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Titel: Local Existence of Weak Solutions to the Quasi-Linear Wave Equation for Large Ini...

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Jahr: 1985

PURL: https://resolver.sub.uni-goettingen.de/purl?266833020_0190|log25

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Math. Z. 190, 249–276 (1985)

Mathematische Zeitschrift
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Local Existence of Weak Solutions to the Quasi-Linear Wave Equation for Large Initial Values

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1. Introduction

We consider the quasilinear hyperbolic initial value problem

$$u_t = f(u)_x, \quad x \in \mathbb{R},\ t \geqq 0, \tag{1.1}$$

$$u(x, 0) = U(x), \tag{1.2}$$

with $u=(u_1, u_2)\colon \mathbb{R} \times \mathbb{R}_0^+ \to \mathbb{R}^2$, $U=(U_1, U_2)\colon \mathbb{R} \to \mathbb{R}^2$, $f(u)=(u_2, \sigma(u_1))$. We assume that $\sigma \in C^2(\mathbb{R}, \mathbb{R})$ satisfies $\sigma'(\tau)>0$ for all $\tau \in \mathbb{R}$, which implies that the system is hyperbolic. This system is genuinly nonlinear if $\sigma''(\tau) \neq 0$ for all $\tau \in \mathbb{R}$, and we assume without restriction of generality that $\sigma''(\tau)>0$. For $i=1, 2$ the i-Riemann invariants $R_i\colon \mathbb{R}^2 \to \mathbb{R}$ are defined by

$$R_1(u_1, u_2) = u_2 - \int_0^{u_1} \sigma'(\tau)^{1/2}\, d\tau, \tag{1.3}$$

$$R_2(u_1, u_2) = u_2 + \int_0^{u_1} \sigma'(\tau)^{1/2}\, d\tau, \tag{1.4}$$

and we assume that the following condition holds:

Condition C. Every level curve of R_1 intersects with any level curve of R_2. Under these assumptions we prove in this paper the following result:

Theorem 1.1. *If the initial data U have compact support and are of bounded variation, then there exists $T>0$ such that a weak solution u of (1.1), (1.2) exists in the domain $\mathbb{R} \times [0, T)$ with*

$$\operatorname{var} u(\circ, t) \leqq C, \quad 0 \leqq t < T.$$

This weak solution satisfies the entropy condition.

A bounded measurable function $u: \mathbb{R} \times [0, T) \to \mathbb{R}^2$ is called weak solution of (1.1), (1.2) if

$$\int_0^T \int_{-\infty}^{\infty} (u\phi_t - f(u)\phi_x)\, dx\, dt + \int_{-\infty}^{\infty} U(x)\phi(x, 0)\, dx = 0$$

for all smooth functions ϕ vanishing on the line $t = T$. A function $\eta: \mathbb{R}^2 \to \mathbb{R}$ is called entropy, if there exists a function $q: \mathbb{R}^2 \to \mathbb{R}$ such that for every continuously differentiable solution u of (1.1) the additional conservation law

$$\eta(u(x, t))_t + q(u(x, t))_x = 0$$

holds. We say that a weak solution u of (1.1), (1.2) satisfies the entropy condition, if for every convex entropy the relation

$$\eta_t + q_x \leqq 0 \tag{1.5}$$

holds in the distributional sense, cf. [6]. An example for a convex entropy is the energy density

$$\eta(u_1, u_2) = \tfrac{1}{2}u_2^2 + \int_0^{u_1} \sigma(\tau)\, d\tau, \qquad q(u_1, u_2) = -u_2\, \sigma(u_1).$$

Condition C. is necessary and sufficient for the result in Theorem 1.1 to hold. For, if this condition is not satisfied, then it is possible to construct initial data U for which a local solution satisfying the entropy condition does not exist.

Glimm and Lax proved in [4] that a global weak solution of (1.1), (1.2) exists under essentially the above assumptions for σ, provided that U has bounded variation and, in addition, small oscillation. This means that $|U(x+) - U(x-)|$ must be less than a given small constant for all $x \in \mathbb{R}$. Recently DiPerna proved in [2] that to all bounded initial data a global weak solution exists if $\sigma'' < 0$ for $x < 0$ and $\sigma'' > 0$ for $x > 0$. The proof uses the results on compensated compactness in [7]. This method can be applied if a priori L_∞-estimates are known, and for the systems considered by DiPerna these estimates follow from the existence of bounded invariant subregions. However, under the above assumptions the system (1.1) does not have bounded invariant subregions, in general. We therefore cannot show in this simple way that a priori estimates hold.

Since the proof of Theorem 1.1 contains many technical details we give in the following a detailed description of the ideas. This introduction is intended to make the paper easily accessible. The reader is advised after having read this introduction to directly study the definitions at the end of Sect. 2 and the proof of Lemma 3.4, which is the main part of the paper.

In Sect. 2 a sequence of approximate solutions $\{u^{(n)}\}$ to problem (1.1), (1.2) is constructed. To this end a sequence $\{U^{(n)}\}$ of step functions $U^{(n)}: \mathbb{R} \to \mathbb{R}$ with $|U^{(n)} - U|_\infty \to 0$ and $\operatorname{var} U^{(n)} \to \operatorname{var} U$ is chosen. $u^{(n)}$ is defined as solution of the Riemann initial value problem to the initial values $U^{(n)}$, and we choose for $u^{(n)}$ a solution that does not contain rarefaction waves, but is a piecewise constant function. Rarefaction waves are approximated by a fan of discontinuities with

small magnitude. These discontinuities violate the entropy condition, but the construction is made such that the magnitude of these discontinuities decreases if n increases, so that the entropy condition is satisfied in the limit. It will be seen that the construction is similar to the one given by DiPerna in [1]. If two discontinuities intersect, then the solution is continued by again solving the Riemann initial value problem. After repeated solution of Riemann problems the function $u^{(n)}$ is defined on a strip $\mathbb{R} \times [0, T_n)$. It is discontinuous along a net of line segments $x = \lambda(t)$. The set of all these line segments is denoted by L, the family of line segments with $\partial_t \lambda < 0$ is denoted by L_1, the family with $\partial_t \lambda > 0$ is denoted by L_2. By small shifts it can be arranged that $u^{(n)}$ only contains binary interactions of discontinuities, and it is shown that the assumptions for σ imply that the construction can be made such that only two types of interactions occur. Either two discontinuities of different families interact, in which case exactly one discontinuity of family 1 and one discontinuity of family 2 are issued from the point of interaction, or two discontinuities of the same family interact, in which case exactly one discontinuity of the family of the interacting discontinuities and one or more discontinuities of the opposite family are issued from the point of interaction. For these interactions the well known interaction estimates hold, cf. [3]. These estimates are collected in the appendix in Lemmas A3 and A4. For example if $\lambda_1^- \in L_1$, $\lambda_2^- \in L_2$ interact, and if $\lambda_1 \in L_1$, $\lambda_2 \in L_2$ are issued from the point of interaction, then

$$\theta(\lambda_i) \leqq (1 + V\theta(\lambda_j^-))\, \theta(\lambda_i^-), \qquad i \neq j, \tag{1.6}$$

where $\theta(\lambda)$ is the strength of the discontinuity along λ and V is a constant. Essentially, $\theta(\lambda)$ is defined to be the height of the jump. Together with other definitions the precise definition of θ is given at the end of Sect. 2.

In Sect. 3 the solution u is constructed as limit of a subsequence of $\{u^{(n)}\}$. To find a convergent subsequence, it must be shown that the functions $u^{(n)}$ are uniformly of bounded variation, and that $T > 0$ exists with $T_n \geqq T$ for all n. It is shown in the proof of Lemma 3.3 that T exists if the variation of $u^{(n)}$ is bounded uniformly with respect to n, since the number of discontinuities can be estimated by the variation. So the main part of the paper is centered around the estimates (3.21)–(3.25) in the proof of Lemma 3.4, where the bound for the variation is derived. We explain here the idea of the proof under the simplifying assumption that only interactions of discontinuities of different families occur. Note that the properties of interactions of type 1 described above imply that each discontinuity along a line segment can be continued after interaction in a unique way by a discontinuity along a line segment of the same family. Therefore we can subdivide L_i into pairwise disjoint sets $\alpha = \{\alpha_m\}_{m=0,1,\ldots}$ defined by the properties that $\alpha_0 \in L_i$ issues from a jump of $U^{(n)}$ on the line $t = 0$, and that the discontinuity α_{m+1} continues α_m. Let Γ_i be the set of all these sets $\{\alpha_m\}$, and let $\Gamma = \Gamma_1 \cup \Gamma_2$. The set $F(L)$ of all functions $p: L \to \mathbb{R}_0^+$ is a complete metric space with the metric

$$d(p, q) = \sum_{\alpha \in \Gamma} \sup_{\alpha_m \in \alpha} |p(\alpha_m) - q(\alpha_m)|.$$

The function θ belongs to $F(L)$, and since the variation of $u^{(n)}$ is bounded by $d(\theta, 0)$, it suffices to show that there exists a constant K, which can be chosen independently of n, such that

$$d(\theta, 0) \leqq K \tag{1.7}$$

for all n. Note that the above definitions depend on n.

To prove (1.7), define an operator $B: F(L) \to F(L)$ by

$$[Bp](\alpha_0) = \theta(\alpha_0) \tag{1.8}$$

if α_0 issues from the line $t=0$, and by

$$[Bp](\alpha_{m+1}) = (1 + Vp(\lambda))\, Bp(\alpha_m), \tag{1.9}$$

if $\lambda \in L_j$ interacts with $\alpha_m \in L_i$, $i \neq j$, and if $\alpha_{m+1} \in L_i$ continues the discontinuity α_m. Here V is the constant of (1.6). This definition of B and (1.6) imply that $d(\theta, 0) \leqq d(\omega, 0)$ for every fixed point ω of B. Therefore it suffices to show that for every n the operator B possesses a fixed point ω with $d(\omega, 0) \leqq K$. Such a fixed point is obtained as follows. Formula (1.9) implies for every $\alpha = \{\alpha_m\} \in \Gamma$ that

$$Bp(\alpha_m) \leqq \Pi (1 + Vp(\lambda))\, Bp(\alpha_0) \leqq e^{V \Sigma p(\lambda)}\, Bp(\alpha_0),$$

where the summation and the product are over all $\lambda \in L$ which interact with α in between α_0 and α_m. Since every $\beta \in \Gamma$ contains at most one such λ, and hence interacts with α at most once, we obtain from this inequality and from (1.8) that

$$\sup_{\alpha_m \in \alpha} Bp(\alpha_m) \leqq \exp\Big[V \sum_{\beta \in \Gamma} \sup_{\beta_l \in \beta} p(\beta_l)\Big]\, Bp(\alpha_0) = e^{Vd(p,\,0)}\, \theta(\alpha_0).$$

Summation over $\alpha \in \Gamma$ yields

$$d(Bp, 0) \leqq e^{Vd(p,\,0)} \sum_{\alpha \in \Gamma} \theta(\alpha_0) \leqq e^{Vd(p,\,0)}\, V^* \operatorname{var}(U^{(n)}),$$

since the sum over the strength of all discontinuities issuing from the line $t=0$ is bounded by $V^* \operatorname{var} U^{(n)}$ with a suitable constant V^*. This inequality shows that if K and $\operatorname{var}(U^{(n)})$ are sufficiently small, then B maps the set of all $p \in F(L)$ with $d(p, 0) \leqq K$ into itself. In the same way it is shown that B is a contraction mapping on this set, and therefore possesses a fixed point ω within this set. Since $\operatorname{var}(U^{(n)})$ tends to $\operatorname{var}(U)$, it is clear that K can be chosen independent of n if $\operatorname{var}(U)$ is sufficiently small.

Of course, this argument is not sufficient for the proof of Theorem 1.1, since we must assume that $\operatorname{var}(U)$ is small. We therefore apply these arguments locally and only consider the discontinuities issuing from a small interval Ω on the line $t=0$. Also this modification is not quite sufficient, since the variation does not tend to zero if the interval shrinks, but we have the following. For every $x_0 \in \mathbb{R}$ and every $\varepsilon > 0$ there exists $\delta > 0$ with

$$\operatorname{var}(U \,|\, [x_0 - \delta, x_0)) + \operatorname{var}(U \,|\, (x_0, x_0 + \delta]) < \varepsilon.$$

Thus, we have to cope with the situation that all the discontinuities issuing from the interval $[x_0-\delta, x_0+\delta]$ are weak with the exception of one or several discontinuities issuing from the point $(x_0, 0)$. The preceding ideas are not sufficient to deal with this situation, and we need as an essential new tool the interaction estimate (A8) saying that there exists a constant $V_1<1$ such that if two discontinuities λ_1^-, λ_2^- from the same family L_i with $|\partial_t\lambda_1^-|>|\partial_t\lambda_2^-|$ interact, then

$$\sum_{l=1}^{m}\theta(\lambda^l)\leqq V_1\,\theta(\lambda_1^-),$$

where $\lambda^1,\ldots,\lambda^m\in L_j$ are the discontinuities from the opposite family issuing from the point of interaction. Vividly we can say that reflection of discontinuities is connected with attenuation. The fact that also interactions of type 2 appear makes it necessary to define the operator B on a more general space $F(\Omega)$ and to use besides (1.6) also inequalities (A4) and (A8) for its definition.

In Sect. 4 it is finally shown that the entropy condition is satisfied.

2. The Approximate Solutions

We construct the approximate solutions by solving the Riemann initial value problem by piecewise constant functions. A piecewise constant weak solution of (1.1) must satisfy the Rankine-Hugoniot relation

$$s(u-v)=-(f(u)-f(v)), \tag{2.1}$$

where $u=(u_1, u_2)$, $v=(v_1, v_2)$ are the constant values of the solution on either side of the curve of discontinuity $x=x(t)$, and where $s=dx/dt$ is the speed of the discontinuity. It follows that

$$s=-\frac{u_2-v_2}{u_1-v_1} \tag{2.2}$$

and that $u_2=v_2\pm[(\sigma(u_1)-\sigma(v_1))(u_1-v_1)]^{1/2}$. Define the functions c_{1v}, $c_{2v}\colon \mathbb{R}\to\mathbb{R}$ by

$$c_{1v}(\tau)=\begin{cases}v_2-[(\sigma(\tau)-\sigma(v_1))(\tau-v_1)]^{1/2}, & \tau<v_1\\ v_2+[(\sigma(\tau)-\sigma(v_1))(\tau-v_1)]^{1/2}, & \tau>v_1\end{cases} \tag{2.3a}$$

$$c_{2v}(\tau)=\begin{cases}v_2+[(\sigma(\tau)-\sigma(v_1))(\tau-v_1)]^{1/2}, & \tau<v_1\\ v_2-[(\sigma(\tau)-\sigma(v_1))(\tau-v_1)]^{1/2}, & \tau>v_1.\end{cases} \tag{2.3b}$$

We call the discontinuity a 1-wave, if s is negative, otherwise a 2-wave. Thus, the discontinuity is a 1-wave if u lies on the graph of c_{1v}, and a 2-wave if u lies on the graph of c_{2v}. If v is the state on the left of a 1-wave or the state on the right of a 2-wave, then, with the terminology of Lax [5], the discontinuity is a shock if $u_1>v_1$, otherwise a "rarefaction discontinuity". This follows from Lemma A1(i) in the appendix and from the assumption $\sigma''(\tau)>0$. We now define a weak solution u of the Riemann initial value problem of (1.1) to the

initial values

$$u(x,0)=\begin{cases}u^l, & x<0\\ u^r, & x>0.\end{cases} \tag{2.4}$$

For $i=1,2$, n a nonnegative integer, and $v\in\mathbb{R}^2$ first define the function $c_{iv}^n\colon \mathbb{R}\to\mathbb{R}$ as follows: Let

$$c_{iv}^0=c_{iv}.$$

For $n>0$ let

$$v^0=v \quad \text{and} \quad c_{iv}^n(\tau)=c_{iv}(\tau), \quad v_1\leqq\tau. \tag{2.5}$$

If $v^m\in\mathbb{R}^2$ has been defined, then let

$$c_{iv}^n(\tau)=c_{iv^m}(\tau), \qquad v_1^{m+1}\leqq\tau\leqq v_1^m, \tag{2.6}$$

where $v^{m+1}=(v_1^{m+1}, v_2^{m+1})\in\mathbb{R}^2$ is the point on the graph of c_{iv^m} such that the arclength of this graph in between v^m and v^{m+1} is equal to $1/n$. These points subdivide the graph of c_{iv}^n into pieces of arclength $1/n$. We now define solutions of (1.1), (2.4).

In Lemma A2 it is shown that the graphs of $c_{1u^l}^n$ and $c_{2u^r}^k$ intersect in exactly one point v^+. We define points $w^0,\ldots,w^j$ on these graphs as follows. Let $w^0=u^l$, and if $v^1,\ldots,v^{m-1}$ are the points subdividing the graph of $c_{1u^l}^n$ in between u^l and v^+, let $w^i=v^i$, $i=1,\ldots,m-1$, and $w^m=v^+$. Of course, it might be that $m-1=0$. If $m\geqq 2$ then the definition of v^i implies that $w_1^i<w_1^{i-1}$ for $i=1,\ldots,m$. In the same way we choose $w^{m+1},\ldots,w^{j-1}$ to be the points subdividing graph $c_{2u^r}^k$ in between v^+ and $w^j=u^r$, ordered however such that $w_1^i>w_1^{i-1}$ for $i=m+1,\ldots,j-1$. Again, it might be that such points subdividing the graph do not exist, in which case we have $j=m+1$. Now define the solution of (1.1), (2.4) by

$$u(x,t)=\begin{cases}u^l, & x\leqq s_1 t\\ w^i, & s_i t<x\leqq s_{i+1}t, \quad 1\leqq i\leqq j-1\\ u^r, & s_j t<x,\end{cases} \tag{2.7}$$

where

$$s_i=-\frac{w_2^i-w_2^{i-1}}{w_1^i-w_1^{i-1}}, \quad 1\leqq i\leqq j. \tag{2.8}$$

Lemma A1(i) together with the assumption $\sigma''(\tau)>0$ implies that $s_i<s_{i+1}$. (2.6) and (2.8) imply that u satisfies the Rankine-Hugoniot condition (2.1), and therefore is a weak solution of (1.1).

Definition 2.1. We call the weak solution u of (1.1), (2.4) constructed above resolution of the discontinuity $[u^l, u^r]$ of type (n, k).

Now we are ready to construct the sequence $\{u^{(n)}\}$. Choose a sequence $\{U^{(n)}\}$ of step functions $U^{(n)}\colon \mathbb{R}\to\mathbb{R}^2$, each one having finitely many steps, such that the following properties are satisfied:

$$\operatorname{supp} U^{(n)}\subseteq\{x\in\mathbb{R}\mid \operatorname{dist}(x,\operatorname{supp} U)\leqq 1\} \tag{2.9}$$

$$|U^{(n)}-U|_\infty \to 0 \tag{2.10}$$

$$\operatorname{var} U^{(n)} \to \operatorname{var} U, \tag{2.11}$$

where U denotes the given initial data. We choose $U^{(n)}$ such that all the intervals on which $U^{(n)}$ is constant contain more than one point, which is possible, since we can assume that U is continuous from the left. We next use the following lemma.

Lemma 2.2. *If $h: \mathbb{R} \to \mathbb{R}^2$ is a function of bounded variation then to every $\varepsilon>0$ and every $x \in \mathbb{R}$ there exists $\delta>0$ with* $\operatorname{var}[h|(x-\delta, x)]$, $\operatorname{var}[h|(x, x+\delta)] < \varepsilon$.

The proof is obvious. (2.10) and (2.11) imply that for every interval I we have $\operatorname{var}(U^{(n)}|I) \to \operatorname{var}(U|I)$. From Lemma 2.2 it thus follows that to every $\varepsilon>0$ and to every $x \in \mathbb{R}$ there exists $\delta>0$ with

$$\operatorname{var}[U^{(n)}|(x-\delta, x)], \quad \operatorname{var}[U^{(n)}|(x, x+\delta)] < \varepsilon \tag{2.12}$$

for all n sufficiently large.

Next we define a set $\{t_k\}_{k \geqq 0}$ and simultaneously the function $u^{(n)}$ on the domains $\mathbb{R} \times [t_k, t_{k+1})$ by induction. Let $t_0=0$, and solve the initial value problem

$$u_t^{(n)} = f(u^{(n)})_x$$
$$u^{(n)}(x, 0) = U^{(n)}(x)$$

by resolution of the discontinuities of $U^{(n)}$ of type (n, n), and continue the solution $u^{(n)}(x, t)$ as long as the discontinuities of $u^{(n)}$ do not intersect. Let t_1 be the time where the first such intersection occurs. Thus, $u^{(n)}$ is defined on $\mathbb{R} \times [t_0, t_1)$.

Assume that times $t_0 < \ldots < t_m$ have been defined, and that $u^{(n)}$ is defined on $\mathbb{R} \times [0, t_m)$. The line $\mathbb{R} \times \{t_m\}$ is intersected by discontinuities of $u^{(n)}$ at finitely many points $(x_1, t_m), \ldots, (x_k, t_m)$ with $x_l < x_{l+1}$. Assume that at the point (x_j, t_m) there intersect the ν lines of discontinuity

$$x = \lambda_i(t) = s_i(t-t_m)+x_j, \quad t_{m-1} \leqq t < t_m, \quad i=1, \ldots, \nu, \tag{2.13}$$

with $s_i > s_{i+1}$ and $\nu \geqq 1$. We first continue the functions $\lambda_i: [t_{m-1}, t_m) \to \mathbb{R}$ defined by (2.13) to the interval $[t_{m-1}, t_m]$. Choose $\varepsilon>0$ small enough such that

$$\varepsilon < \frac{1}{n}(t_m - t_{m-1}) \tag{2.14}$$

and

$$[x_l-\varepsilon, x_l+\varepsilon] \subseteq (\tfrac{1}{2}(x_{l-1}+x_l), \tfrac{1}{2}(x_l+x_{l+1})), \quad l=1, \ldots, k,$$

where we define $x_0=-\infty$, $x_{k+1}=\infty$. Now choose $\lambda_1(t_m) \in (x_j-\varepsilon, x_j+\varepsilon)$. If $\lambda_i(t_m)$ is defined, then let $\lambda_{i+1}(t_m)=\lambda_i(t_m)$ if $i+1$ is even, else choose $\lambda_{i+1}(t_m) \in (x_j-\varepsilon, x_j+\varepsilon)$ with $\lambda_{i+1}(t_m) > \lambda_i(t_m)$. The functions λ_i are called segments of discontinuity. If $\lambda_i(t_m)=\lambda_{i+1}(t_m)$, then the point $(\lambda_i(t_m), t_m)$ is called point of interaction.

Now let $v^0, \ldots, v^\nu$ be the constant values of $u^{(n)}$ in between the discontinuities along the lines $x=\lambda_i(t)$, where

$$u^{(n)}(x,t)=v^i, \quad \lambda_i(t)<x<\lambda_{i+1}(t), \quad i=1,\ldots,\nu-1.$$

Define the function v on the interval $[\frac{1}{2}(x_{j-1}+x_j), \frac{1}{2}(x_j+x_{j+1})]$ by

$$v(x)=\begin{cases} v^0, & x<\lambda_1(t_m) \\ v^i, & \lambda_i(t_m)<x<\lambda_{i+1}(t_m), \quad i=1,\ldots,\nu-1 \\ v^\nu, & \lambda_\nu(t)<x. \end{cases}$$

If this is done for every j, then v is defined on all of $\mathbb{R}$ and is discontinues only at the points $\lambda(t_m)$, where λ is a segment of discontinuity defined on $[t_{m-1}, t_m]$. Non continue $u^{(n)}$ as solution of (1.1) to the initial values

$$u^{(n)}(x,t_m)=v(x)$$

as follows: If v is discontinues at $x=\xi$, then choose the type of the resolution of the discontinuity at ξ according to the

Condition R. If (ξ, t_m) is the point of interaction of two segments of discontinuity λ, μ, then

a) choose type $(0, n)$, if the discontinuities along $x=\lambda(t)$ and $x=\mu(t)$ are both 1-waves.

b) Choose type $(n, 0)$ if these discontinuities are both 2-waves.

c) Choose type $(0, 0)$ if one discontinuity is a 1-wave, the other one a 2-wave.

If (ξ, t_m) is not a point of interaction, then use resolution of type $(0, 0)$.

This construction implies that $u^{(n)}(x, t_m+)$ differs from $u^{(n)}(x, t_m-)$ only on the intervals $(x_j-\varepsilon, x_j+\varepsilon)$, $j=1, \ldots, k$. Therefore we can choose ε small enough such that

$$\int_{-\infty}^{\infty} |u^{(n)}(x,t_m+)-u^{(n)}(x,t_m-)|\,dx \leqq \frac{1}{n}\,2^{-m}. \tag{2.15}$$

Continue $u^{(n)}$ until two discontinuities intersect, and let t_{m+1} be the time where this happens. If such intersections do not occur, then define $t_{m+1}=\infty$.

In this way the function $u^{(n)}$ is defined now on the domain $\mathbb{R}\times[0, T_n)$, with

$$T_n=\sup_{m\geqq 0} t_m. \tag{2.16}$$

In Sect. 3 it is shown that $T>0$ exists with $T_n \geqq T$. In the remainder of this section we introduce some definitions and notations. These definitions depend on n, but we do not denote this dependence explicitly.

Let L_1 be the set of all segments of discontinuity with $\partial_t\lambda<0$, and let L_2 be the set of all segments of discontinuity with $\partial_t\lambda>0$. Let $L=L_1\cup L_2$. We say that $\lambda_1, \lambda_2 \in L$ interact, if $\lambda_1 \neq \lambda_2$ and if λ_1, λ_2 are both defined on the same interval $[t_k, t_{k+1}]$ with $\lambda_1(t_{k+1})=\lambda_2(t_{k+1})$. We say that $\lambda\in L$ issues from the point (x, t_k), if λ is defined on $[t_k, t_{k+1}]$ with $x=\lambda(t_k)$. Let $\Omega\subseteq\mathbb{R}$. We say that $\lambda\in L$ can be connected to Ω, if there exist $\lambda_0, \ldots, \lambda_m \in L$ with $\lambda_j: [t_j, t_{j+1}]\to\mathbb{R}$,

$\lambda_0(0)\in\Omega$, $\lambda_j(t_{j+1})=\lambda_{j+1}(t_{j+1})$, and with $\lambda_m=\lambda$. By $L_i(\Omega)$ we denote the set of all $\lambda\in L_i$ which cannot be connected to $\mathbb{R}\setminus\Omega$.

For $i=1, 2$ *let* $\Gamma_i(\Omega)$ be the set of all subsets $\alpha=\{\alpha_m\}_{m\in I}\subseteq L_i(\Omega)$ defined by the following properties:

a) I is a set of nonnegative integers with $0\in I$.

b) Either α_0 issues from the line $\mathbb{R}\times\{0\}$, or there exist interacting λ_1, $\lambda_2\in L_j$, $j\neq i$, such that α_0 issues from the point of interaction of λ_1 and λ_2.

c) If α_m is defined on $[t_k, t_{k+1}]$, then by conditionR.) above there exists at most one $\lambda\in L_i(\Omega)$ defined on $[t_{k+1}, t_{k+2}]$ with $\lambda(t_{k+1})=\alpha_m(t_{k+1})$. If such λ exists, then $m+1\in I$ and $\alpha_{m+1}=\lambda$. If such λ does not exist, then $I=\{0, 1, \ldots, m\}$.

Let $\Gamma(\Omega)=\Gamma_1(\Omega)\cup\Gamma_2(\Omega)$ and $\Gamma_i=\Gamma_i(\mathbb{R})$. We say that $\alpha, \beta\in\Gamma(\Omega)$ interact, if there exist $\alpha_m\in\alpha$, $\beta_j\in\beta$ which interact. The point of interaction of α_m and β_j is also called point of interaction of α and β. Note that to every couple $\alpha, \beta\in\Gamma(\Omega)$ there exists at most one point of interaction. This is clear if $\alpha\in\Gamma_1(\Omega)$ and $\beta\in\Gamma_2(\Omega)$. If α and β belong to the same family, then α and β coincide from the point of interaction on. It is clear that $\alpha\in\Gamma(\Omega)$, which is by definition a subset of L, also defines a function $\alpha\colon[t_k, t_m]\to\mathbb{R}$ for suitable $t_k<t_m$, and we use both meanings interchangeably.

Let $\Delta(\Omega)=\{(\alpha, \lambda)\in\Gamma(\Omega)\times \mathrm{L}\,(\Omega)\mid\lambda\in\alpha\}$. If $\alpha, \beta\in\Gamma(\Omega)$ interact, then define

$$\alpha<\beta=(\alpha, \alpha_m)\in\Delta(\Omega), \qquad \alpha>\beta=(\alpha, \alpha_{m+1})\in\Delta(\Omega), \tag{2.17}$$

where $\alpha_m\in\alpha$ interacts with $\beta_j\in\beta$.

For $\lambda\in L_i$ let u^l and u^r be the constant values of $u^{(n)}$ on the left and right hand sides of the straight line $x=\lambda(t)$. Then either $u^l, u^r\in\operatorname{graph} c_{1u^l}$, if $i=1$, or $u^l, u^r\in\operatorname{graph} c_{2u^r}$, if $i=2$. The arclength $\theta(\lambda)$ of the graph between u^l and u^r is called strength of the discontinuity along λ.

3. A Priori Estimates

Theorem 3.1. *There exist constants* $T, \rho, V_2, V_3, C>0$, *independent of* n, *such that for all* n *the function* $u^{(n)}$ *is defined on* $\mathbb{R}\times[0, T)$ *and satisfies*

$$|u^{(n)}(x, t)|\leqq\rho, \qquad (x, t)\in\mathbb{R}\times[0, T) \tag{3.0}$$

$$\operatorname{var}[u^{(n)}(\circ, t)]\leqq V_2, \qquad 0\leqq t<T \tag{3.1}$$

$$\int_{-\infty}^{\infty}|u^{(n)}(x, t)-u^{(n)}(x, t')|\,dx\leqq V_3(t-t')+\frac{C}{n}, \qquad 0\leqq t'\leqq t<T. \tag{3.2}$$

This theorem will be proved below.

Corollary 3.2. *There exists a subsequence* $\{u^{(n_i)}\}$ *and a weak solution* $u\in\mathscr{L}_1(\mathbb{R}\times(0, T))$ *of* (1.1), (1.2) *with*

$$\begin{aligned}&|u(x, t)|\leqq\rho, \qquad \operatorname{var}[u(\circ, t)]\leqq V_2,\\ &\lim_{i\to\infty}\int_{-\infty}^{\infty}|u(x, t)-u^{(n_i)}(x, t)|\,dx=0\\ &\lim_{i\to\infty}\int_{-\infty}^{\infty}|f(u(x, t))-f(u^{(n_i)}(x, t))|\,dx=0.\end{aligned} \tag{3.3}$$

Proof of the Corollary. It has been shown in [3] that the estimates in Theorem 3.1 imply the existence of the subsequence and of $u \in \mathscr{L}_1(\mathbb{R} \times (0, T))$ such that the estimates and limits in the corollary hold. To see that u is a weak solution let ϕ be a smooth function vanishing on the line $t = T$. Then by partial integration and the fact that $u^{(n)}$ is a weak solution on every strip $\mathbb{R} \times [t_k, t_{k+1})$ we obtain together with (2.10) and (2.15) that

$$\begin{aligned}
&\left| \int_0^T \int_{-\infty}^{\infty} (u\phi_t - f(u)\phi_x)\, dx\, dt + \int_{-\infty}^{\infty} U(x)\phi(x,0)\, dx \right| \\
&\quad = \left| \lim_{i\to\infty} \left[\int_0^T \int_{-\infty}^{\infty} (u^{(n_i)}\phi_t - f(u^{(n_i)})\phi_x)\, dx\, dt + \int_{-\infty}^{\infty} U^{(n_i)}(x)\phi(x,0)\, dx \right] \right| \\
&\quad = \left| \lim_{i\to\infty} \sum_{k\geq 1} \int_{-\infty}^{\infty} [u^{(n_i)}(x, t_k-) - u^{(n_i)}(x, t_k+)]\, \phi(x, t_k)\, dx \right| \leqq C \lim_{i\to\infty} \frac{1}{n_i} = 0.
\end{aligned}$$

In the proof of Theorem 3.1 we can assume without restriction of generality that a constant $M > 0$ exists with

$$4M^{-2} \leqq \sigma'(\tau) \leqq \tfrac{1}{4}M^2, \qquad \tau \in \mathbb{R}. \tag{3.4}$$

For, in the following proof the constant ρ of (3.0) and (3.3) is chosen in (3.14a), and this choice is independent of the behaviour of $\sigma(\tau)$ for large values of $|\tau|$. As we assumed that $\sigma'(\tau) > 0$, $\sigma''(\tau) > 0$, we therefore can modify $\sigma(\tau)$ for large values of τ such that (3.4) is satisfied. Theorem 3.1 is a consequence of the following local result:

Lemma 3.3. *For every $x_0 \in \mathbb{R}$ there exist constants $\delta, \rho, V_2, V_3 > 0$, independent of n, such that for all n the points of interaction do not accumulate in the triangle*

$$D(x_0, \delta) = \{(x, t) \in \mathbb{R} \times \mathbb{R}_0^+ \mid x_0 - \delta + Mt < x < x_0 + \delta - Mt\},$$

where M is the constant defined in (3.4). *Also,*

$$|u^{(n)}(x, t)| \leqq \rho, \qquad (x, t) \in D(x_0, \delta) \cap (\mathbb{R} \times [0, T_n)) \tag{3.5}$$

$$\operatorname{var}[u^{(n)}(\circ, t) \mid (x_0 - \delta + Mt, x_0 + \delta - Mt)] \leqq V_2, \qquad 0 \leqq t < T_n. \tag{3.6}$$

$$\int_{x_0 - \delta + Mt}^{x_0 + \delta - Mt} |u^{(n)}(x, t) - u^{(n)}(x, t')|\, dx \leqq \frac{1}{n} + V_3(t - t'), \qquad 0 \leqq t' \leqq t < T_n. \tag{3.7}$$

Proof of Theorem 3.1. Note first that if $\lambda \in L$ is defined on $[t_k, t_{k+1}]$, then (2.8), (2.13), Lemma A1(i) and (3.4) imply

$$2M^{-1} \leqq |\partial_t \lambda(t)| \leqq \tfrac{1}{2}M, \qquad t_k \leqq t < t_{k+1}.$$

(2.14) and the definition of $\lambda(t_{k+1})$ thus imply

$$M^{-1}(t - t_k) \leqq |\lambda(t) - \lambda(t_k)| \leqq M(t - t_k), \qquad t_k \leqq t \leqq t_{k+1}, \tag{3.8}$$

for all sufficiently large n. This inequality and (2.9) show that $u^{(n)}$ vanishes on the set $D = \{(x, t) \in \mathbb{R} \times \mathbb{R}_0^+ \mid \operatorname{dist}(x, \operatorname{supp} U) > 1 + Mt\}$. Theorem 3.1 thus follows from Lemma 3.3 by covering the compact set $\{x \mid \operatorname{dist}(x, \operatorname{supp} U) \leqq 1\}$ by finitely

many intervals $(x_i-\delta_i, x_i+\delta_i)$ chosen as in Lemma 3.3. Then there exists $T>0$ with $\mathbb{R}\times[0, T)\subseteq D\cup\bigcup D(x_i, \delta_i)$, which implies that $\mathbb{R}\times[0, T-\varepsilon]$ contains at most finitely many points of interaction for $\varepsilon>0$, hence $T_n\geqq T$, T_n being the number defined in (2.16). It is clear that (3.0)–(3.2) follow from (3.5)–(3.7).

It thus remains to prove Lemma 3.3. This lemma follows from the following basic result:

Lemma 3.4. *For every $x_0\in\mathbb{R}$ there exist constants $\delta, \rho, V_2, V_3>0$, independent of n, and for every $n\in\mathbb{N}$ a function $\omega: \Delta(\Omega)\to\mathbb{R}_0^+$, where $\Omega=(x_0-\delta, x_0+\delta)$, such that*

$$\theta(\lambda)\leqq\sum_{\substack{\alpha\in\Gamma(\Omega)\\ \text{with } \lambda\in\alpha}}\omega(\alpha, \lambda), \quad \text{for all } \lambda\in L(\Omega) \tag{3.9}$$

$$\theta(\lambda)\leqq\frac{1}{n}V_3, \tag{3.10}$$

if the discontinuity along $\lambda\in L(\Omega)$ is a rarefaction wave.

$$\sum_{\alpha\in\Gamma(\Omega)}\sup_{\lambda\in\alpha}\omega(\alpha, \lambda)\leqq V_2. \tag{3.11}$$

$$|u^l(\lambda)|, \quad |u^r(\lambda)|\leqq\rho, \quad \text{for all } \lambda\in L(\Omega). \tag{3.12}$$

Here $u^l(\lambda)$ and $u^r(\lambda)$ denote the constant values of $u^{(n)}$ on the left and right hand side of graph λ.

Proof of Lemma 3.4. I. For every $n\in\mathbb{N}$ let $F(\Omega)$ be the set of all functions $p: \Delta(\Omega)\to\mathbb{R}_0^+$. Of course, $F(\Omega)$ depends on n, but we do not denote this dependence explicitly. We define an operator $B: F(\Omega)\to F(\Omega)$ and show that B possesses a fixed point ω satisfying (3.9) and (3.11) for δ sufficiently small, with the constants V_2 and δ independent of n. Note that (2.10) implies $\lim_{n\to\infty} U^{(n)}(x\pm)=U(x_0\pm)$. It follows from this that if the resolution of the discontinuity $[U(x_0-), U(x_0+)]$ contains an i-shock wave, then $\Gamma_i(\Omega)$ contains an i-shock ε_i issuing from $(x_0, 0)$ for all n sufficiently large. Similarly, if the resolution of $[U(x_0-), U(x_0+)]$ contains i-rarefaction waves, then $\Gamma_i(\Omega)$ contains i-rarefaction waves issuing from $(x_0, 0)$ for all n sufficiently large. We first assume that the resolution of $[U(x_0-), U(x_0+)]$ consists of a 1- and a 2-shock wave, and therefore that for all n sufficiently large $\varepsilon_1\in\Gamma_1(\Omega)$ and $\varepsilon_2\in\Gamma_2(\Omega)$ exist issuing from $(x_0, 0)$. In part VII we discuss the modifications necessary to treat the other cases.

For the definition of B note that by (2.10) and the fact that $\operatorname{var} U$ is finite there exists a compact set $W\subseteq\mathbb{R}^2$ with $U^{(n)}(\mathbb{R})\subseteq W$ for all n. According to Lemma A2 in the appendix there exists a constant V^* with

$$l_1^j(w, u)+l_2^m(w, v)\leqq V^*|u-v| \tag{3.13}$$

for all $u, v\in W$. Choose ρ sufficiently large such that

$$\{w\in\mathbb{R}^2\,|\operatorname{dist}(w, W)\leqq V_2\}\subseteq\{w\in\mathbb{R}^2\,||w|<\rho\}, \tag{3.14a}$$

where

$$V_2 = V^*(\operatorname{var} U + 1) + 1. \tag{3.14b}$$

Also, let V be the larger one of the two constants belonging to the compact set

$$\{|w| \leqq \rho\} \times [0, V_2] \times [0, V_2] \subseteq \mathbb{R}^4,$$

whose existence is assured by Lemma A3(ii) and by Lemma A4(iii), and let $V_1 < 1$ be the constant belonging to the compact set

$$\{|w| \leqq \rho\} \times [0, V_2] \subseteq \mathbb{R}^3$$

according to Lemma A4(iv).

We define B as follows. Let $\alpha = \{\alpha_m\}_{m \in I} \in \Gamma(\Omega)$. We first define $Bp(\alpha, \alpha_0)$. There are two possibilities.

1. If α issues from $\Omega \times \{0\}$, then

$$Bp(\alpha, \alpha_0) = \theta(\alpha_0). \tag{3.15}$$

2. Assume that $\alpha \in \Gamma_i(\Omega)$ issues from the point of interaction of $\lambda_1, \lambda_2 \in L_j(\Omega)$, $j \neq i$. Let $\{\alpha^1, \ldots, \alpha^m\} \subseteq \Gamma_i(\Omega)$ be the set of all the discontinuities issuing from the point of interaction of λ_1 and λ_2. This set contains α, of course. Assume that

$$|\partial_t \alpha_0^m| < \ldots < |\partial_t \alpha_0^1|, \qquad |\partial_t \lambda_1| > |\partial_t \lambda_2|,$$

and let m_0 be the largest nonnegative integer with $m_0 < m$ and with $m_0/m \leqq \Lambda$, where

$$\Lambda = \begin{cases} V\Big[\sum\limits_{\substack{\beta \in \Gamma(\Omega) \\ \text{with } \lambda_1 \in \beta}} p(\beta, \lambda_1)\Big]\Big[\sum\limits_{\substack{\gamma \in \Gamma(\Omega) \\ \text{with } \lambda_2 \in \gamma}} p(\gamma, \lambda_2)\Big], & \text{if } \lambda_2 \notin \varepsilon_j \\ V_1 \sum\limits_{\substack{\beta \in \Gamma(\Omega) \\ \text{with } \lambda_1 \in \beta}} p(\beta, \lambda_1), & \text{if } \lambda_2 \in \varepsilon_j. \end{cases} \tag{3.16}$$

Then define

$$Bp(\alpha^l, \alpha_0^l) = \begin{cases} 1/n, & 1 \leqq l \leqq m_0 \\ \Lambda - m_0/n, & l = m_0 + 1 \\ 0, & m_0 + 2 \leqq l \leqq m. \end{cases} \tag{3.17}$$

In the situation above we call $\alpha^1, \ldots, \alpha^m \in \Gamma_i(\Omega)$ reflected at ε_j if $\lambda_2 \in \varepsilon_j$. The point of interaction of λ_1 and λ_2 is also called point of reflection. Now we define $Bp(\alpha, \alpha_{l+1})$ if $Bp(\alpha, \alpha_l)$ is known. If no interaction takes place in between α_l and α_{l+1}, then let

$$Bp(\alpha, \alpha_{l+1}) = Bp(\alpha, \alpha_l). \tag{3.18}$$

If $\beta^1, \ldots, \beta^m \in \Gamma(\Omega)$ interact with α at the same point of interaction, then let

$$Bp(\alpha > \beta^1) = \prod_{j=1}^{m} [1 + Vp(\beta^j < \alpha)]\, Bp(\alpha < \beta^1), \tag{3.19}$$

where the symbols $<$, $>$ are defined in (2.17). Note that $Bp(\alpha, \lambda)$ is uniquely defined by (3.16)–(3.19) for all $(\alpha, \lambda) \in \Delta(\Omega)$, since $\lambda \in L(\Omega)$ cannot be connected to $\mathbb{R} \setminus \Omega$.

II. We next show that B is a contraction mapping if the interval $\Omega = (x_0 - \delta, x_0 + \delta)$ is chosen sufficiently small. Let $0 < \kappa_0 < 1$. $F(\Omega)$ is a complete metric space with the metric

$$d(p, q) = \sum_{\alpha \in \Gamma(\Omega)} \sup_{\lambda \in \alpha} \kappa(\lambda) |p(\alpha, \lambda) - q(\alpha, \lambda)|, \tag{3.20}$$

where

$$\kappa(\lambda) = \begin{cases} \kappa_0, & \text{if } \lambda \in \varepsilon_1 \cup \varepsilon_2 \\ 1, & \text{if } \varepsilon_1(t) \leqq \lambda(t) \leqq \varepsilon_2(t) \text{ for all } t, \text{ but } \lambda \notin \varepsilon_1 \cup \varepsilon_2 \\ \kappa_0^{-1}, & \text{else.} \end{cases}$$

The proof is based on the following four inequalities for $p, q \in F(\Omega)$ and $\alpha = \{\alpha_m\}_{m \in I} \in \Gamma(\Omega)$

$$\kappa(\alpha_m) Bp(\alpha, \alpha_m) \leqq e^{Vd(p, 0)} \kappa(\alpha_0) Bp(\alpha, \alpha_0) \tag{3.21}$$

$$\begin{aligned} \kappa(\alpha_m) |Bp(\alpha, \alpha_m) - Bq(\alpha, \alpha_m)| &\leqq Vd(p, q)\, e^{V[d(p, 0) + d(0, q)]} \kappa(\alpha_0) Bp(\alpha, \alpha_0) \\ &\quad + e^{Vd(0, q)} \kappa(\alpha_0) |Bp(\alpha, \alpha_0) - Bq(\alpha, \alpha_0)| \end{aligned} \tag{3.22}$$

$$\sum_{\substack{\alpha \in \Gamma(\Omega) \\ \alpha \text{ reflected} \\ \text{at } \varepsilon_1 \cup \varepsilon_2}} |Bp(\alpha, \alpha_0) - Bq(\alpha, \alpha_0)| \leqq V_1 d(p, q) \tag{3.23}$$

$$\sum_{\substack{\alpha \in \Gamma(\Omega) \\ \alpha \text{ not reflected} \\ \text{at } \varepsilon_1 \cup \varepsilon_2}} \kappa(\alpha_0) |Bp(\alpha, \alpha_0) - Bq(\alpha, \alpha_0)| \leqq V[d(p, 0) + d(0, q)]\, d(p, q). \tag{3.24}$$

Together with the estimate

$$\sum_{\substack{\alpha \in \Gamma(\Omega) \\ \alpha \text{ issuing from} \\ (\Omega \setminus \{x_0\}) \times \{0\}}} |Bp(\alpha, \alpha_0)| \leqq V^* [\operatorname{var}(U^{(n)} | (x_0 - \delta, x_0)) + \operatorname{var}(U^{(n)} | (x_0, x_0 + \delta))], \tag{3.25}$$

which is an immediate consequence of (3.13) and (3.15), it follows from these inequalities that B is a contraction mapping for δ sufficiently small. For, (3.21) yields

$$\begin{aligned} d(Bp, 0) &= \sum_{\substack{\alpha \in \Gamma(\Omega) \\ \alpha \neq \varepsilon_1, \varepsilon_2}} \sup_{\lambda \in \alpha} \kappa(\lambda) Bp(\alpha, \lambda) + \kappa_0 \sum_{i=1}^{2} \sup_{\lambda \in \varepsilon_i} Bp(\varepsilon_i, \lambda) \\ &\leqq e^{Vd(p, 0)} \Big[\sum_{\substack{\alpha \text{ reflected} \\ \text{at } \varepsilon_1 \cup \varepsilon_2}} \kappa(\alpha_0) Bp(\alpha, \alpha_0) + \kappa_0 \sum_{i=1}^{2} Bp(\varepsilon_i, \varepsilon_{i0}) \\ &\quad + \sum_{\substack{\alpha \text{ not reflected} \\ \text{at } \varepsilon_1 \cup \varepsilon_2}} \kappa(\alpha_0) Bp(\alpha, \alpha_0) + \sum_{\substack{\alpha \text{ issuing from} \\ (\Omega \setminus \{x_0\}) \times \{0\}}} \kappa(\alpha_0) Bp(\alpha, \alpha_0) \Big], \end{aligned}$$

where $\varepsilon_i=\{\varepsilon_{im}\}_{m\in I}$. (3.23)–(3.25) with $q=0$ now imply together with (3.13) and the fact that $\kappa(\alpha_0)=1$ if α is reflected at $\varepsilon_1\cup\varepsilon_2$

$$
\begin{aligned}
(3.26)\qquad d(Bp,0)\leqq e^{Vd(p,0)}\{&V_1\,d(p,0)+\kappa_0\,V^*|U^{(n)}(x_0+)-U^{(n)}(x_0-)|\\
&+Vd(p,0)^2+V^*\kappa_0^{-1}[\mathrm{var}(U^{(n)}|(x_0-\delta,x_0))\\
&+\mathrm{var}(U^{(n)}|(x_0,x_0+\delta))]\}.
\end{aligned}
$$

Note that by (2.11),

$$
(3.27)\qquad |U^{(n)}(x_0+)-U^{(n)}(x_0-)|\leqq \mathrm{var}\,U^{(n)}\leqq C
$$

for every $n\in\mathbb{N}$. Since $V_1<1$ we can choose $K_1>0$ such that $V_1+K_1V^*C<1$ for this constant C. We choose now $K>0$ sufficiently small such that

$$
(3.28)\qquad N=e^{VK}(V_1+K_1V^*C+VK)K<K.
$$

Finally, by (2.12) we can choose $\delta>0$ small enough such that

$$
(3.29)\qquad N+V^*\kappa_0^{-1}e^{VK}[\mathrm{var}(U^{(n)}|(x_0-\delta,x_0))+\mathrm{var}(U^{(n)}|(x_0,x_0+\delta))]\leqq K
$$

for all sufficiently large n. We set $\kappa_0=\mathrm{K}_1\mathrm{K}$ and choose K and δ still smaller if necessary such that $\kappa_0\leqq 1$. It then follows for this choice of κ_0, K, and δ from (3.26)–(3.29) that for all $p\in F(\Omega)$ with $d(p,0)\leqq K$ we have $d(Bp,0)\leqq K$, which shows that the operator B maps the closed set $F_K=\{p\in F(\Omega)\,|\,d(p,0)\leqq K\}$ into itself.

In the same way we conclude from (3.22)–(3.24) that for $p,q\in F_K$

$$
\begin{aligned}
d(Bp,Bq)\leqq e^{2VK}V\{&V_1K+\kappa_0V^*|U^{(n)}(x_0+)-U^{(n)}(x_0-)|+VK^2\\
&+V^*\kappa_0^{-1}[\mathrm{var}(U^{(n)}|(x_0-\delta,x_0))+\mathrm{var}(U^{(n)}|(x_0,x_0+\delta))]\}\,d(p,q)\\
&+e^{VK}(V_1+2VK)\,d(p,q).
\end{aligned}
$$

Since $V_1<1$, we obtain with the same choice of K_1 as above, but by further decreasing K and δ if necessary, that $d(Bp,Bq)<\eta d(p,q)$ with a suitable constant $\eta<1$, which shows that B possesses a fixed point $\omega\in F_K$, hence $d(\omega,0)\leqq K$. Since the choice of κ_0, K, and δ is independent of n, (3.11) follows from (3.20). In part IV of this proof it is shown that for V_2 in (3.11) we can choose the constant defined in (3.14b).

III. It remains to prove (3.21)–(3.24). (3.19) yields for $\alpha\in\Gamma(\Omega)$

$$
\begin{aligned}
(3.30)\qquad &\kappa(\alpha_m)\,|Bp(\alpha,\alpha_m)-Bq(\alpha,\alpha_m)|\\
&\quad=|\Pi\kappa^*(\beta)[1+Vp(\beta<\alpha)]\,\kappa(\alpha_0)\,Bp(\alpha,\alpha_0)\\
&\qquad-\Pi\kappa^*(\beta)[1+Vq(\beta<\alpha)]\,\kappa(\alpha_0)\,Bq(\alpha,\alpha_0)|
\end{aligned}
$$

where the products are taken over all $\beta\in\Gamma(\Omega)$ which interact with α in between α_0 and α_m, and where $\kappa^*(\beta)=\kappa(\alpha_{l+1})/\kappa(\alpha_l)$ if β interacts with α in between α_l and α_{l+1}. In the derivation of (3.30) we use the fact every $\beta\in\Gamma(\Omega)$ interacts with α at most once, and that $\kappa(\alpha_0)\,\Pi\kappa^*(\beta)=\kappa(\alpha_m)$. Note also that by construction $\alpha\in\Gamma(\Omega)$ cannot pass from the interior of the region bounded by graph

$\varepsilon_1 \cup \text{graph}\, \varepsilon_2$ into the exterior. Therefore $\kappa^*(\beta)$ always takes one of the values 1, κ_0, κ_0^2, hence $\kappa^*(\beta) \leqq 1$. We now apply the mean value theorem to (3.30) and obtain

(3.31)

$$\begin{aligned}
&\kappa(\alpha_m)\,|Bp(\alpha, \alpha_m) - Bq(\alpha, \alpha_m)| \\
&\quad \leqq |\Pi\, \kappa^*(\beta)[1 + Vp(\beta < \alpha)] - \Pi\, \kappa^*(\beta)[1 + Vq(\beta < \alpha)]|\, \kappa(\alpha_0)\, Bp(\alpha, \alpha_0) \\
&\qquad + \Pi\, \kappa^*(\beta)[1 + Vq(\beta < \alpha)]\, \kappa(\alpha_0)\, |Bp(\alpha, \alpha_0) - Bq(\alpha, \alpha_0)| \\
&\quad \leqq V e^{V \sum \kappa^*(\beta)[p(\beta < \alpha) + q(\beta < \alpha)]} \sum \kappa^*(\beta)\, |p(\beta < \alpha) - q(\beta < \alpha)|\, \kappa(\alpha_0)\, Bp(\alpha, \alpha_0) \\
&\qquad + e^{V \sum \kappa^*(\beta)\, q(\beta < \alpha)}\, \kappa(\alpha_0)\, |Bp(\alpha, \alpha_0) - Bq(\alpha, \alpha_0)|.
\end{aligned}$$

(3.20) and the fact that $\kappa^*(\varepsilon_i) = \kappa_0$ or $= \kappa_0^2 \leqq \kappa_0$ imply

$$\sum \kappa^*(\beta)\, |p(\beta < \alpha) - q(\beta < \alpha)| \leqq d(p, q). \tag{3.32}$$

Insertion of this relation into (3.31) yields (3.22). (3.21) is proved in the same way. (3.24) is obtained as follows. Let (x, t) be the point of interaction of $\lambda_1, \lambda_2 \in L_i(\Omega)$, and assume that (x, t) is not a point of reflection. Then finitely many $\alpha = \{\alpha_m\}_{m \in I} \in \Gamma_j(\Omega)$, $j \neq i$, issue from (x, t). We then have

$$\hat{\kappa} = \kappa(\alpha_0) \leqq \kappa(\lambda_1)\, \kappa(\lambda_2). \tag{3.33}$$

For, if $(x, t) \notin \text{graph}\, \varepsilon_1 \cup \text{graph}\, \varepsilon_2$, then $\kappa(\lambda_1) = \kappa(\lambda_2) = \kappa(\alpha_0) \geqq 1$. If $(x, t) \in \text{graph}\, \varepsilon_1 \cup \text{graph}\, \varepsilon_2$ with $|\partial_t \lambda_1| > |\partial_t \lambda_2|$, then $\lambda_1 \in \text{graph}\, \varepsilon_1 \cup \text{graph}\, \varepsilon_2$, since (x, t) is not a point of reflection, hence $\kappa(\alpha_0) = 1 = \kappa_0^{-1}\, \kappa_0 = \kappa(\lambda_2)\, \kappa(\lambda_1)$. Since for all $\alpha \in \Gamma_j(\Omega)$ issuing from (x, t) the difference $Bp(\alpha, \alpha_0) - Bq(\alpha, \alpha_0)$ has the same sign, we thus obtain from (3.16), (3.17) that

$$\begin{aligned}
&\sum_{\substack{\alpha \text{ issuing} \\ \text{from } (x, t)}} \kappa(\alpha_0)\, |Bp(\alpha, \alpha_0) - Bq(\alpha, \alpha_0)| = \hat{\kappa} \Big| \sum Bp(\alpha, \alpha_0) - Bq(\alpha, \alpha_0) \Big| \\
&\quad = \hat{\kappa} \Big| V \Big[\sum_{\substack{\beta \in \Gamma(\Omega) \\ \text{with } \lambda_1 \in \beta}} p(\beta, \lambda_1) \Big] \Big[\sum_{\substack{\gamma \in \Gamma(\Omega) \\ \text{with } \lambda_2 \in \gamma}} p(\gamma, \lambda_2) \Big] \\
&\qquad - V \Big[\sum_{\substack{\beta \in \Gamma(\Omega) \\ \text{with } \lambda_1 \in \beta}} q(\beta, \lambda_1) \Big] \Big[\sum_{\substack{\gamma \in \Gamma(\Omega) \\ \text{with } \lambda_2 \in \gamma}} q(\gamma, \lambda_2) \Big] \Big| \\
&\quad \leqq \hat{\kappa} V \Big[\sum_{\substack{\beta \in \Gamma(\Omega) \\ \lambda_1 \in \beta}} p(\beta, \lambda_1) \Big] \sum_{\substack{\gamma \in \Gamma(\Omega) \\ \lambda_2 \in \gamma}} |p(\gamma, \lambda_2) - q(\gamma, \lambda_2)| \\
&\qquad + \hat{\kappa} V \Big[\sum_{\substack{\gamma \in \Gamma(\Omega) \\ \lambda_2 \in \gamma}} q(\gamma, \lambda_2) \Big] \sum_{\substack{\beta \in \Gamma(\Omega) \\ \lambda_1 \in \beta}} |p(\beta, \lambda_1) - q(\beta, \lambda_1)| \\
&\quad \leqq V \sum_{\beta \in \Gamma(\Omega)} \sum_{\substack{\gamma \in \Gamma(\Omega) \\ \gamma \text{ interacts} \\ \text{with } \beta \text{ at } (x, t)}} \kappa(\beta < \gamma) [p(\beta < \gamma) \\
&\qquad + q(\beta < \gamma)]\, \kappa(\gamma < \beta)\, |p(\gamma < \beta) - q(\gamma < \beta)|.
\end{aligned}$$

where we used that $\hat{\kappa} \leqq \kappa(\beta < \gamma)\, \kappa(\gamma < \beta)$, which is (3.33), by a slight abuse of notation. Summation over all non-reflecting points of interaction of couples

$\lambda, \mu \in L$ from the same family yields

$$\begin{aligned}
&\sum_{\substack{\alpha\in\Gamma(\Omega)\\ \alpha \text{ not reflected}\\ \text{at } \varepsilon_1\cup\varepsilon_2}} \kappa(\alpha_0)\,|Bp(\alpha,\alpha_0)-Bq(\alpha,\alpha_0)| \\
&\quad\leqq V \sum_{\beta\in\Gamma(\Omega)} \sum_{\substack{\gamma\in\Gamma(\Omega)\\ \gamma \text{ and } \beta \text{ interact}}} \kappa(\beta<\gamma)[p(\beta<\gamma)+q(\beta<\gamma)] \\
&\qquad\cdot \kappa(\gamma<\beta)\,|p(\gamma<\beta)-q(\gamma<\beta)| \\
&\quad\leqq V \sum_{\beta\in\Gamma(\Omega)} [\sup_{\nu\in\beta} \kappa(\nu)\,p(\beta,\nu)+\sup_{\nu\in\beta}\kappa(\nu)\,q(\beta,\nu)] \\
&\qquad\cdot \sum_{\gamma\in\Gamma(\Omega)} \sup_{\mu\in\gamma} \kappa(\mu)|p(\gamma,\mu)-q(\gamma,\mu)| \\
&\quad\leqq V[d(p,0)+d(0,q)]\,d(p,q).
\end{aligned}$$

This proves (3.24). Finally, (3.23) is obtained in the same way using the second definition of Λ in (3.16).

IV. For $p, q\in F(\Omega)$ *let*

$$d'(p,q)=\sum_{\substack{\alpha\in\Gamma(\Omega)\\ \alpha\neq\varepsilon_1,\,\varepsilon_2}} \sup_{\lambda\in\alpha} \kappa(\lambda)\,|p(\alpha,\lambda)-q(\alpha,\lambda)|. \tag{3.34}$$

We now show that $d'(\omega,0)$ can be made arbitrarily small by decreasing δ. It is immediately seen that (3.23) is valid with d replaced by d', and that (3.24) is valid if the right hand side is replaced by

$$Vd'(p,0)\,d(p,q)+Vd(0,q)\,d'(p,q).$$

Using this, inspection of the proof of (3.26) shows at once that

$$\begin{aligned}
d'(Bp,0)\leqq e^{Vd(p,0)}[&V_1 d'(p,0)+Vd(p,0)\,d'(p,0) \\
&+V^*\kappa_0^{-1}[\operatorname{var}(U^{(n)}|(x_0-\delta,x_0))+\operatorname{var}(U^{(n)}|(x_0,x_0+\delta))]].
\end{aligned} \tag{3.35}$$

The choice of K_1 and K in (3.28) implies that

$$C_1=e^{VK}(V_1+VK)<1,$$

C_1 being independent of n. (3.35) thus implies

$$(1-C_1)\,d'(\omega,0)\leqq V^*\kappa_0^{-1}\,e^{VK}[\operatorname{var}(U^{(n)}|(x_0-\delta,x_0))+\operatorname{var}(U^{(n)}(x_0,x_0+\delta))],$$

from which it follows by (2.12) that we can make $d'(\omega,0)$ arbitrarily small by decreasing δ.

Similarly, if $\alpha=\varepsilon_i$ then (3.21) is valid with d replaced by d'. Since $\kappa(\varepsilon_{im})=\kappa(\varepsilon_{i0})$ it therefore follows from (3.13) and (3.15) that

$$\begin{aligned}
\sum_{i=1}^{2}\omega(\varepsilon_i,\varepsilon_{im})=\sum_{i=1}^{2}B\omega(\varepsilon_i,\varepsilon_{im})&\leqq e^{Vd'(\omega,0)}\,V^*|U^{(n)}(x_0+)-U^{(n)}(x_0-) \\
&\leqq e^{Vd'(\omega,0)}\,V^*(\operatorname{var}U+1),
\end{aligned}$$

for n sufficiently large. This formula, (3.34), and the discussion above show that by choosing δ sufficiently small we can achieve that

$$\sum_{\alpha\in\Gamma(\Omega)} \sup_{\lambda\in\alpha} \omega(\alpha,\lambda)\leqq V^*(\operatorname{var} U+1)+1=V_2. \tag{3.36}$$

In the following we assume that this estimate holds.

V. We next show that (3.9) and (3.12) are satisfied for this fixed point ω. Assume first that (3.9) is true. We show that we have $|u^l(\lambda)|$, $|u^r(\lambda)|\leqq\rho$ for all $\lambda\in L(\Omega)$, where ρ is defined in (3.14). Assume that λ is defined on $[t_k, t_{k+1}]$, and choose $(y,\tau)\in\mathbb{R}\times(t_k, t_{k+1})$ to the left of graph λ such that $u^{(n)}(y,\tau)=u^l(\lambda)$. Define a function $h\colon[0,\tau]\to\mathbb{R}$ as follows. Let

$$h(t)=y,\qquad t_k\leqq t\leqq\tau.$$

Choose $\lambda_1, \lambda_2\in L(\Omega)$ defined on $[t_{k-1}, t_k]$ with

$$\lambda_1(t_k)=\max\{\lambda(t_k)\,|\,\lambda\in L(\Omega),\ \lambda\colon[t_{k-1},t_k]\to\mathbb{R},\ \lambda(t_k)\leqq h(t_k)\},$$
$$\lambda_2(t_k)=\min\{\lambda(t_k)\,|\,\lambda\in L(\Omega),\ \lambda\colon[t_{k-1},t_k]\to\mathbb{R},\ \lambda(t_k)\geqq h(t_k)\},$$

and choose $\xi\in\mathbb{R}$ with

$$\lim_{t\to t_k-}\lambda_1(t)\leqq\xi\leqq\lim_{t\to t_k-}\lambda_2(t).$$

If such λ_1 and λ_2 do not exist, then let $\xi=h(t_k)$. Now define

$$h(t)=\xi,\qquad t_{k-1}\leqq t\leqq t_k.$$

Continue in this way. For technical convenience we can choose y and ξ such that the graph of h does not contain a point of interaction. It then follows that

$$|u^l(\lambda)-U^{(n)}(h(0))|=|u^{(n)}(h(\tau),\tau)-U^{(n)}(h(0))|=\sum\theta(\lambda),$$

where the summation is over all $\lambda\in L(\Omega)$ whose graph intersect with the graph of h. From (3.9), (3.36), and from the fact that graph α intersects with graph h at most once for every $\alpha\in\Gamma(\Omega)$ it follows that

$$|u^l(\lambda)-U^{(n)}(h(0))|\leqq V^*(\operatorname{var} U+1)+1,$$

and therefore $|u^l(\lambda)|\leqq\rho$, by definition of ρ in (3.14). $u^r(\lambda)$ can be treated in the same way.

It thus remains to show that (3.9) is satisfied. Let $\lambda\in L(\Omega)$ be defined on $[0, t_1]$. Then (3.15) implies that (3.9) is satisfied. Assume next that (3.9) is satisfied for all $\mu\in L(\Omega)$ defined on $[t_{l-1}, t_l]$ with $l\leqq k$. We show that then (3.9) is also true for $\lambda\in L(\Omega)$ defined on $[t_k, t_{k+1}]$. For such λ we have to distinguish between three possibilities:

1. $\lambda\in L_i(\Omega)$ issues from the point of interaction of $\lambda_1, \lambda_2\in \mathrm{L}_j(\Omega)$ with $j\neq i$. Then there exists a finite set $\{\alpha^1,\ldots,\alpha^m\}\subseteq\Gamma_i(\Omega)$ of discontinuities issuing from

this point of interaction, and for one $\alpha=\{\alpha_l\}_{l\in I}\in\{\alpha^1,\ldots,\alpha^m\}$ we have $\lambda=\alpha_0$. Condition R) implies that the discontinuity of $u^{(n)}$ at this point of interaction is resolved by resolution of type $(n,0)$ if $i=1$, and of type $(0,n)$ if $i=2$. We first assume that this point of interaction is not a point of reflection. From Lemma A4(iii) in the appendix it follows that

$$\sum_{l=1}^{m}\theta(\alpha_0^l)\leqq V\theta(\lambda_1)\,\theta(\lambda_2), \tag{3.37}$$

$$\theta(\alpha_0^l)=\frac{1}{n},\qquad 1\leqq l\leqq m-1, \tag{3.38}$$

if we assume that $|\partial_t\alpha_0^m|<\ldots<|\partial_t\alpha_0^l|$. V is the constant corresponding to the compact set

$$\{|w|\leqq\rho\}\times[0,V_2]\times[0,V_2],$$

where ρ and V_2 are defined in (3.14a) and (3.14b), since by the induction hypotheses and (3.36) we have $\theta(\lambda_1)$, $\theta(\lambda_2)\leqq V_2$, and since by the induction hypotheses and the preceding discussion we have $|u^{l,r}(\lambda_1)|$, $|u^{l,r}(\lambda_2)|\leqq\rho$. (3.37), the first definition of Λ in (3.16), (3.17), and the induction hypotheses imply

$$\sum_{l=1}^{m}\theta(\alpha_0^l)\leqq\sum_{l=1}^{m}B\omega(\alpha^l,\alpha_0^l),$$

from which it follows by (3.17) and (3.38) that

$$\omega(\alpha^l,\alpha_0^l)=B\omega(\alpha^l,\alpha_0^l)=\frac{1}{n}=\theta(\alpha_0^l),\qquad l=1,\ldots,m-1,$$

$$\omega(\alpha^m,\alpha_0^m)\geqq\theta(\alpha_0^m),$$

which proves (3.9) in this situation. The case that the point of interaction is a point of reflection is treated in the same way, using Lemma A4(iv) and the second definition of Λ in (3.16). In the application of this lemma we must assume that $\theta(\lambda_1)$ is sufficiently small, if the notation is chosen such that $|\partial_t\lambda_1|>|\partial_t\lambda_2|$. But since

$$\theta(\lambda_1)\leqq\sum_{\substack{\beta\in\Gamma(\Omega)\\ \lambda_1\in\beta}}\omega(\beta,\lambda_1)\leqq d'(\omega,0),$$

by (3.34) and the induction hypotheses, it follows from part IV above that $\theta(\lambda_1)$ can be made arbitrarily small by choosing δ sufficiently small.

2. Assume that $\lambda\in L_i(\Omega)$ issues from the point of interaction of $\lambda_1\in L_1$, $\lambda_2\in L_2$. Condition R) then implies that the discontinuity of $u^{(n)}$ at the point of interaction is resolved by resolution of type $(0,0)$, and thus Lemma A3(ii),

(3.19), and the induction hypotheses yield for $j \neq i$

$$\theta(\lambda) \leqq [1+V\theta(\lambda_j)]\,\theta(\lambda_i) \leqq [1+V \sum_{\substack{\beta\in\Gamma(\Omega)\\ \lambda_j\in\beta}} \omega(\beta,\lambda_j)] \sum_{\substack{\alpha\in\Gamma(\Omega)\\ \lambda_i\in\alpha}} \omega(\alpha,\lambda_i)$$

$$\leqq \sum_{\substack{\alpha\in\Gamma(\Omega)\\ \lambda_i\in\alpha}} \prod_{\substack{\beta\in\Gamma(\Omega)\\ \lambda_j\in\beta}} [1+V\omega(\beta,\lambda_j)]\,\omega(\alpha,\lambda_i)$$

$$= \sum_{\substack{\alpha\in\Gamma(\Omega)\\ \lambda_i\in\alpha}} B\omega(\alpha,\lambda) = \sum_{\substack{\alpha\in\Gamma(\Omega)\\ \lambda\in\alpha}} \omega(\alpha,\lambda),$$

proving (3.9).

3. If λ does not issue from a point of interaction, then (3.9) is trivially satisfied, by (3.18) and condition R). The proof of (3.9) and (3.12) is complete.

VI. For the proof of (3.10) assume that the discontinuity along $\lambda\in L_i(\Omega)$ is a rarefaction wave, and let $\alpha=\{\alpha_m\}_{m\in I}\in\Gamma_i(\Omega)$ with $\lambda=\alpha_m$. Lemma A3(i) and Lemma A4(ii) then imply that the discontinuities along $\alpha_0, \ldots, \alpha_m$ all are rarefaction waves, that α does not interact with any other $\beta\in\Gamma_i$ in between α_0 and α_m, and thus that α is the unique element in Γ with $\lambda\in\alpha$. Since the discontinuity along α_0 is a rarefaction wave, we have $\theta(\alpha_0)\leqq\frac{1}{n}$, by construction of $u^{(n)}$. Therefore Lemma A3(ii) yields

$$\theta(\lambda)=\theta(\alpha_m)=\Pi[1+V\theta(\mu)]\,\theta(\alpha_0)\leqq e^{V\Sigma\theta(\mu)}\,\frac{1}{n},$$

where the product and the summation are over all $\mu\in L(\Omega)$ which interact with α in between α_0 and α_m. The assertion follows from this inequality and from

$$\sum\theta(\mu)\leqq \sum_{\substack{\beta\in\Gamma(\Omega)\\ \text{with } \mu\in\beta}} \omega(\beta,\mu)\leqq \sum_{\beta\in\Gamma(\Omega)} \sup_{\nu\in\beta}\omega(\beta,\nu)\leqq C\,d(\omega,0),$$

by (3.9).

VII. We finally discuss the modifications necessary in the case that the resolution of the discontinuity $[U(x_0-), U(x_0+)]$ consists of 1- and 2-rarefaction waves. It is obvious how to treat the case where this resolution contains both rarefaction and shock waves. Assume that for $i=1,2$ the discontinuities $\gamma_i^1, \ldots, \gamma_i^{m_i}\in\Gamma_i(\Omega)$ issue from $(x_0, 0)$, and let

$$\varepsilon_i^l=\gamma_i^l-\bigcup_{\alpha\in\Gamma(\Omega)-\{\gamma_i^1,\ldots,\gamma_i^{m_i}\}}\alpha.$$

The meaning of this definition is that the γ_i^l are cut off at the first point of interaction with $\alpha\in\Gamma_i(\Omega)$. Note that by construction of $u^{(n)}$ and by Lemmas A3(i) and A4(ii) the discontinuity along ε_i^l must be a rarefaction wave, which implies that couples ε_i^l, ε_i^k do not interact with one another, by Lemma A4(i).

Now let

$$\Gamma^*(\Omega)=(\Gamma(\Omega)-\{\gamma_1^1,\ldots,\gamma_2^{m_2}\})\cup\{\varepsilon_1^1,\ldots,\varepsilon_2^{m_2}\},$$
$$\Delta^*(\Omega)=\{(\alpha,\lambda)\in\Gamma^*(\Omega)\times L(\Omega)\mid\lambda\in\alpha\},$$

and let $F^*(\Omega)$ be the set of all functions $p: \Delta^*(\Omega) \to \mathbb{R}_0^+$. Let $0 < \kappa_0 \leqq 1$. The metric on $F^*(\Omega)$ is defined by

$$d(p, q) = \sum_{\alpha \in \Gamma^*(\Omega)} \sup_{\lambda \in \alpha} \kappa(\lambda) |p(\alpha, \lambda) - q(\alpha, \lambda)|,$$

where $\kappa(\lambda) = \kappa_0$ if $\lambda \in \bigcup_{i,l} \varepsilon_i^l$, else $\kappa(\lambda) = 1$. $B: F^*(\Omega) \to F^*(\Omega)$ is defined by (3.15)–(3.19). However, in the case where ε_i^l interacts with a discontinuity of the same family, Conditions (3.16) and (3.19) are modified. We define

$$\text{(3.16a)} \quad \Lambda = \begin{cases} V[\sum_{\substack{\beta \in \Gamma^*(\Omega) \\ \text{with } \lambda_1 \in \beta}} p(\beta, \lambda_1)][\sum_{\substack{\gamma \in \Gamma^*(\Omega) \\ \text{with } \lambda_2 \in \gamma}} p(\gamma, \lambda_2)], & \text{if } \lambda_1, \lambda_2 \notin \bigcup_{i,l} \varepsilon_i^l \\ V\kappa_0[\sum_{\substack{\beta \in \Gamma(\Omega) \\ \text{with } \lambda_j \in \beta}} p(\beta, \lambda_j)]\, p(\varepsilon_i^l, \lambda_i), & \text{if } \lambda_i \in \varepsilon_i^l,\ j \neq i. \end{cases}$$

If $\alpha \in \Gamma_i^*(\Omega)$ interacts with ε_i^l, then let

$$\text{(3.19a)} \qquad Bp(\alpha > \varepsilon_i^l) = [1 + V\kappa_0\, p(\varepsilon_i^l < \alpha)]\, Bp(\alpha < \varepsilon_i^l).$$

With these definitions, (3.23) and (3.24) are replaced by

$$\sum_{\alpha \in \Gamma^*(\Omega)} |Bp(\alpha, \alpha_0) - Bq(\alpha, \alpha_0)| \leqq V[d(p, 0) + d(0, q)]\, d(p, q).$$

(3.21) and (3.22) are replaced by

$$Bp(\alpha, \alpha_m) \leqq e^{\hat{\kappa} V d(p, 0)}\, Bp(\alpha, \alpha_0)$$

and

$$\begin{aligned} &|Bp(\alpha, \alpha_m) - Bq(\alpha, \alpha_m)| \\ &\quad \leqq \hat{\kappa} V d(p, q)\, e^{\hat{\kappa} V [d(p, 0) + d(0, q)]}\, Bp(\alpha, \alpha_0) \\ &\qquad + e^{\hat{\kappa} V d(0, q)} |Bp(\alpha, \alpha_0) - Bq(\alpha, \alpha_0)|, \end{aligned}$$

where $\hat{\kappa} = \kappa_0^{-1}$ if $\alpha \in \Gamma_i(\Omega)$ interacts with at least one ε_j^l, $j \neq i$, in between α_0 and α_m, else $\hat{\kappa} = 1$. From these formulas we obtain by analogous considerations as above that B possesses a fixed point ω. The choice of the constant $V\kappa_0$ in (3.16a) and (3.19a) and the cutting off of the rarefaction discontinuities γ_i^l is justified in the proof of (3.9) for ω, where we use formulas (A5) and (A6) in Lemma A4(iii), since we can assume that $\theta(\lambda_1^-) + \theta(\lambda_2^-) \leqq \kappa_0$ if λ_j^- interacts with $\lambda_i^- \in \varepsilon_i^l$. For, just as in part IV of this proof it can be seen that $\theta(\lambda_j^-)$ can be made arbitrarily small by decreasing δ, and just as in part VI it can be seen that $\theta(\lambda_i^-)$ can be made arbitrarily small by increasing n, since the discontinuity along ε_i^l is a rarefaction wave. The proof of Lemma 3.4 is complete.

Proof of Lemma 3.3. We first show that the points of interaction of couples $\lambda_1, \lambda_2 \in L$ do not accumulate in the triangle $D(x_0, \delta)$, δ being the constant of Lemma 3.4. Let J be the set of all points of interaction of couples $\mu_1, \mu_2 \in L(\Omega)$ from which more than two segments of discontinuity issue. We show that J is finite. Let $\lambda_1, \ldots, \lambda_{m-1} \in L_j(\Omega)$ issue from the point of interaction of $\mu_1, \mu_2 \in L_i(\Omega)$ with $i \neq j$ and $m \geqq 3$. By formulas (A6) and (A7) of Lemma A4 we

have

$$V\theta(\mu_1)\,\theta(\mu_2)\geqq\sum_{l=1}^{m-1}\theta(\lambda_l)>\frac{1}{n}.$$

From (3.9) and (3.11) it follows for $l=1, 2$ that

$$\theta(\mu_l)\leqq\sum_{\substack{\alpha\in\Gamma(\Omega)\\ \text{with } \mu_1\in\alpha}}\omega(\alpha,\mu_l)\leqq\sum_{\alpha\in\Gamma(\Omega)}\sup_{\lambda\in\alpha}\omega(\alpha,\lambda)\leqq V_2.$$

From these two relation and from (3.9) we conclude that

$$\mu_l\in N=\left\{\lambda\in L(\Omega)\,\middle|\,\sum_{\substack{\alpha\in\Gamma(\Omega)\\ \text{with } \lambda\in\alpha}}\sup_{\nu\in\alpha}\omega(\alpha,\nu)>\frac{1}{nVV_2}\right\}.$$

We introduce a partial ordering $\leqq$ on L as follows. For $\nu, \mu\in L$ we have $\nu\leqq\mu$ if there exists $\alpha=\{\alpha_m\}_{m\in I}\in\Gamma$ such that $\nu=\alpha_l$, $\mu=\alpha_k$ with $l\leqq k$. Let N_0 be the set of minimal elements of N with respect to this partial ordering. Note that to every $\mu\in N$ there exists $\nu\in N_0$ with $\nu\leqq\mu$. We now show that N_0 is finite. If this has been shown it follows that J is finite. For, if $(x, t_k)\in J$, then there exist $\mu_1, \mu_2\in N$ which interact at (x, t_k), and $\nu_1, \nu_2\in N_0$ with $\nu_1\leqq\mu_1$, $\nu_2\leqq\mu_2$, and therefore also $\alpha, \beta\in\Gamma(\Omega)$ with $\nu_1\in\alpha$, $\nu_2\in\beta$, such that (x, t_k) is the point of interaction of α and β. Thus

$$J\subseteq\{(x,t)\,|\,\text{There exist } \alpha,\beta\in\Gamma(\Omega) \text{ with } \alpha\cap N_0\neq\emptyset,\ \beta\cap N_0\neq\emptyset,\ \text{which interact at } (x,t)\}.$$

The set of all $\alpha\in\Gamma$ with $\alpha\cap N_0\neq\emptyset$ is finite if N_0 is finite. Therefore, since any couple $\alpha, \beta\in\Gamma$ interact at most once, it follows that J is finite.

N_0 must be finite, since any $\alpha\in\Gamma$ contains at most one $\nu\in N_0$, hence, by definition of N_0 and by (3.11),

$$|N_0|\,\frac{1}{nVV_2}=\sum_{\nu\in N_0}\frac{1}{nVV_2}\leqq\sum_{\nu\in N_0}\sum_{\substack{\alpha\in\Gamma(\Omega)\\ \text{with } \nu\in\alpha}}\sup_{\mu\in\alpha}\omega(\alpha,\mu)$$
$$\leqq\sum_{\alpha\in\Gamma}\sup_{\mu\in\alpha}\omega(\alpha,\mu)\leqq V_2,$$

which implies that N_0 and therefore J is finite. This means that for every k the number of the elements $\lambda\in L(\Omega)$ defined on $[t_k, t_{k+1}]$ is bounded, uniformly with respect to k, and that there exists k_0 such that from any point of interaction (x, t_k) with $k\geqq k_0$ there issue exactly one $\lambda_1\in L_1(\Omega)$ and one $\lambda_2\in L_2(\Omega)$. Note further that it follows from (3.8) that

$$\text{graph } \lambda\cap D(x_0,\delta)=\emptyset \tag{3.39}$$

for all $\lambda\in L-L(\Omega)$, since $\Omega=(x_0-\delta, x_0+\delta)$. Remember that $D(x_0, \delta)$ is defined in Lemma 3.3. This implies that all the points of interaction contained in $D(x_0, \delta)$ are points of interaction of couples $\lambda_1, \lambda_2\in L(\Omega)$. But it follows from a

simple argument that these points cannot accumulate, using (3.8) and the above results about the set $L(\Omega)$. This proves the first assertion of Lemma 3.3.

(3.5) is an immediate corollary of (3.39) and (3.12).

For the proof of (3.6) note that (3.9) implies

$$\text{(3.40)}\quad \operatorname{var}[u^{(n)}(\circ, t)\,|\,(x_0-\delta+Mt, x_0+\delta-Mt)] \leqq \sum \theta(\lambda) \leqq \sum \sum_{\substack{\alpha\in\Gamma(\Omega)\\ \text{with } \lambda\in\alpha}} \omega(\alpha, \lambda)$$

for $t_k < t \leqq t_{k+1}$, where the summation is over all $\lambda \in L(\Omega)$ defined on $[t_k, t_{k+1}]$. For such λ_1, λ_2 the sets $\{\alpha\in\Gamma(\Omega)\,|\,\lambda_1\in\alpha\}$ and $\{\alpha\in\Gamma(\Omega)\,|\,\lambda_2\in\alpha\}$ are disjoint if $\lambda_1 \neq \lambda_2$, and therefore (3.6) follows from (3.40) and (3.11).

For the proof of (3.7) let $t' \leqq t_k < \ldots < t_l \leqq t$, and let $a = x_0 - \delta + Mt$, $b = x_0 + \delta - Mt$. Then

$$\begin{aligned}\text{(3.41)}\qquad \int_a^b |u^{(n)}(x,t)-u^{(n)}(x,t')|\,dx &\leqq \int_a^b |u^{(n)}(x,t)-u^{(n)}(x,t_l)|\,dx\\ &+\sum_{i=k}^{l}\int_a^b |u^{(n)}(x,t_i)-u^{(n)}(x,t_i-)|\,dx\\ &+\sum_{i=k+1}^{l}\int_a^b |u^{(n)}(x,t_i-)-u^{(n)}(x,t_{i-1})|\,dx\\ &+\int_a^b |u^{(n)}(x,t_k-)-u^{(n)}(x,t')|\,dx.\end{aligned}$$

Let $t_i < s' < s < t_{i+1}$, and $s < t$. Then, by definition of $u^{(n)}$ in the strip $\mathbb{R}\times[t_i, t_{i+1})$ and by (3.8)

$$|u^{(n)}(x,s)-u^{(n)}(x,s')| \leqq \operatorname{var}[u^{(n)}(\circ, s')\,|\,[x-M(s-s'), x+M(s-s')]],$$

hence, with $a(m) = \min[\max(a, M(s-s')m)), b]$, $b(m) = a(m+1)$,

$$\begin{aligned}\int_a^b |u^{(n)}(x,s)-u^{(n)}(x,s')|\,dx &= \sum_{m=-\infty}^{\infty}\int_{a(m)}^{b(m)} |u^{(n)}(x,s)-u^{(n)}(x,s')|\,dx\\ &\leqq \sum_{m=-\infty}^{\infty}\int_{a(m)}^{b(m)} \operatorname{var}[u^{(n)}(\circ,s')\,|\,[x-M(s-s'), x+M(s-s')]]\,dx\\ &\leqq M(s-s')\sum_{m=-\infty}^{\infty} \operatorname{var}[u^{(n)}(\circ,s')\,|\,[a(m)-M(s-s'), b(m)+M(s-s')]]\\ &\leqq M(s-s')\,3\operatorname{var}[u^{(n)}(\circ,s')\,|\,[a-M(s-s'), b+M(s-s')]]\\ &\leqq M(s-s')\,3V_2.\end{aligned}$$

In the last step we used (3.6) and the fact that $[a-M(s-s'), b+M(s-s')] \subseteq (x_0 - \delta + Ms', x_0+\delta-Ms')$, since $s<t$. Letting tend s to t_{i+1} or to t and s' to t_i or to t', (3.7) follows from this inequality, from (3.41), and from (2.15). This proves Lemma 3.3.

4. The Entropy Condition

Lemma 4.1. *The weak solution u constructed in the preceding section satisfies the entropy condition* (1.5).

Proof. It is clear that it suffices to show that

$$\int_0^T \int_{-\infty}^{\infty} [\eta(u^{(n)}(x,t))\,\phi_t(x,t) + q(u^{(n)}(x,t))\,\phi_x(x,t)]\,dx\,dt \geqq -\frac{C}{n} \tag{4.1}$$

for every convex entropy pair (η, q), for every function $\phi \geqq 0$ with compact support in $\mathbb{R} \times (0, T)$, and for a constant C independent of n. The definition of $u^{(n)}$ in Sect. 2 implies that

$$\begin{aligned} \int_0^T \int_{-\infty}^{\infty} (\eta\phi_t + q\phi_x)\,dx\,dt &= -\sum_{\lambda \in L} \int_{\text{graph } \lambda} (\nu_2[\eta] + \nu_1[q])\,\phi\,ds \\ &\quad + \sum_{k \geqq 1} \int_{-\infty}^{\infty} [\eta(u^{(n)}(x, t_k-)) - \eta(u^{(n)}(x, t_k+))]\,\phi(x, t_k)\,dx, \end{aligned} \tag{4.2}$$

with $[\eta] = \eta(u^r) - \eta(u^l)$, $[q] = q(u^r) - q(u^l)$, u^r and u^l being the constant values of $u^{(n)}$ to the right and to the left of graph λ, and with $\nu = (\nu_1, \nu_2)$ being the unit normal of graph λ with $\nu_1 > 0$. Since

$$-\sum_{\lambda \in L} \int_{\text{graph } \lambda} (\nu_2[\eta] + \nu_1[q])\,\phi\,ds = \sum_{\lambda \in L} \int_{\text{graph } \lambda} (\partial_t \lambda[\eta] - [q])\,\nu_1\,\phi\,ds,$$

(4.1) results from (4.2) and from the following three estimates

$$\left| \sum_{k \geqq 1} \int_{-\infty}^{\infty} [\eta(u^{(n)}(x, t_k-)) - \eta(u^{(n)}(x, t_k+))]\,\phi(x, t_k)\,dx \right| \leqq \frac{C}{n}. \tag{4.3}$$

Let L' denote the set of all $\lambda \in L$ along which the discontinuity of $u^{(n)}$ is a rarefaction wave. Then

$$\left| \sum_{\lambda \in L'} \int_{\text{graph } \lambda} (\partial_t \lambda[\eta] - [q])\,\nu_1\,\phi\,ds \right| \leqq \frac{C'}{n^2}, \tag{4.4}$$

for a suitable constant independent of n. Along $\lambda \in L - L'$ we have

$$\partial_t \lambda[\eta] - [q] \geqq 0. \tag{4.5}$$

It thus remains to prove (4.3)–(4.5). (3.0) yields

$$|[\eta(u^{(n)}(x, t_k-)) - \eta(u^{(n)}(x, t_k+))]\,\phi(x, t_k)| \leqq C\,|u^{(n)}(x, t_k-) - u^{(n)}(x, t_k+)|,$$

which together with (2.15) implies (4.3). For the proof of (4.4) note that it has been shown in [6] that

$$|\partial_t \lambda[\eta] - [q]| \leqq C\theta(\lambda)^3.$$

(3.9), (3.10), (3.11), and the fact that $\operatorname{supp} u^{(n)} \cap (\mathbb{R} \times (0, T))$ is covered by finitely many triangles $D(x_i, \delta_i)$ imply

$$\begin{aligned}
\sum_{\lambda \in L'} \int_{\text{graph } \lambda} |(\partial_t \lambda [\eta] - [q])\, v_1 \phi|\, ds &\leqq \sum_{\lambda \in L'} C\theta(\lambda)^3 \int_{\text{graph } \lambda} ds \\
&\leqq \frac{C'}{n^2} \sum_{\lambda \in L'} \theta(\lambda) \int_{\text{graph } \lambda} ds \leqq \frac{C'}{n^2} \sum_{\lambda \in L'} \Big[\Big(\sum_{\substack{\alpha \in \Gamma \\ \text{with } \lambda \in \alpha}} \omega(\alpha, \lambda) \Big) \int_{\text{graph } \lambda} ds \Big] \\
&\leqq \frac{C'}{n^2} \sum_{\alpha \in \Gamma} \Big(\sum_{\lambda \in \alpha} \Big(\omega(\alpha, \lambda) \int_{\text{graph } \lambda} ds \Big) \Big) \leqq \frac{C'}{n^2} \sum_{\alpha \in \Gamma} \Big(\int_{\text{graph } \alpha} ds \sup_{\lambda \in \alpha} \omega(\alpha, \lambda) \Big) \\
&\leqq \frac{C'}{n^2} (1 + M^2)^{1/2}\, T \sum_{\alpha \in \Gamma} \sup_{\lambda \in \alpha} \omega(\alpha, \lambda) \leqq \frac{C''}{n^2}.
\end{aligned}$$

Here we used $\int_{\text{graph } \alpha} ds \leqq (1 + M^2)^{1/2}\, T$ which follows from $|\partial_t \lambda| \leqq \frac{1}{2} M$, implied by (3.4) and Lemma A1(i). This proves (4.4). For the proof of (4.5) we assume that $\lambda \in L_1 \setminus L'$. The other case is treated in the same way. Let w be the state on the left of λ, and let v be the state on the right of λ. Since the discontinuity along λ is a shock, we have $v \in \operatorname{graph} c_{1w}$ with $v_1 > w_1$. For the proof of (4.5) it therefore suffices to show that

$$\frac{d}{dv_1} (s[\eta] - [q]) \geqq 0, \tag{4.6}$$

with $s = \partial_t \lambda$. Proceeding as in [6], we have with $v' = (v_1, v_2)' = (1, c'_{1w}(v_1))$

$$\begin{aligned}
\frac{d}{dv_1} (s[\eta] - [q]) &= \frac{d}{dv_1} [s(\eta(v) - \eta(w)) - (q(v) - q(w))] \\
&= s'(\eta(v) - \eta(w)) + s \nabla \eta(v)\, v' - \nabla q(v)\, v'.
\end{aligned} \tag{4.7}$$

Using the fact that $\nabla q(v) = -\nabla \eta(v) f'(v)$, cf. [6], we obtain

$$\nabla q(v)\, v' = -\nabla \eta(v) f'(v)\, v'. \tag{4.8}$$

From the Rankine-Hugoniot condition (2.1) we obtain by differentiation

$$s'(v - w) + s v' = -f'(v)\, v',$$

which together with (4.8) yields

$$\nabla q(v)\, v' = s' \nabla \eta(v)(v - w) + s \nabla \eta(v)\, v'.$$

Combining this equation with (4.7) gives

$$\frac{d}{dv_1} (s[\eta] - [q]) = -s'(\eta(w) - \eta(v) - \nabla \eta(v)(w - v)).$$

Since η is convex, we have

$$\eta(w) - [\eta(v) + \nabla \eta(v)(w - v)] \geqq 0.$$

(4.6) thus follows from $s' < 0$, by Lemma A1(ii).

Appendix

Lemma A1. *Let $v=(v_1, v_2)$ and $u=(u_1, c_{iv}(u_1))$ be the constant values on either side of the discontinuity curve $x=x(t)$ of an i-discontinuity.*

(i) *There exists τ between v_1 and u_1 such that*

$$s=-\frac{c_{iv}(u_1)-c_{iv}(v_1)}{u_1-v_1}=(-1)^i\sigma'(\tau)^{1/2},$$

where $s=s(u_1, v_1)=dx/dt$ is the speed of the discontinuity.

(ii) $$(-1)^{i-1}\partial_{u_1}s(u_1, v_1)<0$$

(iii) $$c'_{iv}(\tau)=(-1)^{i-1}\sigma'(\tau)^{1/2}\frac{1}{2}\left[\frac{\sigma'(\tau)^{1/2}|\tau-v_1|^{1/2}}{|\sigma(\tau)-\sigma(v_1)|^{1/2}}+\frac{|\sigma(\tau)-\sigma(v_1)|^{1/2}}{\sigma'(\tau)^{1/2}|\tau-v_1|^{1/2}}\right]$$
$$=(-1)^{i-1}\frac{1}{2}\left[\frac{\sigma'(\tau)}{|s(\tau, v_1)|}+|s(\tau, v_1)|\right]$$

(iv) $$|c'_{iv}(\tau)|\geqq\sigma'(\tau)^{1/2}.$$

Proof. (i) follows from (2.3) and the mean value theorem.

(iii) Follows from (2.3) by calculation, (iv) follows from (iii) and the formula $a+1/a=(a^2+1)/a\geqq 2$ if $a>0$. For the proof of (ii) note that

(A1) $$\partial_{u_1}s(u_1, v_1)=(c_{iv}(u_1)-c_{iv}(v_1))/(u_1-v_1)^2-c'_{iv}(u_1)/(u_1-v_1)$$
$$=(s+c'_{iv}(u_1))/(v_1-u_1).$$

From (i) and the fact that $\sigma''>0$ it follows that $\sigma'(u_1)<s^2$ for $u_1<v_1$ and $\sigma'(u_1)>s^2$ for $u_1>v_1$. Using this fact, (iii), and (2.3) we conclude that

$$(-1)^{i-1}c'_{iv}(u_1)=\frac{1}{2}\left(\frac{\sigma'(u_1)}{|s|}+|s|\right)\begin{cases}<|s|, & \text{for } u_1<v_1\\ >|s|, & \text{for } u_1>v_1.\end{cases}$$

As $|s|=-(-1)^{i-1}s$, (ii) follows from this relation and from (A1).

Lemma A2. (i) *For $v=(v_1, v_2)\in\mathbb{R}^2$ let $\eta_{1v}, \eta_{2v}\colon \mathbb{R}\to\mathbb{R}$ be defined by*

$$\eta_{1v}(\tau)=v_2+\int_{v_1}^{\tau}\sigma'(s)^{1/2}\,ds, \qquad \eta_{2v}(\tau)=v_2-\int_{v_1}^{\tau}\sigma'(s)^{1/2}\,ds.$$

Then to every compact set $[r, v_1]$ there exists a constant C, independent of n, with

$$|(c^n_{iv})'(\tau)-\eta'_{iv}(\tau)|\leqq\frac{C}{n^2}, \qquad |c^n_{iv}(\tau)-\eta_{iv}(\tau)|\leqq\frac{C}{n^2}|\tau-v_1|,$$

for every $r\leqq\tau\leqq v_1$ with the exception of the isolated points where the derivative of c^n_{iv} does not exist.

(ii) *For every $u, v\in\mathbb{R}^2$ and every nonnegative integers m, n the graphs of c^n_{1u} and c^m_{2v} intersect at exactly one point w.*

(iii) *For every compact set $W\subseteq\mathbb{R}^2$ there exists a constant V, independent of n, m, with*

$$l^n_1(w, u)+l^m_2(w, v)\leqq V|u-v|,$$

for every $u, v \in W$, where $l_1^m(w, u)$ and $l_2^m(w, v)$, respectively, denote the arclength of graph c_{1u}^n between u and w, and the arclength of graph c_{2v}^m between v and w, respectively.

Sketch of Proof. Note that the graph of η_{iv} is equal to the level curve of the i-Riemann invariant defined in (1.3), (1.4) through v. In [5] it is proved that

$$c_{iv}(\tau) - \eta_{iv}(\tau) = O(|\tau - v_1|^3), \qquad c'_{iv}(\tau) - \eta'_{iv}(\tau) = O(|\tau - v_1|^2).$$

(i) Follows from this relation, from the fact that η'_{iv} is independent of v_2, and from the definition of the function c_{iv}^n in (2.5), (2.6), where it is constructed out of functions c_{iv^k} for suitable $v^k \in \mathbb{R}^2$. It also follows from this construction and from Lemma A1(iv) that

$$(A2) \qquad (c_{1u}^n)'(\tau) \geqq \eta'_{1u}(\tau), \qquad (c_{2v}^m)'(\tau) \leqq \eta'_{2v}(\tau).$$

(ii) Follows from Condition C) and from (A2) by some simple geometrical arguments. We leave the details to the reader.

Also the statements of the following lemmas are consequences of elementary properties of the graphs of the functions c_{iv}. The interaction estimates in these lemmas are mainly corollaries of the estimates in [3], with the exception of (A8), which we believe to be new. Figure 1 shows the interaction of type 1 studied in Lemma A3, and Fig. 2 shows the interaction of type 2 studied in Lemma A4.

Lemma A3. *Let $\lambda_1^- \in L_1$, $\lambda_2^- \in L_2$ interact, assume that the discontinuity at the point of interaction is resolved by resolution of type (0, 0), and let $\lambda_1 \in L_1$, $\lambda_2 \in L_2$ issue from the point of interaction.*

(i) *Then the discontinuity along λ_i is a rarefaction wave if and only if the discontinuity along λ_i^- is a rarefaction wave, for $i = 1, 2$.*

(ii) *Let u^- be the constant state in between λ_1^- and λ_2^-. Then there exists to every compact set $W \times [0, \varepsilon_1] \times [0, \varepsilon_2] \subset \mathbb{R}^2 \times \mathbb{R}_0^+ \times \mathbb{R}_0^+$ a constant $V > 0$ with*

$$(A3) \qquad \vartheta(\lambda_1) \leqq (1 + V\theta(\lambda_2^-))\,\theta(\lambda_1^-), \qquad \theta(\lambda_2) \leqq (1 + V\theta(\lambda_1^-))\,\theta(\lambda_2^-)$$

for all $(u^-, \theta(\lambda_1^-), \theta(\lambda_2^-)) \in W \times [0, \varepsilon_1] \times [0, \varepsilon_2]$.

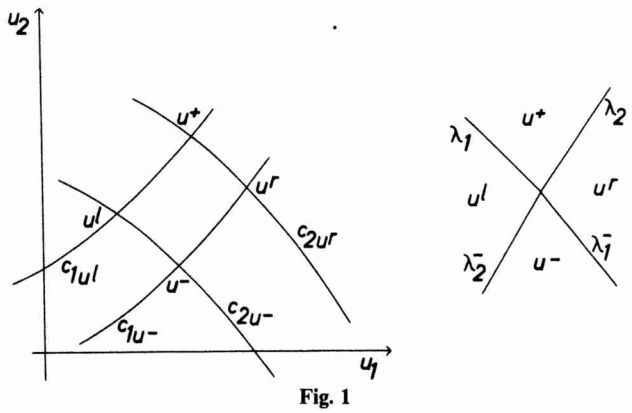

Fig. 1

Lemma A4. *Let λ_1^-, $\lambda_2^- \in L_i$ interact, and assume that the discontinuity at the point of interaction is resolved by resolution of type $(0, n)$ if $i=1$, and by resolution of type $(n, 0)$ if $i=2$. Let $\lambda \in L_i$, $\lambda^1, \ldots, \lambda^m \in L_j$, $j \neq i$, issue from the point of interaction with $|\partial_t \lambda^m| < \ldots < |\partial_t \lambda^1|$.*

(i) *Then the discontinuities along λ_1^- and λ_2^- cannot both be rarefaction waves.*

(ii) *The discontinuity along λ is a shock.*

(iii) *To every compact set $W \times [0, \varepsilon_1] \times [0, \varepsilon_2] \subseteq \mathbb{R}^2 \times \mathbb{R}_0^+ \times \mathbb{R}_0^+$ there exists a constant $V > 0$, independent of n, such that for every $(u^-, \theta(\lambda_1^-), \theta(\lambda_2^-)) \in W \times [0, \varepsilon_1] \times [0, \varepsilon_2]$*

$$\theta(\lambda) \leqq \theta(\lambda_1^-) + \theta(\lambda_2^-) + V \min(1, \theta(\lambda_1^-) + \theta(\lambda_2^-))\, \theta(\lambda_1^-)\, \theta(\lambda_2^-) \tag{A4}$$

if both discontinuities along λ_1^- and λ_2^- are shocks,

$$\theta(\lambda) \leqq \theta(\lambda_1^-) + V \min(1, \theta(\lambda_1^-) + \theta(\lambda_2^-))\, \theta(\lambda_2^-))\, \theta(\lambda_1^-) \tag{A5}$$

if the discontinuity along λ_2^- is a rarefaction wave,

$$\sum_{l=1}^{m} \theta(\lambda^l) \leqq V \min(1, \theta(\lambda_1^-) + \theta(\lambda_2^-))\, \theta(\lambda_1^-)\, \theta(\lambda_2^-) \tag{A6}$$

and

$$\theta(\lambda^l) = \frac{1}{n},\ l = 1, \ldots, m-1, \qquad \theta(\lambda^m) \leqq \frac{1}{n}. \tag{A7}$$

(iv) *Assume that $|\partial_t \lambda_1^-| > |\partial_t \lambda_2^-|$. Then to every compact set $W \times [0, \varepsilon_2] \subseteq \mathbb{R}^2 \times \mathbb{R}_0^+$ there exist constants $0 < V_1 < 1$, $\varepsilon_1 > 0$, independent of n, such that for all $(u^-, \theta(\lambda_2^-)) \in W \times [0, \varepsilon_2]$*

$$\sum_{l=1}^{m} \theta(\lambda^l) \leqq V_1\, \theta(\lambda_1^-) \quad \text{if } \theta(\lambda_1^-) \leqq \varepsilon_1. \tag{A8}$$

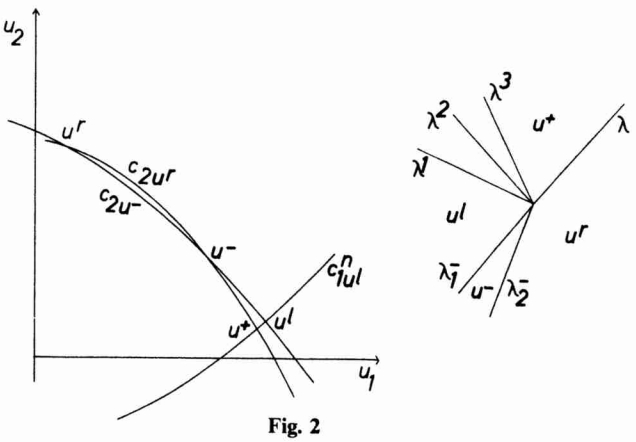

Fig. 2

Proof. We consider the case $i=2$ and assume that $\partial_t \lambda_1^- > \partial_t \lambda_2^-$. u^l, u^-, and u^r are then the states on the left hand side of λ_1^-, in between λ_1^- and λ_2^-, and on the right hand side of λ_2^-. u^+ is the state on the left hand side of λ. We have

$$u^- \in \text{graph}\, c_{2u^r}, \quad u^l \in \text{graph}\, c_{2u^-}, \tag{A9}$$

$$u^+ \in \text{graph}\, c_{2u^r} \cap \text{graph}\, c^n_{1u^l}. \tag{A10}$$

Since the discontinuities $[u^-, u^r]$ along λ_2^- and $[u^l, u^-]$ along λ_1^- interact, it follows for the speeds of these discontinuities that

$$s(u_1^l, u_1^-) > s(u_1^-, u_1^r).$$

Since $s(u_1^-, u_1^r) = s(u_1^r, u_1^-)$, by (2.2), it follows from Lemma A1(ii) that

$$u_1^l > u_1^r, \tag{A11}$$

which implies that not both discontinuities $[u^-, u^r]$ and $[u^l, u^-]$ can be rarefaction waves. By (A9) we have $u^- \in \text{graph}\, c_{2u^r}$, hence, by (2.3b), $u^r \in \text{graph}\, c_{2u^-}$. Since also $u^l \in \text{graph}\, c_{2u^-}$, by (A9), (A11) implies for $\tau < u_1^r$

$$c_{2u^r}(\tau) > c_{2u^r}(u_1^r) = u_2^r = c_{2u^-}(u_1^r) > c_{2u^-}(u_1^l) = u_2^l > c^n_{1u^l}(\tau),$$

since c_{2u^r}, c_{2u^-} are monotonically decreasing, $c^n_{1u^l}$ is monotonically increasing. This shows that graph c_{2u^r} and graph $c^n_{1u^l}$ cannot intersect to the left of u^r. (A10) thus yields $u_1^+ > u_1^r$, which means that the discontinuity $[u^+, u^r]$ along λ is a shock.

The remaining parts of the proof are left to the reader.

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Received June 7, 1984