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The Splitting Relation for Köthe Spaces

Jörg Krone and Dietmar Vogt

Bergische Universität, Gesamthochschule Wuppertal, Fachbereich Mathematik, Gaußstr. 20, D-5600 Wuppertal 1, Federal Republic of Germany

In the present note we will give a complete characterization in terms of the defining matrices of those pairs $(\lambda(A), \lambda(B))$ of Köthe spaces, for which every exact sequence $0 \to \lambda(B) \to G \to \lambda(A) \to 0$ of Fréchet spaces splits. This can be expressed by the equation $\mathrm{Ext}^1(\lambda(A), \lambda(B)) = 0$, where $\mathrm{Ext}^1(\lambda(A), \cdot)$ is the first derived functor of the functor $L(\lambda(A), \cdot)$ acting from the category of Fréchet spaces to the category of linear spaces (over the scalar field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). Hence we can also say that we will give a characterization of those pairs of Köthe spaces, for which $\mathrm{Ext}^1(\lambda(A), \lambda(B)) = 0$. From the general properties of the Ext^1-functor it follows that for these spaces and any exact sequence $\lambda(B) \to G \xrightarrow{q} \lambda(A) \to 0$ of Fréchet spaces q has a right inverse.

The condition, which is characteristic (called (S^*) in this note), is the condition (S_2^*) defined in [14], §3 (cf. [13]), where it is shown to be necessary for $\mathrm{Ext}^1(E, F) = 0$ for any two Fréchet spaces E and F. We show that it is also sufficient in the case of two Köthe spaces. We give also a proof for the necessity in this case, since it is, in contrast to the general case, quite elementary.

General sufficient conditions for $\mathrm{Ext}^1(E, F) = 0$ have been given by several authors: condition (S) in [1], condition (S_1^*) in [14] and the condition in [5], Theorem 4.1. The first two turn out not to be necessary, since (S_1^*) and (S_2^*) are in general inequivalent (see [4]). An example is contained in §3.

In §2 we show that (S^*) also determines the topological properties of the space $L_b(\lambda(A), \lambda(B))$. It is shown that this space is bornological (or barrelled) if and only if $(\lambda(A), \lambda(B))$ satisfy (S^*). This result is in strong connection with a classical result of Grothendieck ([3], II, §4, n^0 3 Theorem 15 (f)). In fact our proof can be used to considerably simplify the proof of Grothendieck. (S^*) also simplifies his condition. In 3.1 the equivalence is shown.

In §3 also we discuss the relation between (S^*), (S_1^*) and the condition given in [12] for $L(\lambda(B), \lambda(A)) = LB(\lambda(B), \lambda(A))$. This can be used to give an example separating (S^*) and (S_1^*).

Preliminaries. We use the common notation for locally convex spaces (see [6, 11]). For sequence spaces see also [2], for concepts of homological algebra [10].

$A=(a_{\nu,n})_{\nu,n\in\mathbb{N}}$ and $B=(b_{j,k})_{j,k\in\mathbb{N}}$ always denote infinite matrices which satisfy $\sup_n a_{\nu,n}>0$, $0\leqq a_{\nu,n}\leqq a_{\nu,n+1}$ for all ν, n, resp. an analoguous condition for B (Köthe matrices). We define

$$\lambda(A)=\{x=(x_1,x_2,\ldots)\colon \|x\|_k=\sum_j |x_j|\, a_{j,k}<+\infty \text{ for all } k\}$$

$$\lambda^\infty(A)=\{x=(x_1,x_2,\ldots)\colon \|x\|_k=\sup_j |x_j|\, a_{j,k}<+\infty \text{ for all } k\}.$$

Equipped with the seminorms $\|\ \|_k$ these are Fréchet spaces. If A has the form $a_{j,k}=e^{\rho_k\alpha_j}$, where $\rho_k\nearrow r$, $r\in\{0,+\infty\}$ and $\alpha=(\alpha_1,\alpha_2,\ldots)$, $\alpha_j\nearrow+\infty$, then $\Lambda_r(\alpha)\colon =\lambda(A)$ is called power series space, of finite (resp. infinite) type for $r=0$ (resp. $r=+\infty$). We put:

$$\lambda^x(A)=\left\{x=(x_1,x_2,\ldots)\colon \|x\|_{-k}=\sum_j |x_j|\frac{1}{a_{j,k}}<+\infty \text{ for some } k\right\}$$

where $\frac{\alpha}{0}=+\infty$ for $\alpha>0$. Equipped with the inductive limit topology of the $\|\ \|_{-k}$ this is a (DF)-space.

A step space of $\lambda(A)$ is a space of the form $\{(x_{n(j)})_j\colon x\in\lambda(A)\}$ for some strictly increasing sequence $(n(j))_j$ in $\mathbb{N}$, i.e. $\lambda(A)$ with $\tilde{A}=(a_{n(j),k})_{j,k}$.

By b_k we denote the sequence $(b_{j,k})_{j\in\mathbb{N}}$. We put $J_k=\{j\in\mathbb{N}\colon b_{j,k}>0\}$ and define

$$l^1(b_k)=\{x=(x_j)_{j\in J_k}\colon \|x\|_k=\sum_{j\in J_k}|x_j|\, a_{j,k}<+\infty\}.$$

We define $\mathrm{Ext}^1(\lambda(A),\cdot)$ as the first derived functor of the functor $L(\lambda(A),\cdot)$. From [13], Theorem 1.6 (which obviously holds also for $E=\lambda(A)$, since Corollary 1.2 is true then) or [14], Theorem 1.2 we get the following concrete representation:

$$\mathrm{Ext}^1(\lambda(A),\lambda(B))=\prod_k L(\lambda(A),l^1(b_k))/B(\lambda(A),B)$$

where

$$B(\lambda(A),B)=\{(A_k)_k\in L(\lambda(A),l^1(b_k))\colon \text{there exists } (B_k)_k\in\prod_k L(\lambda(A),l^1(b_k))$$
$$\text{such that } A_k=\rho_{k+1,k}\circ B_{k+1}-B_k \text{ for all } k\}.$$

Here $\rho_{k+1,k}$ denotes the restriction map $x\to x|_{J_k}$ for $x\in l^1(b_{k+1})$. $\mathrm{Ext}^1(\lambda(A),\lambda(B))=0$ if and only if every exact sequence

$$0\to\lambda(A)\to G\to\lambda(B)\to 0$$

of Fréchet spaces splits. It is also equivalent to more general lifting and extension properties (see [14] Theorem 1.8). It implies moreover (see [14], Proposition 1.6) that every exact sequence

$$\lambda(A)\to G\to\lambda(B)\to 0$$

splits, that means: every surjective continuous linear map from a Fréchet space G onto $\lambda(B)$, the kernel of which is a quotient of $\lambda(A)$, has a right inverse.

From the above representation we derive the following

Criterion. $\operatorname{Ext}^1(\lambda(A), \lambda(B))=0$ if and only if for every sequence $A_k \in L(\lambda(A), l^1(b_k))$, $k=1,2,\ldots$, we can find another sequence $B_k \in L(\lambda(A), l^1(b_k))$, $k=1,2,\ldots$, such that $(A_k x)_j=(B_{k+1}x)_j-(B_k x)_j$ for all $j \in J_k$.

Notice that the representation and hence the criterion depends on B only up to equivalence, where B is equivalent to $\tilde{B}$ iff $\lambda(B)=\lambda(\tilde{B})$.

By $L_b(E,F)$ we denote the space of continuous linear maps from E to F with the topology of uniform convergence on bounded subsets of E, by $LB(E,F)$ the space of bounded linear maps from E to F, i.e. maps φ with φU bounded for some open U.

We always make use of the following

Definition. For $a, b, a', b' \in [0, +\infty)$: $\frac{a}{b} \leqq \frac{a'}{b'} \Leftrightarrow ab' \leqq a'b$.

1. In this section we consider fixed Köthe spaces $\lambda(A)$ and $\lambda(B)$, where $A=(a_{\nu,n})_{\nu,n\in\mathbb{N}}$, $B=(b_{j,k})_{j,k\in\mathbb{N}}$ are Köthe matrices. We define:

Definition. $(\lambda(A), \lambda(B))$ satisfy condition (S^*) iff we have:

(S^*) $\quad \forall \mu \quad \exists n_0, k \quad \forall K, m \quad \exists n, S \quad \forall \nu, j$:

$$\frac{a_{\nu,m}}{b_{j,k}} \leqq S \max\left(\frac{a_{\nu,n}}{b_{j,K}}, \frac{a_{\nu,n_0}}{b_{j,\mu}}\right).$$

We will prove the following theorem:

1.1. Theorem. $\operatorname{Ext}^1(\lambda(A), \lambda(B))=0$ *if and only if* $(\lambda(A), \lambda(B))$ *satisfy condition* (S^*).

The proof of the sufficiency part is given in Proposition 1.5, the proof of the necessity part in Proposition 1.7.

We start by rewriting the condition in an apparently strengthened form (cf. [14], 3.3):

1.2. Lemma. *If* $(\lambda(A), \lambda(B))$ *satisfy* (S^*) *then either* $\lambda(A) \cong l^1$ *or* $(\lambda(A), \lambda(B))$ *satisfy the following condition:*

$(S^*)_0$ $\quad \forall \mu \quad \exists n_0, k \quad \forall K, m, \quad R>0 \quad \exists n, S \quad \forall \nu, j$:

$$\frac{a_{\nu,m}}{b_{j,k}} \leqq \max\left(S\frac{a_{\nu,n}}{b_{j,K}}, \frac{1}{R}\frac{a_{\nu,n_0}}{b_{j,\mu}}\right).$$

Proof. We assume that $\lambda(A) \ncong l^1$. We choose $\bar{n}_0$, $\bar{k}$ according to (S^*) and determine $\bar{m}$, such that

$$\inf\left\{\frac{a_{l,\bar{n}_0}}{a_{l,\bar{m}}} : l \in \mathbb{N}, a_{l,\bar{m}} \neq 0\right\}=0.$$

This is possible since $\lambda(A)$ is not a Banach space. Hence we have with $L_{\bar{m}}=\{l : a_{l,\bar{m}} \neq 0\}$:

$\forall K \quad \exists \bar{n}, \bar{S} \quad \forall l \in L_{\bar{m}}, \quad j \in \mathbb{N}$:

$$\frac{1}{b_{j,\bar{k}}} \leqq \bar{S} \max\left(\frac{a_{l,\bar{n}}}{a_{l,\bar{m}}} \cdot \frac{1}{b_{j,K}}, \frac{a_{l,\bar{n}_0}}{a_{l,\bar{m}}} \cdot \frac{1}{b_{j,\mu}}\right).$$

From this one derives by choosing $l = l(\varepsilon)$ appropriately and putting $C = \bar{S}\frac{a_{l,\bar{n}}}{a_{l,\bar{m}}}$:

$$\forall K, \varepsilon > 0 \quad \exists C \quad \forall j: \frac{1}{b_{j,k}} \leqq \max\left(C\frac{1}{b_{j,K}}, \frac{1}{b_{j,\mu}}\right).$$

We apply again (S^*) with $\mu = \bar{k}$ and obtain k, n_0 such that:

$\forall K, m \quad \exists n, S \quad \forall \nu, j$:

$$\frac{a_{\nu,m}}{b_{j,k}} \leqq S \max\left(\frac{a_{\nu,n}}{b_{j,K}}, \frac{a_{\nu,n_0}}{b_{j,\bar{k}}}\right).$$

If additionally $R > 0$ is given, we apply the previous with $\varepsilon = \frac{1}{SR}$ and obtain assuming $n \geqq n_0$, $C > 1$:

$$\frac{a_{\nu,m}}{b_{j,k}} \leqq \max\left(S C \frac{a_{\nu,n}}{b_{j,K}}, \frac{1}{R}\frac{a_{\nu,n_0}}{b_{j,\mu}}\right).$$

The next lemma serves merely for simplification:

1.3. Lemma. *If* $(\lambda(A), \lambda(B))$ *satisfy* $(S^*)_0$ *then we may without restriction of generality assume that there is a sequence* $n_0(k)$ *in* $\mathbb{N}$ *such that:*

$\forall k, m, R > 0 \quad \exists n, S \quad \forall \nu, j$:

$$\frac{a_{\nu,m}}{b_{j,k}} \leqq \max\left(S\frac{a_{\nu,n}}{b_{j,k+1}}, \frac{1}{R}\frac{a_{\nu,n_0(k-1)}}{b_{j,k-1}}\right).$$

Proof. We can determine a function $g\colon \mathbb{N} \to \mathbb{N}$, $g(\mu) > \mu$, such that for any μ the choice of $k = g(\mu)$ is in accordance with (S^*). We put $h(1) = 1$, $h(k+1) = g(h(k))$ and replace the matrix $B = (b_{j,k})_{j,k}$ by the equivalent matrix $\tilde{B} = (b_{j,h(k)})_{j,k}$.

1.4. Lemma. *If* $(\lambda(A), \lambda(B))$ *satisfy the condition in* 1.3, *then for any sequences* $(m(k))_k$, $(C_k)_k$ *there exists sequences* $(n(k))_k$, $(S(k))_k$ *such that* $n(k)$ *is strictly increasing and for all* k, ν, j *we have:*

(i) $C_k a_{\nu,m(k)} \leqq S_k a_{\nu,n(k)}$.

(ii) $$\frac{S_k a_{\nu,n(k)}}{b_{j,k}} \leqq \max\left(\frac{S_{k+1} a_{\nu,n(k+1)}}{b_{j,k+1}}, 2^{-k}\frac{S_{k-1} a_{\nu,n(k-1)}}{b_{j,k-1}}\right).$$

Proof. We determine $n(k)$, S_k inductively. We put $n(1) = \max(m(1), n_0(1))$, $S_1 = C_1$ and $n(2) = \max(m(2), n_0(2))$, $S_2 = C_2$. Let $n(1), \ldots, n(k)$ and $S_1, \ldots, S_k$ be determined, and assume (i'): $n(j) \geqq \max(m(j), n_0(j))$, $S_j \geqq C_j$ for $j = 1, \ldots, k$. We apply Lemma 1.3 to $k, m = n(k)$, $R = 2^k\frac{S_k}{S_{k-1}}$ and obtain $n = \tilde{n}(k+1)$, $S = \tilde{S}_{k+1}$, such that

(1) $$\frac{a_{\nu,n(k)}}{b_{j,k}} \leqq \max\left(\tilde{S}_{k+1} \frac{a_{\nu,\tilde{n}(k+1)}}{b_{j,k+1}}, 2^{-k} \frac{S_{k-1}}{S_k} \frac{a_{\nu,n_0(k-1)}}{b_{j,k-1}}\right).$$

We put $n(k+1) = \max(\tilde{n}(k+1), \quad m(k+1), \quad n_0(k+1), \quad n(k)+1)$. $S_{k+1} = \max(C_{k+1}, S_k \tilde{S}_{k+1})$. Hence we have (i′) for $k+1$. From (1) and (i′) we obtain (ii). (i′) for k implies (i).

We can now prove the sufficiency part of 1.1 which is the main result of this section.

1.5. Proposition. *If* $(\lambda(A), \lambda(B))$ *satisfies* (S^*) *then* $\operatorname{Ext}^1(\lambda(A), \lambda(B)) = 0$.

Proof. If $\lambda(A) \cong l^1$, then we are ready, since l^1 is projective. Hence, according to Lemma 1.2 we may assume that $(\lambda(A), \lambda(B))$ satisfy $(S^*)_0$.

We use the representation of $\operatorname{Ext}^1(\lambda(A), \lambda(B))$ explained in the preliminary section. Let $A_k \in L(\lambda(A), l^1(b_k))$, $k = 1, 2, \ldots$. For every k we have C_k, $m(k)$ such that

$$\|A_k x\|_k \leqq C_k \|x\|_{m(k)}$$

for all $x \in \lambda(A)$.

We choose sequences according to Lemma 1.4. Changing A into an equivalent matrix (namely $(S_k a_{j,n(k)})_{j,k}$) we may assume:

(i) $\|A_k x\|_k \leqq \|x\|_k$.

(ii) $\dfrac{a_{\nu,k}}{b_{j,k}} \leqq \max\left(\dfrac{a_{\nu,k+1}}{b_{j,k+1}}, 2^{-k} \dfrac{a_{\nu,k-1}}{b_{j,k-1}}\right)$ for all ν, j.

We claim that:

(1) $\dfrac{a_{\nu,k}}{b_{j,k}} \leqq \dfrac{a_{\nu,k+1}}{b_{j,k+1}}, \quad j \in J_k$ implies $\dfrac{a_{\nu,l}}{b_{j,l}} \leqq \dfrac{a_{\nu,l+1}}{a_{j,l+1}}, \quad j \in J_l$ for all $l \geqq k$

(2) $\dfrac{a_{\nu,k}}{b_{j,k}} \leqq 2^{-k} \dfrac{a_{\nu,k-1}}{b_{j,k-1}}, \quad 2 \leqq l \leqq k, \quad j \in J_l$ implies $\dfrac{a_{\nu,l}}{b_{j,l}} \leqq 2^{-l} \dfrac{a_{\nu,l-1}}{b_{j,l-1}}$.

We have to prove this only for $l = k+1$ (resp. $l = k-1$). Then it follows immediately from (ii) applied to $k+1$ (resp. $k-1$) instead of k, since by assumption the max is attained at the first (resp. second) term.

For A_k we have a matrix representation

$$A_k x = \Big(\sum_\nu t_{j,\nu}^{(k)} x_\nu\Big)_{j \in J_k} \quad \text{for } x = (x_1, x_2, \ldots) \in \lambda(A).$$

Since by (i) ($e_\nu = \nu$-th unit vector):

$$\sum_{j \in J_k} |t_{j,\nu}^{(k)}| b_{j,k} = \|A_k e_\nu\|_k \leqq \|e_\nu\|_k = a_{\nu,k}$$

we can represent the matrix in the form:

$$t_{j,\nu}^{(k)} = \lambda_{j,\nu}^{(k)} \frac{a_{\nu,k}}{b_{j,k}} \quad \text{for } j \in J_k,$$

where

$$\sum_j |t_{j,\nu}^{(k)}| \leqq 1.$$

We put $t_{j,\nu}^{(k)} = \lambda_{j,\nu}^{(k)} = 0$ for $j \notin J_k$.

We define

$$t_{j,\nu}^{(k),+} = \begin{cases} t_{j,\nu}^{(k)} & \text{for } j \in J_k, \quad \dfrac{a_{\nu,k}}{b_{j,k}} \leqq \dfrac{a_{\nu,k+1}}{b_{j,k+1}} \\ 0 & \text{otherwise} \end{cases}$$

$$t_{j,\nu}^{(k),-} = t_{j,\nu}^{(k)} - t_{j,\nu}^{(k),+}.$$

For $x = (x_1, x_2, \ldots) \in \lambda(A)$ we put:

$$A_k^+ x = \Big(\sum_\nu t_{j,\nu}^{(k),+} x_\nu\Big)_{j \in \mathbb{N}}$$

$$A_k^- x = \Big(\sum_\nu t_{j,\nu}^{(k),-} x_\nu\Big)_{j \in \mathbb{N}}.$$

We obtain for $l \geqq k$

$$\begin{aligned} \|A_k^+ x\|_1 &= \sum_j b_{j,l} \Big|\sum_\nu t_{j,\nu}^{(k),+} x_\nu\Big| \\ &\leqq \sum_{j,\nu}{}^{+} b_{j,l} |\lambda_{j,\nu}^{(k)}| \frac{a_{\nu,k}}{b_{j,k}} |x_\nu| \\ &\leqq \sum_{j,\nu} b_{j,l} |\lambda_{j,\nu}^{(k)}| \frac{a_{\nu,l}}{b_{j,l}} |x_\nu| \\ &\leqq \sum_\nu a_{\nu,l} |x_\nu| = \|x\|_l \end{aligned}$$

and for $l < k$

$$\begin{aligned} \|A_k^- x\|_l &= \sum_{j \in J_l} b_{j,l} \Big|\sum_\nu t_{j,\nu}^{(k),-} x_\nu\Big| \\ &\leqq \sum_{\nu, j \in J_l}{}^{-} b_{j,l} |\lambda_{j,\nu}^{(k)}| \frac{a_{\nu,k}}{b_{j,k}} |x_\nu| \\ &\leqq 2^{-k} \sum_{\nu, j \in J_l} b_{j,l} |\lambda_{j,\nu}^{(k)}| \frac{a_{\nu,l}}{b_{j,l}} |x_\nu| \\ &\leqq 2^{-k} \|x\|_l. \end{aligned}$$

In both cases the second estimate comes from (1) or (2) respectively (notice that (2) gives even the factor $2^{-k-(k-1)-(k-2)-\cdots}$). Σ^+, Σ^- indicates summation over the non zero terms of $t_{j,\nu}^{(k),+}$, $t_{j,\nu}^{(k),-}$.

In particular we obtain that $A_k^+ \in L(\lambda(A), \lambda(B))$ for all $k \in \mathbb{N}$. Obviously A_k^- defines a map in $L(\lambda(A), l^1(b_k))$ for all $k \in \mathbb{N}$. We put for $x \in \lambda(A)$:

$$B_k x = \sum_{l=1}^{k-1} A_l^+ x - \sum_{\nu=k}^{\infty} A_\nu^- x.$$

This defines by restriction to J_k a map $B_k \in L(\lambda(A), l^1(b_k))$. The series converges because of the previous estimate.

We obtain
$$B_{k+1}x|_{J_k}-B_kx=(A_k^+x+A_k^-x)_{J_k}=A_kx.$$

From the proof of 1.5 we draw the following:

1.6. Corollary. *If* (S^*) *is satisfied, then for every equicontinuous set* $M_1\subset L(\lambda(A),\prod_k l^1(b_k))=\prod_k L(\lambda(A),l^1(b_k))$ *there is an equicontinuous set* $M_2\subset L(\lambda(A),\prod_k l^1(b_k))=\prod_k L(\lambda(A),l^1(b_k))$ *such that for any* $(A_k)_k\in M_1$ *there is* $(B_k)_k\in M_2$ *with* $B_{k+1}x|_{J_k}-B_kx=A_kx$ *for all* k.

The necessity part of Theorem 1.1 is contained in [14], Theorem 3.9. Since, however, in the case of Köthe spaces there is an elementary proof, we include it here.

1.7. Proposition. If $\mathrm{Ext}^1(\lambda(A),\lambda(B))=0$ *then* $(\lambda(A),\lambda(B))$ *satisfy* (S^*).

Proof. If $\lambda(A)$ does not have a continuous norm, then it contains ω as a complemented subspace, hence $\mathrm{Ext}^1(\omega,\lambda(B))=0$. From [13], Lemma 3.2 we then know, that the $\lambda(B)/\ker\|\ \|$ is a Banach space for any continuous seminorm $\|\ \|$ on $\lambda(B)$. This means that eiter $\lambda(B)\cong l^1$ or $\lambda(B)\cong(l^1)^{\mathbb{N}}$. In both cases (S^*) is trivially satisfied.

So we assume now that $\lambda(A)$ admits a continuous norm. In this case we may assume that $a_{\nu,m}>0$ for all $\nu,m\in\mathbb{N}$.

Let (S^*) not be satisfied. This means:

$$\exists\mu\quad\forall n_0,k\quad\exists K,m\quad\forall n,S\quad\exists\nu,j:$$
$$\frac{a_{\nu,m}}{b_{j,k}}>S\max\left(\frac{a_{\nu,n}}{b_{j,K}},\frac{a_{\nu,n_0}}{b_{j,\mu}}\right).$$

We may assume $\mu=1$. We apply the above to k and $n_0=k-1$. Then we obtain sequences $K(k)$, $m_0(k)$, which may be assumed strictly increasing, such that

(1) $\quad\forall k,n,S\quad\exists\nu,j:$
$$\frac{a_{\nu,m_0(k)}}{b_{j,k}}>S\max\left(\frac{a_{\nu,n}}{b_{j,K(k)}},\frac{a_{\nu,k-1}}{b_{j,1}}\right).$$

By an equivalent change of matrices we may assume $m_0(k)=k$, $K(k)=k+1$. By the assumption on the matrix $(a_{\nu,k})$ a $j\in\mathbb{N}$ chosen according to (1) is automatically even in J_1.

So we can find double indexed sequences $\nu(k,n)$ in $\mathbb{N}$, $j(k,n)$ in J_1 such that for $\nu=\nu(k,n)$, $j=j(k,n)$:
$$\frac{a_{\nu,k}}{b_{j,k}}>2^n\cdot\max\left(\frac{a_{\nu,n}}{b_{j,k+1}},\frac{a_{\nu,k-1}}{b_{j,1}}\right).$$

By choosing the sequences inductively in k for fixed k and by applying (1) in each step with large enough $S\geqq 2^n$ we can achieve that $\nu(k,n_1)\neq\nu(k,n_2)$ or $j(k,n_1)\neq j(k,n_2)$ for $n_1\neq n_2$.

We put

$$(A_k x)_j = \frac{1}{b_{j,k}} \sum_{\substack{n \\ j=j(k,n)}} a_{v(k,n),k} x_{v(k,n)}$$

for $j \in J_k$. An empty sum is counted as 0. Obviously this defines a map $A_k \in L(\lambda(A), l^1(b_k))$. Assume we have maps $B_k \in L(\lambda(A), l^1(b_k))$ such that

$$B_{k+1} x|_{J_k} - B_k = A_k$$

for all $k \in \mathbb{N}$.

B_k corresponds to a matrix $(t_{j,v}^{(k)})_{j,v}$ (see proof of 1.5) with

$$|t_{j,v}^{(k)}| \leqq C_k \frac{a_{v,m(k)}}{b_{j,k}}$$

for all $v \in \mathbb{N}$, $j \in J_k$ with appropriate C_k, $m(k)$.

We fix $v = v(k,n)$, $j = j(k,n) \in J_1$. For $1 \leqq l \leqq k$ we have

$$\varepsilon_l \frac{a_{v,l}}{b_{j,l}} = t_{j,v}^{(l+1)} - t_{j,v}^{(l)},$$

where $\varepsilon_l = 1$ or $\varepsilon_l = 0$ and $\varepsilon_k = 1$. Addition yields:

$$\sum_{l=1}^{k} \varepsilon_l \frac{a_{v,l}}{b_{j,l}} = t_{j,v}^{(k+1)} - t_{j,v}^{(1)}.$$

If moreover $k > m(1)$ we obtain

$$\begin{aligned} \frac{a_{v,k}}{b_{j,k}} &\leqq \sum_{l=1}^{k-1} \frac{a_{v,l}}{b_{j,l}} + C_{k+1} \frac{a_{v,m(k+1)}}{b_{j,k+1}} + C_1 \frac{a_{v,m(1)}}{b_{j,1}} \\ &\leqq C \max\left(\frac{a_{v,m(k+1)}}{b_{j,k+1}}, \frac{a_{v,k-1}}{b_{j,1}}\right) \end{aligned}$$

with some $C = C(k)$. For $2^n \geqq \max(C, m(k+1))$ this yields a contradiction.

2. It is a surprising fact that the condition (S^*), which we found as solution of the "splitting problem" resembles very much to a condition which Grothendieck obtained, when he investigated topological properties of tensor products of (F)- and (DF)-spaces (see [3], II, §4). In fact, we will show in §3 directly, that they are equivalent. The connection is cleared up in [15] in an abstract context. Motivated by all this we will now turn to an investigation of the topological properties of the space $L_b(\lambda(A), \lambda(B))$. We will prove the following theorem which is an analogue to Grothendieck [3], II, §4, n°3, Theorem 15. The proof however and the condition are much more transparent. We assume $\lambda(A)$ to be a Schwartz-space.

2.1. Theorem. *The following are equivalent:*

(1) *$(\lambda(A), \lambda(B))$ satisfy condition (S^*)*

(2) *$L_b(\lambda(A), \lambda(B))$ is bornological*

(3) *$L_b(\lambda(A), \lambda(B))$ is barrelled (quasi-barrelled)*

(4) *The strong dual of $L_b(\lambda(A), \lambda(B))$ is sequentially complete.*

Proof. First of all we note that for any complete locally convex space X we have the following implications

$$X \text{ bornological} \Rightarrow X \text{ barrelled} \Rightarrow X \text{ quasi-barrelled} \Rightarrow X'_b \text{ sequentially complete.}$$

Hence we only have to show (1) ⇒ (2), (4) ⇒ (1). This is the content of the following two propositions. For the first one cf. [15].

2.2. Proposition. *If* $(\lambda(A), \lambda(B))$ *satisfy* (S^*), *then* $X := L_b(\lambda(A), \lambda(B))$ *is bornological.*

Proof. If $\lambda(A) \cong l^1$ then we are ready since then X is a Fréchet space. Hence, according to Lemma 1.2 we may assume that $(\lambda(A), \lambda(B))$ satisfy $(S^*)_0$.

Let $M \subset X$ be an absolutely convex set which absorbs all bounded (= equicontinuous) subsets in $X = L_b(\lambda(A), \lambda(B))$. We claim that there is a k such that M absorbs all sets $B_{k,m} := \{\varphi \in X: \|\varphi x\|_k \leqq \|x\|_m\}$ for $m = 1, 2, \ldots$.

Assume, that this is not the case. Then there is a sequence $m(k)$ such that M does not absorb $B_{k,m(k)}$ for all k. We may assume (by changing A and B into equivalent matrices) that $m(k) = k$ and A and B satisfy condition (ii) in the proof of Proposition 1.5. Then the construction in the proof of Proposition 1.5 tells us that we can split up any $\varphi \in B_{k,k}$ as $\varphi = \varphi^+ + \varphi^-$ where $\varphi^+ \in B_{l,l}$ for all $l \geqq k$, $\varphi^- \in B_{l,l}$ for all $l < k$. Hence

$$B_{k,k} \subset \bigcap_{l \geqq k} B_{l,l} + \bigcap_{l < k} B_{l,l}$$

Since the first set on the right hand is bounded, M does not absorb $\bigcap_{l<k} B_{l,l}$ for all k. Choose $\varphi_k \in \bigcap_{l<k} B_{l,l}$, $\varphi_k \notin kM$ for all k. Then $\{\varphi_k: k = 1, 2, \ldots\}$ is equicontinuous, hence absorbed by M, which is a contradiction.

Put $\tilde{M} = \overline{\bigcup_m m\rho_k(M \cap B_{k,m})}$, where ρ_k is the canonical map $X \to L_b(\lambda,(A), l^1(b_k))$. Then $\tilde{M}$ absorbs all bounded sets in $L_b(\lambda(A), l^1(b_k))$. Since $\lambda(A)$ is a Schwartz-space $L_b(\lambda(A), l^1(b_k))$ is separable. Moreover it is isomorphic to the dual of the Fréchet space $\lambda(A) \hat{\otimes}_\pi c_0$ (or even $\lambda(A)^m$ for some m). Hence it is bornological (see [6], p. 403), and therefore $\tilde{M}$ is a neighbourhood of zero. So $M = \rho_k^{-1} \tilde{M}$ is a neighbourhood of zero.

The converse direction is an adaptation of the proof of [13], Proposition 4.4 to the present situation, where we can avoid the complicated construction of a biorthogonal sequence. Again we put $X = L_b(\lambda(A), \lambda(B))$.

2.3. Proposition. *If* X'_b *is sequentially complete then* $(\lambda(A), \lambda(B))$ *satisfy* (S^*).

Proof. If $\lambda(A)$ does not have a continuous norm, then it contains ω as a complemented subspace and by assumption we know that $(L_b(\omega, \lambda(B)))'_b$ is sequentially complete. Let $\| \ \|$ be a continuous seminorm on $\lambda(B)$ and put $E = \lambda(B)/\ker \| \ \|$. Then $L_b(\omega, E) \cong \bigoplus_{\mathbb{N}} E$ algebraically. The topology is given by the norms $\|\bigoplus_k x_k\|_n = \sup_k \|x_k\|_n$, where $\| \ \|_n$ is a fundamental system of norms on E. The strong dual of this space is again sequentially complete. This implies that

E is a Banach space. Concluding as in the proof of 1.7 we see that (S^*) is satisfied.

So we may assume that $\lambda(A)$ has a continuous norm and therefore that $a_{\nu,m}>0$ for all ν, m. Assuming now that (S^*) is not satisfied we proceed as in the proof of Proposition 1.7 and choose $\nu(k,n)\in\mathbb{N}$, $j(k,n)\in J_1$ such that for $\nu=\nu(k,n)$, $j=j(k,n)$:

$$\frac{a_{\nu,k}}{b_{j,k}}>2^n \max\left(\frac{a_{\nu,n}}{b_{j,k+1}}, \frac{a_{\nu,k-1}}{b_{j,1}}\right).$$

We consider the series:

$$\sum_{\substack{k,n\\ n\geqq k}} n\frac{b_{j(k,n)}}{a_{\nu(k,n)}} f_{j(k,n)}\otimes e_{\nu(k,n)}$$

where f_j (resp. e_ν) are the canonical unit vectors in $\lambda(B)'$ (resp. $\lambda(A)$). We have to show that it is a Cauchy series in X'_b.

Let $B=\{\varphi\in X\colon \|\varphi x\|_k\leqq C_k\|x\|_{m(k)}$ for all $k\}$ be a bounded set in X. $\varphi\in B$ can be represented by a matrix $(t_{j,\nu})$ with $|t_{j,\nu}|b_{j,k}\leqq C_k a_{\nu,m(k)}$ for all j,ν,k. We obtain with $l=m(1)$ and $m=:\max(m(l+1),l)$:

$$\begin{aligned}
&\sum_{\substack{k,n\\ n\geqq k}} n\frac{b_{j(k,n),k}}{a_{\nu(k,n),k}}|\langle f_{j(k,n)}, \varphi\, e_{\nu(k,n)}\rangle|\\
&\quad=\sum_{\substack{k,n\\ n\geqq k}} n\frac{b_{j(k,n),k}}{a_{\nu(k,n),k}}|t_{j(k,n),\nu(k,n)}|\\
&\quad\leqq\sum_{n=1}^{\infty}\sum_{k=l+1}^{n} n\frac{b_{j(k,n),k}}{a_{\nu(k,n),k}} C_1\frac{a_{\nu(k,n),k-1}}{b_{j(k,n),1}}\\
&\quad\quad+\sum_{k=1}^{l}\sum_{n=m}^{\infty} n\frac{b_{j(k,n),k}}{a_{\nu(k,n),k}} C_{l+1}\frac{a_{\nu(k,n),n}}{b_{j(k,n),k+1}}\\
&\quad\quad+\text{finite sum}\\
&\quad\leqq C_1\sum_{n=1}^{\infty} n^2 2^{-n}+C_{l+1}\, l\sum_{n=1}^{\infty} n2^{-n}+\dots.
\end{aligned}$$

Since the first estimate was termwise and the sums converge, the first series converges uniformly for $\varphi\in B$.

Assume that the series converges to some element $y\in X'$. Let $\tilde f_\nu$ (resp. $\tilde e_j$) be the canonical unit vectors in $\lambda(A)'$ (resp. $\lambda(B)$).

Consider

$$\varphi_{k,n}=\frac{a_{\nu(k,n),k}}{b_{j(k,n),k}}\tilde f_{\nu(k,n)}\otimes\tilde e_{j(k,n)}\in X.$$

Let $U=\{\varphi\colon \varphi B\subset U_k\}$, B bounded in $\lambda(A)$, $U_k=\{x\in\lambda(B)\colon \|x\|_k\leqq 1\}$ be a neighbourhood of zero in X. Then $\{\varphi_{k,n}\colon n=1,2,\dots\}\subset\lambda U$ for some $\lambda>0$. But $\langle y,\varphi_{k,n}\rangle=n$. Hence y is not bounded on any neighbourhood of zero in X, which is a contradiction.

Remark. Let X be any locally convex space with $\lambda(A)'_b\otimes_\pi\lambda(B)\overset{\imath_1}{\hookrightarrow}X\overset{\imath_2}{\hookrightarrow}L_b(\lambda(A),\lambda(B))$, where $\imath_1$, $\imath_2$ are continuous maps and $\imath_2\circ\imath_1$ is the natural inclusion. The previous proof shows that if X'_b is sequentially complete, then $(\lambda(A), \lambda(B))$ satisfy (S^*).

3. We want now to compare condition (S^*) directly with Grothendieck's condition. We first describe his result. We put $P=\lambda^\times(A)\hat{\otimes}_\pi\lambda(B)$. Then P can be considered as a space of infinite matrices. The dual P' can be identified with the space of all matrices belonging to bounded linear maps from $\lambda(B)$ to $\lambda^\infty(A)$. By P^* we denote the Köthe dual of P, i.e. the space of all matrices $v=(v_{i,j})_{i,j}$ such that $\sum_{i,j}|u_{i,j}v_{i,j}|<+\infty$ for all $u=(u_{i,j})_{i,j}\in P$. $\lambda^\infty_+(A)$ denotes the set of all nonnegative real sequences in $\lambda^\infty(A)$. We can put the essential part (with respect to our considerations) of [3], II, §4, n° 3, Theorem 15 into the following form:

3.1. Theorem (Grothendieck). *The following are equivalent*

(a) *P is bornological*

(b) *P is barrelled*

(c) $P'=P^*$

(d) $\forall n_0 \quad \exists m_0 \quad \forall m, \lambda\in\lambda^\infty_+(A) \quad \exists R>0 \quad \forall i,j$:

$$\frac{a_{i,m}}{b_{j,m_0}}\leqq R\max\left(\frac{1}{\lambda_i b_{j,m}}, \frac{a_{i,m_0}}{b_{j,n_0}}\right).$$

We will now show:

3.2 Proposition. *Condition* 3.1 (d) *and* (S^*) *are equivalent.*

Proof. Since obviously we can write (S^*) in the following form

$\forall n_0 \quad \exists m_0\geqq n_0 \quad \forall m \quad \exists n, R>0 \quad \forall \nu, j$:

$$\frac{a_{i,m}}{b_{j,m_0}}\leqq R\max\left(\frac{a_{i,n}}{b_{j,m}}, \frac{a_{i,m_0}}{b_{j,n_0}}\right)$$

it is easy to see that (S^*) implies 3.1 (d).

For the converse direction we first choose n_0, $m_0>n_0$ and assume that there exists i such that $a_{i,m_0}=0$. We choose $m_1\geqq m_0$ such that $a_{i,m_1}>0$ and λ such that $\lambda_i>0$. Then for all $m\geqq m_1$ and $j\in J_{m_0}$ we have $b_{j,m}\leqq\frac{R}{\lambda_i a_{i,m}}b_{j,m_0}$. Consequently $\lambda(B)\cong(l^1)^{\mathbb{N}}$ or $\lambda(B)\cong l^1$ and (S^*) is trivially satisfied.

Hence we assume that $a_{i,m}>0$ for all i and m. We assume $n_0\leqq m_0$ and m fixed in accordance with 3.1 (d).

Assume, that there does not exist n and R according to (S^*). Then we can find sequences $i(n)\in\mathbb{N}$, $j(n)\in J_{m_0}$ such that

$$\frac{a_{i(n),m}}{b_{j(n),m_0}}>n\cdot\max\left(\frac{a_{i(n),n}}{b_{j(n),m}}, \frac{a_{i(n),m_0}}{b_{j(n),n_0}}\right).$$

As in the proofs of 1.7 and 2.3 we may assume that $n\to i(n)$ is injective. We put

$$\lambda_i = \begin{cases} \dfrac{1}{a_{i(n),n}} & \text{for } i = i(n) \\ 0 & \text{otherwise.} \end{cases}$$

Then $\lambda \in \lambda_+^\infty(A)$ yields a contradiction to 3.1 (d).

Now [12], Proposition 7.3 says that $L(\lambda(B), \lambda(A)) = LB(\lambda(B), \lambda(A))$ (at least under reasonable assumptions) should imply Grothendieck's condition, hence (S^*). We give a direct proof. First we restate [12], Satz 1.5 under slightly changed assumptions.

3.3. Theorem. *The following are equivalent:*

(1) $L(\lambda(B), \lambda(A)) = LB(\lambda(B), \lambda(A))$

(2) *for every sequence $K(N)$ there exists L such that for each m we have N_0 and C with*

$$\frac{a_{\nu,m}}{b_{j,L}} \leqq C \max_{N=1,\dots,N_0} \frac{a_{\nu,N}}{b_{j,K(N)}}$$

for all ν, j.

The *proof* of (1) $\Rightarrow$ (2) is exactly as in [12]. For the converse we have to represent the matrix of a map in $L(\lambda(B), \lambda(A))$ in an analogous way as in the proof of 2.3 and then to proceed as in [12].

We obtain by use of this characterization:

3.4. Proposition. *If $L(\lambda(B), \lambda(A)) = LB(\lambda(B), \lambda(A))$ then $(\lambda(A), \lambda(B))$ satisfy (S^*).*

Proof. Assume, that they do not satisfy (S^*). Then we have μ and for every N a $k(N)$ such that for all n and S there exist ν, j with

$$\frac{a_{\nu,K(N)}}{b_{j,N}} > S \max\left(\frac{a_{\nu,n}}{b_{j,K(N)}}, \frac{a_{\nu,N}}{b_{j,\mu}}\right).$$

We may assume $\mu \leqq K(1) \leqq K(2) \leqq \dots$.

We apply condition 3.2 (2) to the sequence $K(N)$. We obtain L such that (for $m = K(L)$) we have N_0 and C with

$$\frac{a_{\nu,K(L)}}{b_{j,L}} \leqq C \max_{N=1,\dots,N_0} \frac{a_{\nu,N}}{b_{j,K(N)}} \leqq C \max\left(\frac{a_{\nu,N_0}}{b_{j,K(L)}}, \frac{a_{\nu,L}}{b_{j,\mu}}\right)$$

for all ν, j. This is a contradiction.

The following condition was used in [1] and [14] as a sufficient condition for $\mathrm{Ext}^1(\lambda(A), \lambda(B)) = 0$.

(S_1^*) $\exists n_0 \ \forall \mu \ \exists k \ \forall K, m \ \exists n, S \ \forall \nu, j$:

$$\frac{a_{\nu,m}}{b_{j,k}} \leqq S \max\left(\frac{a_{\nu,n}}{b_{j,K}}, \frac{a_{\nu,n_0}}{b_{j,\mu}}\right).$$

We recall that $\lambda(B)$ has property $(\bar{\Omega})$ (or is of type (d_2)) if the following holds:

$$\forall p \quad \exists q \quad \forall k \quad \exists C>0 \quad \forall j \in \mathbb{N}: \ b_{j,p} b_{j,k} \leqq C b_{j,q}^2.$$

Any power series space of finite type satisfies $(\bar{\Omega})$. In fact $(\bar{\Omega})$ implies that $\lambda(B)$ is a quotient of such spaces (s. [16]).

We quote from [13] (in a slightly modified form):

3.5 Lemma. *If $\lambda(B)$ has property $(\bar{\Omega})$, then (S_1^*) implies $L(\lambda(B), \lambda(A)) = LB(\lambda(B), \lambda(A))$.*

From [9] (or [4]) we know that there are power series spaces $\Lambda_1(\alpha)$ with $\operatorname{Ext}^1(\Lambda_1(\alpha), \Lambda_1(\alpha)) = 0$. Take e.g. $\alpha_k = 2^{k^2}$. But $(\Lambda_1(\alpha), \Lambda_1(\alpha))$ certainly does not satisfy (S_1^*) because on account of Lemma 3.4 this would imply that the identical map in $\Lambda_1(\alpha)$ is bounded. For a systematic treatment of (S^*) and (S_1^*) in the context of power series and L_f-spaces see [4].

It should finally be remarked that in [8] it is shown that (S_1^*) implies that either $L(\lambda(B), \lambda(A)) = LB(\lambda(B), \lambda(A))$ or $\lambda(A)$ and $\lambda(B)$ have a common step space. If $\lambda(A)$ or $\lambda(B)$ has property (DN), the conditions (S^*) and (S_1^*) coincide. This means that then the situation described in 3.4 is "perturbed" only by spaces $\lambda(A)$ with $\operatorname{Ext}^1(\lambda(A), \lambda(A)) = 0$. So it is interesting to characterize those $\lambda(A)$. In fact, the space (s) is one of them. Such a characterization (under the assumption that $\lambda(A)$ has property (DN)) has been given in [7]. It would be interesting to give a general characterization.

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