_{j=1}^{2q}|R_{n,q,j}(\lambda)|^2F(d\lambda)$$

as it has been shown by Gaposhkin ([5], the proof of Th. 1).

On the other hand, by (19),

$$\sum_{n=1}^{\infty}\sum_{q=1}^{n}q^2\sum_{j=1}^{2q}B_{n,q,j}\leqq 4\sum_{n=1}^{\infty}\sum_{q=1}^{n}q^2\,2^q\sum_{\substack{t,v=1\\ t\neq v}}^{t_n}|\Phi(x_{n,t}^*x_{n,v})|$$
$$<4\sum_{n=1}^{\infty}n^{-2}<\infty.$$

Then, we have proved (22).

Now, we observe that

$$\sum_{n=1}^{\infty}\sum_{m=1}^{2^n-1}\|\eta_{n,m}\|^2<\infty \tag{24}$$

for, by (15)

$$\|\eta_{n,m}\|^2\leqq n^2\sum_{q=1}^{n}\|\eta_q^{j_q}\|^2<2^{-n}n^{-2}.$$

Analogously, by (18) we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{2^n-1}\|\xi_{n,m}\|^2<\infty. \tag{25}$$

Putting

$$\gamma_{n,m}=\eta_{n,m}+\xi_{n,m}$$

from (24) and (25) we get

$$\sum_{n=1}^{\infty}\sum_{m=1}^{2^n-1}\|\gamma_{n,m}\|^2<\infty. \tag{26}$$

We have just proved that

$$\delta_{n,m}=\gamma_{n,m}+y_{n,m}\,\xi_0 \qquad (m=1, 2, \ldots, 2^n-1) \tag{27}$$

and conditions (26), (21), (22) hold.

The convergence of series (10), (26) and (22) implies that, for every $\varepsilon>0$, there exists a sequence $\varepsilon_n>0$ with $\varepsilon_n\to 0$ such that

$$\sum_{n=1}^{\infty}\varepsilon_n^{-1}\left(\Phi(D_n)+\sum_{m=1}^{2^n-1}\|\gamma_{n,m}\|^2+\|\delta_n\|^2\right)<\varepsilon.$$

By Lemma 1.5 (with $\alpha=\mathrm{id}$) there exists a $p\in\mathrm{Proj}\,M$ such that $\Phi(1-p)\leqq 4\varepsilon$ and

$$\|pD_np\|_\infty\leqq 2\varepsilon_n$$
$$\|\gamma_{n,m}\|_p<5\varepsilon_n^{1/2}, \qquad m=1, 2, \ldots, 2^n-1$$

and

$$\|\delta_n\|_p<5\varepsilon_n^{1/2}. \tag{28}$$

Thus, by (21) we get

$$\|y_{n,m}p\|_\infty^2=\|p\,|y_{n,m}|^2p\|_\infty\leqq\|pD_np\|_\infty\leqq 2\varepsilon_n$$

and, by (27)

$$\|\delta_{n,m}\|_p\leqq\|\gamma_{n,m}\|_p+\|y_{n,m}p\|_\infty<7\varepsilon_n^{1/2}. \tag{29}$$

Now, let $2^n\leqq N<2^{n+1}$. We have by (28) and (29)

$$\|\sigma_N-Z(|\lambda|\leqq 2^{-n})\|_p^2\leqq 2\|\sigma_N-\sigma_{2^n}\|_p^2+2\|\sigma_{2^n}-Z(|\lambda|\leqq 2^{-n})\|_p^2<148\varepsilon_n.$$

The proof is completed.

2.3. Remark. If the spectral measure E of a unitary operator u satisfies the condition

$$\int_{-\pi}^{\pi} \log|\lambda| \, F(d\lambda) > -\infty \tag{30}$$

for some $\xi \in H$, then $S_n(\xi) \to \xi'$ *a.s.* where $\xi' = Z(\{0\})$ is given by the mean ergodic theorem.

Indeed, putting $\tau_n = Z(0 < |\lambda| \leqq 2^{-n})$ we get

$$\sum_{n=1}^{\infty} \|\tau_n\|^2 < \infty. \tag{31}$$

In fact,

$$\sum_{n=1}^{\infty} \|\tau_n\|^2 = \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} F(2^{-(k+1)} < |\lambda| \leqq 2^{-k})$$

$$= \sum_{k=1}^{\infty} kF(2^{-(k+1)} < |\lambda| \leqq 2^{-k}) < \infty$$

by virtue of (30).

Like in the proof of Theorem 2.2 we can join (31) with the conditions (10), (26) and (22). Now, by Lemma 1.5, there exists a $p \in \operatorname{Proj} M$ with $\Phi(1-p) \leqq 4\varepsilon$ such that, additionally, we have $\|\tau_n\|_p \to 0$. The estimation

$$\|\sigma_N - \xi'\|_p^2 \leqq 2\,\|\sigma_N - Z(|\lambda| \leqq 2^{-n})\|_p^2 + 2\,\|\tau_n\|_p^2$$

ends the proof.

2.4. Remark. In the case when Φ is a tracial state, obviously, $S_n(\xi) \to \xi'$ *a.s.* if and only if

$$E(0 < |\lambda| \leqq 2^{-n})\xi \to 0 \quad a.s.$$

This follows immediately from the subadditivity of any trace on the lattice Proj M.

3. Ergodic Hilbert Transform

Let $(\Omega, \mathscr{F}, p)$ be a probability space and let $\alpha\colon \Omega \to \Omega$ be a measure preserving automorphism of Ω. M. Cotlar [2] proved (as a part of a more general result) that the limit

$$\lim_{n \to \infty} \sum_{0 < |k| \leqq n} \frac{f(\alpha^k \omega)}{k} \tag{32}$$

exists with probability one, for every function $f \in L_1(\Omega, \mathscr{F}, p)$. This limit is called the ergodic Hilbert transform of f. The most direct and elementary proof of the above result was given recently by K. Petersen [12] (cf. also [13]). In comparison with the (Garsia's) proof of the Birkhoff individual ergodic theorem, the known proofs of the existence of the ergodic Hilbert transform (32) are much

more complicated. In this section we prove an asymptotic formula for the ergodic Hilbert transform analogous to formula (9). It implies some results for tracial states which clarify the connection between the Césaro averages of a unitary operator and the existence of its ergodic Hilbert transform.

3.1. Theorem. *Let u be a unitary operator acting in H with its spectral measure $E(\cdot)$. Let us put $\tilde{S}_n = \sum_{0<|k|\leqq n} (1/k)u^k$ $n=1, 2, \ldots$. Then, for every $\xi \in H$ we have the following asymptotic formula*

$$\tilde{S}_n(\xi) - i\pi a_n\, \xi - a\xi \to 0 \quad a.s.,$$

as $n \to \infty$, where

$$\begin{aligned} a_n &= E(-2^{-[\log_2 n]} \leqq \lambda < 0) - E(0 < \lambda \leqq 2^{-[\log_2 n]}) \\ a &= i \int_{-\pi}^{\pi} (\lambda - \pi \operatorname{sgn} \lambda) E(d\lambda). \end{aligned} \tag{33}$$

Proof. It is enough to modify the proof of Theorem 2.2, so we use similar notations. Let $\tilde{\sigma}_n = \tilde{S}_n(\xi)$, for $n=1, 2, \ldots$. Then by (2),

$$\begin{aligned} \tilde{\sigma}_n &= 2i \int_{-\pi}^{\pi} L_n(\lambda) Z(d\lambda) \\ &= -2i \int_{-\pi}^{\pi} (\tilde{K}_n(\lambda) + \lambda/2 - (\pi/2) \operatorname{sgn} \lambda) Z(d\lambda) \end{aligned}$$

where L_n and $\tilde{K}_n$ are given by (1). Putting, for $n=1, 2, \ldots$

$$\tilde{\delta}_n = \tilde{\sigma}_{2^n} - i\pi \int_{\{2^{-n} < |\lambda| \leqq \pi\}} (\operatorname{sgn} \lambda) Z(d\lambda) + i \int_{-\pi}^{\pi} \lambda Z(d\lambda)$$

and using the estimations (3) and (4) we get that $\sum_{n=1}^{\infty} \| \delta_n \|^2 < \infty$. Next, let for $n=1, 2, \ldots,$

$$\tilde{\delta}_{n,m} = \tilde{\sigma}_{2^n+m} - \tilde{\sigma}_{2^n} \quad (m = 1, 2, \ldots, 2^n - 1).$$

Analogously, as in the proof of Theorem 2.2,

$$\tilde{\delta}_{n,m} = \sum_{q=1}^{n} \varepsilon_q \tilde{\Delta}_q^{j_q}$$

where, this time,

$$\begin{aligned} \tilde{\Delta}_q^j &= -2i \int_{-\pi}^{\pi} (\tilde{K}_{2^n + j2^{n-q}}(\lambda) - \tilde{K}_{2^n + (j-1)2^{n-q}}(\lambda)) Z(d\lambda) \\ &= -2i \int_{-\pi}^{\pi} \tilde{R}_{n,q,j}(\lambda) Z(d\lambda) \end{aligned}$$

$(j = 1, \ldots, 2^q;\ q = 1, \ldots, n)$.

By the conditions (5), (6) and (7) the (continuous) functions $\tilde{R}_{n,q,j}$ are estimated just in the same way as $R_{n,q,j}$ in (23). The rest part of the proof does not need any changes. The equality

$$i\pi \int\limits_{\{2^n<|\lambda|\leqq\pi\}} (\operatorname{sgn}\lambda)\, Z(d\lambda) - i\int\limits_{-\pi}^{\pi} \lambda Z(d\lambda) = i\pi a_n \xi + a\xi$$

completes the proof.

3.2. Remark. In the case when Φ is a tracial state, clearly, $\tilde{S}_n(\xi)\to a\xi$ *a.s.* (where a is given by (33)) if and only if

$$E(-2^{-n}\leqq\lambda<0)\xi - E(0<\lambda\leqq 2^{-n})\xi \to 0 \quad a.s.$$

The next theorem clarifies the connection between the Césaro averages and the ergodic Hilbert transform.

3.3. Theorem. *Let u be a unitary operator on H. Let us assume that Φ is a tracial state. Then the existence of the ergodic Hilbert transform of u*

$$\lim_{n\to\infty} \tilde{S}_n(\xi) = a\xi \quad \text{a.s.}, \tag{34}$$

for every $\xi\in H$, is equivalent to the existence of the limit of the Césaro means

$$\lim_{n\to\infty} S_n(\xi) = \xi' \quad \text{a.s.}, \tag{35}$$

for every $\xi\in H$.

Proof. Put $P=E(-\pi\leqq\lambda<0)$, $Q=E(0<\lambda\leqq\pi)$ and $P_n=E(-2^{-n}\leqq\lambda<0)$, $Q_n=E(0<\lambda\leqq 2^{-n})$. Assume that the limit (34) exists almost surely, for every $\xi\in H$. Then we have, by 3.2

$$P_n\xi = P_n(P\xi) = (P_n - Q_n)P\xi \to 0 \ a.s.$$

In the same way, we show that $Q_n\xi\to 0$ a.s. Consequently, $E(0<|\lambda|\leqq 2^{-n})\,\xi = (P_n+Q_n)\xi\to 0$ *a.s.* which implies, by Remark 2.4 the existence of the limit (35) almost surely. In the similar way we shown the converse implication.

3.4. Corollary. *Let $(\Omega, \mathscr{F}, p)$ be a probability space. For a unitary operator u in $L_2(\Omega, \mathscr{F}, p)$, each of the following conditions:*

(α) $uf\geqq 0$ a.e. *for* $f\geqq 0$ a.e.

(β) *there is a non-negative function $g\in L_1(\Omega, \mathscr{F}, p)$ such that*

$$\int\limits_{\Omega} |u^{-k}f|\,dp \leqq \int\limits_{\Omega} g\,|f|\,dp$$

for all $f\in L_\infty(\Omega, \mathscr{F}, p)$ and $k\geqq 1$,

is sufficient for the existence of the ergodic Hilbert transform

$$\lim_{n\to\infty} \sum_{0<|k|\leqq n} \frac{1}{k} u^k f$$

for all $f\in L_2$.

This follows immediately, by Theorem 3.3, from the well-known ergodic theorems of Stein ([17], p. 87) and R. Duncan ([4]).

4. The Rademacher-Menshov Theorem

The Rademacher-Menshov theorem is one of the most important results in the theory of orthogonal series. In this section we prove an extension of this theorem for orthogonal sequences in $H=L_2(M, \Phi)$. The other versions of this theorem can be found in [7].

4.1. Theorem. *Let (ξ_n) be an orthogonal sequence in H such that*

$$\sum_{n=1}^{\infty} (\log n)^2 \| \xi_n \|^2 < \infty. \tag{36}$$

Put $\sigma_n = \sum_{k=1}^{n} \xi_k$. Then $\sigma_n \to \sigma$ almost surely, where σ is the sum of the series $\sum_{k=1}^{\infty} \xi_k$ in H.

In the proof we shall need the following lemma.

4.2. Lemma. *Let m be a positive integer and let $\{\xi_1, \xi_2, \ldots, \xi_{2^m}\}$ be a sequence of pairwise orthogonal elements in H. Then, for every $\delta>0$, there exist a system $\{x_1, \ldots, x_{2^m}\}$ in M and a positive operator $B \in M$ such that*

$$\| x_i \xi_0 - \xi_i \| < \delta, \qquad (i=1, \ldots, 2^m),$$

$$\left| \sum_{k=1}^{n} x_k \right|^2 \leqq (m+1) B, \qquad (1 \leqq n \leqq 2^m)$$

and

$$\Phi(B) \leqq (m+1) \sum_{k=1}^{2^m} \| \xi_k \|^2 + \delta.$$

Proof. The proof can be obtained as a slight modification of the proof of Proposition 4.4.2 in [8], p. 83. Namely, it suffices to take a good approximation of $\xi_j \in H$ by $x_j \in M$ and repeat the reasoning in [8].

Now, we are in a position to prove Theorem 4.1. Let us remark first that, like in the classical case, we have

$$\sum_{k=1}^{\infty} \| \sigma_{2^k} - \sigma \|^2 < \infty.$$

By Lemma 4.2 we can find two sequences (B_k) of positive operators from M and $(s_n) \subset M$ such that, we have

$$\sum_{n=1}^{\infty} \| \sigma_n - s_n \xi_0 \|^2 < \infty$$

$$|s_n - s_{2^k}|^2 \leqq B_k \qquad \text{for } 2^k < n \leqq 2^{k+1}$$

with

$$\Phi(B_k) \leqq (k+1)^2 \sum_{l=2^k+1}^{2^{k+1}} \|\xi_l\|^2 + \delta_k, \tag{37}$$

where $\sum_{k=1}^{\infty} \delta_k < \infty$. Then (37) and the assumption (36) imply $\sum_{k=1}^{\infty} \Phi(B_k) < \infty$.

Let us arrange the sequences $\{\sigma_k - s_k \xi_0\}$ and $\{\sigma_{2^k} - \sigma\}$ into one sequence $(\tau_1, \tau_2, \ldots) = (\sigma_1 - s_1 \xi_0, \sigma_{2^1} - \sigma, \sigma_2 - s_2 \xi_0, \sigma_{2^2} - \sigma, \ldots)$. Applying Lemma 1.5 to the sequences (τ_k) and (B_k), for every $\varepsilon > 0$, we can find a projection $p \in \operatorname{Proj} M$ with $\Phi(1-p) < \varepsilon$ and such that

$$\begin{aligned} \| p B_k p \|_\infty &\to 0 \\ \| \sigma_k - s_k \xi_0 \|_p &\to 0 \\ \| \sigma_{2^k} - \sigma \|_p &\to 0 \end{aligned} \tag{38}$$

as $k \to \infty$.

We have the following estimation

$$\| \sigma_n - \sigma \|_p \leqq \| \sigma_n - s_n \xi_0 \|_p + \| (s_n - s_{2^k}) \xi_0 \|_p + \| s_{2^k} \xi_0 - \sigma_{2^k} \|_p + \| \sigma_{2^k} - \sigma \|_p .$$

Moreover, for $2^k < n \leqq 2^{k+1}$

$$\| (s_n - s_{2^k}) \xi_0 \|_p \leqq \| (s_n - s_{2^k}) p \|_\infty \leqq \| p B_k p \|_\infty^{1/2} .$$

To finish the proof it is enough to apply (38).

Theorem 4.1 and Kronecker's lemma 1.2 imply the following strong law of large numbers for orthogonal sequences in $L_2(M, \Phi)$.

4.3. Theorem. *Let (ξ_n) be an orthogonal sequence in $L_2(M, \Phi)$ satisfying the condition*

$$\sum_{n=1}^{\infty} \left(\frac{\log n}{n} \right)^2 \| \xi_n \|^2 < \infty .$$

Then

$$\frac{1}{n} \sum_{k=1}^{n} \xi_k \to 0 \quad a.s.$$

(comp. with Theorem 4.4.1 in [8]*).*

4.4. Remark. Keeping the notation from Sect. 2, write the condition

$$\int_{0 < |\lambda| < \lambda_0} \log \log |\lambda| \, F(d\lambda) > -\infty \quad \text{for some } \lambda_0 > 0. \tag{39}$$

Under the assumption that the state Φ is tracial, it is easy to show that (39) is sufficient for the almost sure convergence of the averages $\frac{1}{n} \sum_{k=0}^{n-1} u^k \xi$. Indeed, it is enough to consider the orthogonal series

$$\sum_{k=1}^{\infty} a_k Z_k ,$$

where $a_k = [F(2^{-k-1} \leqq |\lambda| < 2^{-k})]^{1/2}$,

$$Z_k = a_k^{-1} Z(2^{-k-1} \leqq |\lambda| < 2^{-k}),$$

and to apply Theorem 4.1 and Remark 2.4.

Acknowledgement. The authors are indebted to the referee for his suggestions of some improvements in the text of this paper.

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Received September 1, 1985; in final form April 16, 1986