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Filtrations of Principal Indecomposable Modules of Frobenius Kernels of Reductive Groups

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Let G be a simply connected, semi-simple algebraic group over an algebraically closed field K of characteristic $p>0$, and $\mathscr{U}_n$ the hyperalgebra of the n-th Frobenius kernel (It is a $p^{n\dim G}$-dimensional subalgebra of the hyperalgebra U_K of G). Let T be a maximal torus of G, let X be the character group of T, and W the Weyl group. To each dominant weight $\mu\in X^+\subset X$ associate the Weyl module $V(\mu)$ with highest weight μ. To each "restricted" dominant weight $\lambda\in X_n\subset X$ there corresponds a simple $\mathscr{U}_n$-module $M(\lambda,n)$ and a principal indecomposable $\mathscr{U}_n-T$-module $Q(\lambda,n)$ with $M(\lambda,n)$ as its unique top and bottom composition factor. J. Ballard proved for $p\geqq 3h-3$ (where h is the Coxeter number of G) that each $Q(\lambda,n)$ with $\lambda\in X_n$ admits a G-structure (cf. [2]). This bound was improved to $p\geqq 2h-2$ by J. Jantzen in [10], 4.5. He also proved in [10], 5.9 that each $Q(\lambda,n)$ admits a Weyl filtration, i.e. a G-filtration with subquotients of the form $V(\mu)$, $\mu\in X^+$. More generally such filtrations exist also for all $Q(\lambda,n)\otimes V(\nu)^{(p^n)}$ with $\nu\in X^+$, where the exponent (p^n) denotes a Frobenius twist of a module.

The purpose of this paper is to deal with the multiplicity of $V(\mu)$ occuring in these Weyl filtrations. This problem is related to the question of multiplicities of the simple $\mathscr{U}_n-T$-modules $\hat{M}(\mu,n)$, $\mu\in X$ in composition series of the universal $\mathscr{U}_n-T$-modules $\hat{Z}(\lambda,n)$ with highest weight $\lambda\in X$.

In Sect. 1 we introduce the affine Weyl group $W_{a,p}$ of G, discuss its alcove geometry and a partial order on $W_{a,p}$ which induces an order $\uparrow\uparrow$ on X.

In Sect. 2 we mainly discuss the Weyl filtration of $Q(\lambda,n)$. First we define for each $\lambda\in X_n$ the set $\mathscr{S}_{\lambda,n}=\{\mu\in X^+\mid\mu\uparrow\uparrow\lambda^0-\delta\}$ where $\lambda^0=\sigma_0\lambda-\delta+p^n\delta$. (Here $\sigma_0\in W$ is the longest element, and δ is half the sum of all positive roots.) For each $\mu\in\mathscr{S}_{\lambda,n}$ set b_μ^λ equal to the multiplicity of $\hat{M}(\lambda,n)$ as composition factor of $\hat{Z}(\mu+p^n\delta,n)$. We then prove for $p\geqq 2h-2$, for all $\sigma\in W$, $\lambda\in X_n$, $\mu\in\mathscr{S}_{\lambda,n}$, $\nu\in X^+$ with $\sigma\cdot\mu+p^n(\nu+\delta)\in X^+$ that the multiplicity of $V(\sigma\cdot\mu+p^n(\nu+\delta))$ in a Weyl filtration of $Q(\lambda,n)\otimes V(\nu)^{(p^n)}$ is equal to

$$\sum_{\tau\in W}\det(\tau)[\hat{Z}(\sigma\cdot\mu+p^n(\nu+\delta),n):\hat{M}(\lambda+p^n\tau\cdot\nu,n)].$$

(Here we use the dot notation $\sigma\cdot\mu=\sigma(\mu+\delta)-\delta$.)

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This is applied in Sect. 3 in the case $n=1$ to prove the following main theorem:

Suppose $p \geqq 2h-2$. *Let* $\lambda \in X_1$ *be p-regular. Then*

(1) $b^{\lambda}_{\lambda^0-\delta}=1$;

(2) *If* $\mu, \mu' \in \mathscr{S}_{\lambda,1}$ *with* $\mu' \uparrow\uparrow \mu$, *then* $b^{\lambda}_{\mu} \leqq b^{\lambda}_{\mu'}$.

As a corollary we get $b^{\lambda}_{\mu}>0$ for all $\mu \in \mathscr{S}_{\lambda,1}$.

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1. The Affine Weyl Group $W_{a,p}$

Let G be a simply connected, semi-simple algebraic group over an algebraically closed field K of characteristic $p>0$, let T be a maximal torus in G and B a Borel subgroup of G containing T. Let Φ be the root system of (G, T), and let $\Delta=\{\alpha_1, \ldots, \alpha_l\}$ be the set of all simple roots such that B corresponds to the set Φ^+ of positive roots. Set $\Phi^- = -\Phi^+$. We assume that Φ is irreducible and that G, B, T are defined and split over the prime field $\mathbb{F}_p$.

Let W be the Weyl group of G and X the character group of T. Then W acts on X and on $E=X\otimes\mathbb{R}$, and there is on E a positive defined inner product $(\ ,\ )$ invariant under W (and unique up to a scalar multiple with this property). For each root α define the dual root $\alpha^\vee$ as $\alpha^\vee=\dfrac{2\alpha}{(\alpha,\alpha)}$. Define the fundamental weights ω_{α_i} by $(\omega_{\alpha_i}, \alpha_j^\vee)=\delta_{ij}$, $1\leqq i,j\leqq l$. Then we have $X=\sum_{i=1}^{l}\mathbb{Z}\omega_{\alpha_i}$, we call $X^+=\sum_{i=1}^{l}\mathbb{Z}^+\omega_{\alpha_i}$ the set of dominant weights and $X_{rt}=\sum_{i=1}^{l}\mathbb{Z}\alpha_i$ the root lattice of G. Denote the index of connection by $f=[X: X_{rt}]$ (=determinant of the Cartan matrix of Φ), and write $\delta=\sum_{i=1}^{l}\omega_{\alpha_i}$ (=half the sum of positive roots).

For each root β let σ_β be the reflection with $\sigma_\beta(\chi)=\chi-(\chi,\beta^\vee)\beta$ for all $\chi\in E$, and let $P_\beta\subset E$ be the reflecting hyperplane, i.e. $P_\beta=\{\chi\in E|(\chi,\beta^\vee)=0\}=\{\chi\in E|\sigma_\beta(\chi)=\chi\}$. Set $\sigma_i=\sigma_{\alpha_i}$. Then W is a Coxeter group with generators $\{\sigma_1,\ldots,\sigma_l\}$. The connected components of $E-\bigcup_{\beta\in\Phi^+}P_\beta$ are called Weyl chambers. The group W acts simply transitively on the set of Weyl chambers. The closure of any Weyl chamber is a fundamental region for the action of W on E. The open region $E^+=\{\chi\in E|(\chi,\alpha_i^\vee)>0 \text{ for } 1\leqq i\leqq l\}$ is a Weyl chamber which we call the fundamental Weyl chamber. Let $\sigma_0\in W$ be the longest element of W and $\alpha_0\in\Phi^+$ be the highest short root. Denote by h the Coxeter number, $h=(\delta,\alpha_0^\vee)+1$.

Let $W_{a,p}$ be the affine Weyl group of G relative to p, i.e. the group generated by all reflections $S_{\beta,m}$ $(\beta\in\Phi^+,\ m\in\mathbb{Z})$ where $S_{\beta,m}(\chi)=\chi-((\chi,\beta^\vee)$

$-mp)\beta$ (clearly $S_{\beta,0}=\sigma_\beta$). Then $W_{a,p}$ is a Coxeter group on generators $\{S_0, S_1, \ldots, S_l\}$ where $S_i=\sigma_i$, $1\leqq i\leqq l$ and $S_0=S_{\alpha_0,1}$. Denote by T_v for each v the translation on E sending χ to $\chi+v$. Then $W_{a,p}$ is also equal to the semidirect product of W with the normal subgroup (isomorphic to X_{rt}) consisting of all translation T_{pv}, $v\in X_{rt}$. The following properties of the reflections in $W_{a,p}$ are well known and easily proved.

(1.1) **Lemma.** *Suppose* $\beta, v\in\Phi^+$, $m,n\in\mathbb{Z}$, $\sigma\in W$ *and* $v\in X$. *Then*

(1) $S_{\beta,m}^2=1$

(2) $S_{\beta,m}=S_{-\beta,-m}$

(3) $\sigma S_{\beta,m}\sigma^{-1}=S_{\sigma(\beta),m}$

(4) $T_{pv}S_{\beta,m}T_{-pv}=S_{\beta,m+(v,\beta^v)}$

(5) $S_{\gamma,n}S_{\beta,m}S_{\gamma,n}=S_{\sigma_\gamma(\beta),m-n(\gamma,\beta^v)}$.

Define the "dot" action of W and $W_{a,p}$ on E as follows:

$$\sigma\cdot\chi=\sigma(\chi+\delta)-\delta \quad\text{and}\quad w\cdot\chi=w(\chi+\delta)-\delta \qquad \forall\,\chi\in E,\ \sigma\in W,\ w\in W_{a,p}.$$

Let $H_{\beta,m}$ ($\beta\in\Phi^+$, $m\in\mathbb{Z}$) denote the reflecting hyperplane with respect to $S_{\beta,m}$ under the "dot" action. Then the connected components of $E-\bigcup_{\substack{\beta\in\Phi^+\\ m\in\mathbb{Z}}} H_{\beta,m}$ are called alcoves. In particular, the open region C_0 bounded by the $H_{\alpha_i,0}$ ($1\leqq i\leqq l$) and $H_{\alpha_0,1}$ is an alcove which we call the lowest dominant alcove. The closure of any alcove (e.g. $\bar{C}_0$) is a fundamental region for the "dot" action of $W_{a,p}$ on E and $W_{a,p}$ operates simply transitively on the set $\mathscr{C}$ of all alcoves. By sending w to $w\cdot C_0$ we get therefore a $1-1$ correspondence between $W_{a,p}$ and the set $\mathscr{C}$. If $C=w\cdot C_0$ for some $w\in W_{a,p}$ and $\lambda\in C_0\cap X$, then we denote by $\lambda_c=w\cdot\lambda$ the mirror image of λ in C.

Any $w\in W_{a,p}$ can be written in the form $w=S_{i_1}\ldots S_{i_t}$ with $0\leqq i_1,\ldots,i_t\leqq l$. If t is minimal, we call this a reduced decomposition of w and set $l(w)=t$. This length is equal to the number of reflecting hyperplanes that separate C_0 from $w\cdot C_0$. The restriction of $l(\ )$ to W is the usual length function on W.

Let $W_{a,p}^+=\{w\in W_{a,p}\mid l(S_iw)>l(w), 1\leqq i\leqq l\}$. This is a set of distinguished left coset representatives for $W_{a,p}/W$. Let $\mathscr{C}^+=\{C\in\mathscr{C}\mid(\chi+\delta,\alpha_i^v)>0,\ 1\leqq i\leqq l, \forall\,\chi\in C\}$ be the set of all alcoves which lie in $E^+-\delta$. Then the restriction of the $1-1$ correspondence $w\mapsto w\cdot C_0$ to $W_{a,p}^+$ is a $1-1$ correspondence between $W_{a,p}^+$ and $\mathscr{C}^+$.

Let C be an alcove. Then (for each $\beta\in\Phi^+$) the number $n_\beta=\left[\dfrac{(\chi+\delta,\beta^v)}{p}\right]$ is independent of the choice of χ inside C. Therefore

$$d(C)=\sum_{\beta\in\Phi^+} n_\beta \quad\text{and}\quad \bar{d}(C)=\sum_{\beta\in\Phi}|n_\beta|$$

are well defined numbers. For $C\in\mathscr{C}^+$ obviously $d(C)=\bar{d}(C)$. For any $C=w\cdot C_0$ with $w\in W_{a,p}$ one has $\bar{d}(C)=l(w)$ and $\bar{d}(C)$ is equal to the number of reflecting hyperplanes separating C and C_0.

Following Verma (cf. [13]) we introduce partial orders on X, $\mathscr{C}$ and $W_{a,p}$. For $\lambda\in X$ and $\beta\in\Phi^+$, $m\in\mathbb{Z}$ we set $S_{\beta,m}\cdot\lambda\uparrow\lambda$ if $(\lambda+\delta,\beta^v)\geqq mp$ and $\lambda\uparrow S_{\beta,m}\cdot\lambda$ if

$(\lambda+\delta, \beta^\vee) \leqq mp$. Let $\uparrow$ be the partial order on X generated by this, i.e. for $\lambda, \mu \in X$ the relation $\lambda \uparrow \mu$ holds if and only if there is a sequence $\lambda_0=\lambda$, $\lambda_1, \ldots, \lambda_k=\mu$ in X and reflections $S_{\beta_i, m_i} \in W_{a,p}$ with $\lambda_{i-1}=S_{\beta_i, m_i} \cdot \lambda_i \uparrow \lambda_i$ for $1 \leqq i \leqq k$. Obviously $\lambda \uparrow \mu$ implies $\lambda \leqq \mu$ and $\lambda \in W_{a,p} \cdot \mu$.

For any alcove $C \in \mathscr{C}$ and any reflection $S_{\beta,m} \in W_{a,p}$ (with $\beta \in \Phi^+$, $m \in \mathbb{Z}$) we set $S_{\beta,m} \cdot C \underset{A}{\leqq} C$ if $(\chi+\delta, \beta^\vee) > mp$ for all $\chi \in C$ and $C \underset{A}{\leqq} S_{\beta,m} \cdot C$ otherwise. Let $\underset{A}{\leqq}$ be the partial order on $\mathscr{C}$ generated by this, i.e. for $C, C' \in \mathscr{C}$ we have $C \underset{A}{\leqq} C'$ if and only if there is a sequence $C_0=C, C_1, \ldots, C_k=C'$ in $\mathscr{C}$ and reflections $S_{\beta_i, m_i} \in W_{a,p}$ with $C_{i-1}=S_{\beta_i, m_i} \cdot C_i \underset{A}{\leqq} C_i$ for $1 \leqq i \leqq k$. It is easy to show that this coincides with Lusztig's definition in [12].

For $w, w' \in W_{a,p}$ we set $w \underset{A}{\leqq} w'$ if and only if $w \cdot C_0 \underset{A}{\leqq} w' \cdot C_0$.

Let $C \in \mathscr{C}$, $\lambda \in \bar{C} \cap X$ and $w \in W_{a,p}$. The definitions imply easily: If $C \underset{A}{\leqq} w \cdot C$, then $\lambda \uparrow w \cdot \lambda$. If $\lambda \uparrow w \cdot \lambda$, then $C' \underset{A}{\leqq} w \cdot C$ for some alcove C' with $\lambda \in \bar{C}'$. If λ belongs to the upper (resp. lower) closure of C (i.e. if $(\lambda+\delta, \beta^\vee)=mp$ with $\beta \in \Phi^+$ and $m \in \mathbb{Z}$ implies $(\chi+\delta, \beta^\vee) < mp$ resp. $(\chi+\delta, \beta^\vee) > mp$ for all $\chi \in C$), then $C \underset{A}{\leqq} C'$ (resp. $C' \underset{A}{\leqq} C$) for any C' with $\lambda \in \bar{C}'$ by [7], Lemma 6(ii). Thus we get

(1.2) **Lemma.** *If $\lambda \in X$ belongs to the upper (resp. lower) closure of $C \in \mathscr{C}$, then for all $w \in W_{a,p}$*

$$\lambda \uparrow w \cdot \lambda \Leftrightarrow C \underset{A}{\leqq} w \cdot C$$

$$(\textit{resp. } w \cdot \lambda \uparrow \lambda \Leftrightarrow w \cdot C \underset{A}{\leqq} C).$$

The following properties of $\uparrow$ and $\underset{A}{\leqq}$ are well known:

(1.3) **Lemma.** *Suppose $C, C' \in \mathscr{C}$ and $v \in X$, $\beta \in \Phi^+$. Then we have*

(1) $C' \underset{A}{\leqq} C \Leftrightarrow T_{pv} C' \underset{A}{\leqq} T_{pv} C$

(2) $C' \underset{A}{\leqq} C \Leftrightarrow \sigma_0 \cdot C \underset{A}{\leqq} \sigma_0 \cdot C'$

(3) $C \underset{A}{\leqq} T_{p\beta} C$

(4) *If $C \in \mathscr{C}^+$, then $\sigma_0 \cdot C \underset{A}{\leqq} \sigma \cdot C \underset{A}{\leqq} C$ for all $\sigma \in W$.*

(1.4) **Lemma.** *Suppose $\lambda, \mu \in X$ and $v \in X$, $\beta \in \Phi^+$. Then we have*

(1) $\lambda \uparrow \mu \Leftrightarrow T_{pv} \lambda \uparrow T_{pv} \mu$

(2) $\lambda \uparrow \mu \Leftrightarrow \sigma_0 \cdot \mu \uparrow \sigma_0 \cdot \lambda$

(3) $\mu \uparrow T_{p\beta} \mu$

(4) *If $\mu \in X^+ - \delta$, then $\sigma_0 \cdot \mu \uparrow \sigma \cdot \mu \uparrow \mu$ for all $\sigma \in W$.*

The following properties of $\underset{A}{\leqq}$ were proved in [7], Lemma 4 and Corollary 1 to Satz 9.

(1.5) **Lemma.** *Suppose* $C \in \mathscr{C}$.

(1) *If* $S_{\beta,m} \in W_{a,p}$ *is a reflection, then* $S_{\beta,m} \cdot C \underset{\overline{A}}{\leqq} C \Leftrightarrow d(S_{\beta,m} \cdot C) \leqq d(C)$;

(2) *If* $C' \in \mathscr{C}$ *with* $C' \underset{\overline{A}}{\leqq} C$, *then* $d(C') \leqq d(C)$ *and there exist* $C_i \in \mathscr{C}$ *with* $C' = C_1 \underset{\overline{A}}{\leqq} C_2 \underset{\overline{A}}{\leqq} \ldots \underset{\overline{A}}{\leqq} C_k \underset{\overline{A}}{\leqq} C_{k+1} = C$ *and* $d(C_i) = d(C) + i - (k+1)$ *for* $1 \leqq i \leqq k+1$.

It was noted by H. Andersen in [1] that (2) can be sharpened to the following (Remember [13] 1.6)

(1.6) **Lemma.** *If* $C, C' \in \mathscr{C}^+$ *with* $C' \underset{\overline{A}}{\leqq} C$, *then we can choose* $\{C_i\}_{1 \leqq i \leqq k+1}$ *in* (1.5(2)) *such that all* C_i *belong to* $\mathscr{C}^+$.

For each $\lambda \in X$ there is a unique $C \in \mathscr{C}$ such that λ belongs to its lower closure. If $\lambda \in X^+ - \delta$, then $C \in \mathscr{C}^+$. Therefore (1.6) implies

(1.7) **Lemma.** *If* $\lambda, \mu \in X^+ - \delta$ *with* $\lambda \uparrow \mu$, *then there is a sequence* $\lambda = \mu_1 \uparrow \mu_2 \uparrow \ldots \uparrow \mu_k \uparrow \mu_{k+1} = \mu$ *and reflections* $S_{\beta_i, m_i} \in W_{a,p}$ *with* $\beta_i \in \Phi^+$ *and* $m_i > 0$ *such that all* $\mu_i \in X^+ - \delta$ *and* $\mu_{i+1} = S_{\beta_i, m_i} \cdot \mu_i$.

(Note that for $m_i \leqq 0$ we can not have $\mu_i, \mu_{i+1} \in X^+ - \delta$ at the same time except when $\mu_{i+1} = \mu_i$ where we can delete μ_{i+1}.)

Let us consider the following modification of $\uparrow$ and $\underset{\overline{A}}{\leqq}$. For $\lambda \in X$, $\beta \in \Phi^+$ and $m \geqq 0$ we set $S_{\beta,m} \cdot \lambda \uparrow\uparrow \lambda$ if $(\lambda + \delta, \beta^\vee) \geqq mp$ and $\lambda \uparrow\uparrow S_{\beta,m} \cdot \lambda$ if $mp \leqq (\lambda + \delta, \beta^\vee)$, and define $\uparrow\uparrow$ in general as the order relation generated by this. Similarly we define $C \uparrow\uparrow C'$. Obviously $\lambda \uparrow\uparrow \mu$ implies $\lambda \uparrow \mu$ and $C \uparrow\uparrow C'$ implies $C \underset{\overline{A}}{\leqq} C'$. From (1.6) and (1.7) we get the converse in a special case.

(1.8) **Lemma.** (1) *For* $C', C \in \mathscr{C}^+$ *we have* $C' \underset{\overline{A}}{\leqq} C \Leftrightarrow C' \uparrow\uparrow C$.

(2) *For* $\lambda, \mu \in X^+ - \delta$ *we have* $\lambda \uparrow \mu \Leftrightarrow \lambda \uparrow\uparrow \mu$.

The statement in (1.4(3)) has the following counterpart

(1.9) **Lemma.** *Let* $\lambda \in X$ *and* $\beta \in \Phi^+$. *Suppose* $n \in \mathbb{Z}^+$ *satisfies* $np < (\lambda + \delta, \beta^\vee) \leqq (n+1)p$. *Then* $\lambda - mp\beta \uparrow\uparrow \lambda$ *for all* $m \in \mathbb{Z}$ *with* $0 \leqq m \leqq n$.

Proof. For $m > 0$ we have $\lambda - mp\beta = (S_{\beta,n-m} S_{\beta,n}) \cdot \lambda \uparrow\uparrow S_{\beta,n} \cdot \lambda \uparrow\uparrow \lambda$ as

$$(S_{\beta,n} \cdot \lambda + \delta, \beta^\vee) = (\sigma_\beta(\lambda + \delta) + np\beta, \beta^\vee) = 2np - (\lambda + \delta, \beta^\vee)$$
$$\geqq (n-1)p \geqq (n-m)p \geqq 0.$$

Finally we need

(1.10) **Lemma.** (1) *Let* $C, C' \in \mathscr{C}$ *with* $C \uparrow\uparrow C'$. *Let* $\sigma, \sigma' \in W$ *be the unique elements with* $\sigma \cdot C, \sigma' \cdot C' \in \mathscr{C}^+$. *Then* $\sigma \cdot C \uparrow\uparrow \sigma' \cdot C'$.

(2) *Let* $\lambda, \mu \in X$ *with* $\lambda \uparrow\uparrow \mu$. *There are* $\sigma, \sigma' \in W$ *with* $\sigma \cdot \lambda$, $\sigma' \cdot \mu \in X^+ - \delta$. *Then* $\sigma \cdot \lambda \uparrow\uparrow \sigma' \cdot \mu$.

Proof. (1) We may assume $C = S_{\beta,m} \cdot C'$ with $\beta \in \Phi^+$ and $m \geqq 0$. If $m = 0$, then $\sigma \cdot C = \sigma' \cdot C'$ and nothing has to be shown. Suppose $m > 0$. Then $\sigma' \cdot C$

$=\sigma' S_{\beta,m}\sigma'^{-1}\cdot(\sigma'\cdot C')=S_{\sigma'(\beta),m}\cdot(\sigma'\cdot C')$. For any $\chi\in C'$ we have $0<mp<(\chi+\delta,\beta^\vee)=(\sigma'(\chi+\delta),\sigma'(\beta)^\vee)$, hence $\sigma'(\beta)\in\Phi^+$. Therefore [7] Satz 9 implies $\sigma\cdot C=\sigma\sigma'^{-1}\cdot(\sigma'\cdot C)\uparrow\sigma'\cdot C'$ and by (1.8(1)) we get $\sigma\cdot C\uparrow\uparrow\sigma'\cdot C'$.

(2) *Can be proved similarly.*

2. The Weyl Filtration of $Q(\lambda, n)$

Let $\mathfrak{g}$ be the restructed Lie algebra of G, and $\mathscr{U}$ its restricted universal enveloping algebra. Let U_K be the hyperalgebra of G, and $\mathscr{U}_n$ the hyperalgebra of the n-th Frobenius kernel of G. This is a $p^{n\dim G}$-dimensional subalgebra of U_K, we have $\mathscr{U}_1\simeq\mathscr{U}$ and $\mathscr{U}_1\subset\mathscr{U}_2\subset\ldots\subset\mathscr{U}_n\subset\ldots\subset U_K$. The algebras $\mathscr{U}_n$ are Hopf algebras and symmetric Frobenius algebras.

Let $X_n=\{\lambda\in X^+|(\lambda,\alpha_i^\vee)<p^n,\ 1\leqq i\leqq l\}$ be the set of "restricted" dominant weights of G. For each $\lambda\in X_n$ there is an irreducible $\mathscr{U}_n$-module $M(\lambda,n)$ with "highest" weight λ and each irreducible $\mathscr{U}_n$-module is isomorphic to exactly one $M(\lambda,n)$ with $\lambda\in X_n$. Therefore X_n may also be seen as an index set for distinct $\mathscr{U}_n$-PIMs (principal indecomposable module). Denote by $Q(\lambda,n)$ the PIM with $M(\lambda,n)$ as its unique top and bottom composition factor. For each $\lambda\in X_n$ set $Z(\lambda,n)=\mathscr{U}_n\otimes_{\mathscr{b}_n}K_\lambda$ where $\mathscr{b}_n$ is the hyperalgebra of the n-th Frobenius kernel of B and K_λ is the 1-dimensional $\mathscr{b}_n$-module corresponding to $\lambda\in X\simeq X(B)$. The $\mathscr{U}_n$-module $Z(\lambda,n)$ is the "universal highest weight module" with "highest" weight λ. It is indecomposable and has unique irreducible quotient $M(\lambda,n)$. In [8] J. Jantzen proved that $Q(\lambda,n)$ has a $\mathscr{U}_n$-filtration with subquotients $Z(\mu,n)$, the Z-filtration, and that the number of times $Z(\mu,n)$ occurs as a subquotient in such a filtration of $Q(\lambda,n)$ is equal to the multiplicity of $M(\lambda,n)$ as a $\mathscr{U}_n$-composition factor of $Z(\mu,n)$, i.e.

$$(Q(\lambda,n):Z(\mu,n))=[Z(\mu,n):M(\lambda,n)].$$

For $\lambda\in X^+$ let $V(\lambda)$ be the Weyl module with highest weight λ. This is an indecomposable G-module having a unique irreducible quotient denoted by $M(\lambda)$, and it is a universal highest weight module. We write $\mathrm{ch}(\lambda)$ (resp. $p-\mathrm{ch}(\lambda)$) for the formal character of $V(\lambda)$ (resp. $M(\lambda)$). For $\mu\in X^+-\delta$, $\mu\notin X^+$ we set $V(\mu)=0$ and $\mathrm{ch}(\mu)=0$. We extend the definition of $\mathrm{ch}(\mu)$ to all $\mu\in X$: There is $\sigma\in W$ with $\sigma\cdot\mu\in X^+-\delta$. Then set $\mathrm{ch}(\mu)=\det(\sigma)\,\mathrm{ch}(\sigma\cdot\mu)$. (If $\mathrm{ch}(\sigma\cdot\mu)\neq 0$, then σ is unique.)

Let $F_n: G\to G$ be the n-th power of the Frobenius endomorphism (Remember that G is defined over $\mathbb{F}_p$). For each G-module V we introduce a new G-module $V^{(p^n)}$ which we call the n-th Frobenius twist of V. It has the same underlying vector space as V with a new action of G given by $g\circ v=F_n(g)v$ for every $g\in G$ and $v\in V$. Any $\lambda\in X^+$ has a p-adic expansion $\lambda=\lambda_0+p\lambda_1+\ldots+p^k\lambda_k$ where $\lambda_0,\lambda_1,\ldots,\lambda_k\in X_1$. By Steinberg's tensor product theorem we have then

$$M(\lambda)\simeq M(\lambda_0)\otimes M(\lambda_1)^{(p)}\otimes\ldots\otimes M(\lambda_k)^{(p^k)}.$$

For $\lambda\in X_n$ we have $M(\lambda)\simeq M(\lambda,n)$ as a $\mathscr{U}_n$-module, for $\mu\in X_n$ and $\nu\in X^+$ we have $M(\mu+p^n\nu)\simeq M(\mu)^{\dim M(\nu)}$ as a $\mathscr{U}_n$-module. For $\lambda=(p^n-1)\delta$ we have $V((p^n$

$-1)\delta)=M((p^n-1)\delta)$ and this G-module is as $\mathcal{U}_n$-module isomorphic to $Q((p^n-1)\delta, n)=Z((p^n-1)\delta, n)=M((p^n-1)\delta, n)$. We call this module the n-th Steinberg module and denote it by St_n.

There is a bijective map $\lambda\mapsto\lambda^0$ of X_n onto itself given by $\lambda^0=\sigma_0\lambda+(p^n-1)\delta$ $=\sigma_0\cdot(\lambda-\delta)+p^n\delta$ for all $\lambda\in X_n$. Obviously $(\lambda^0)^0=\lambda$ for all λ.

J. Ballard proved for $p\geqq 3h-3$ that each $Q(\lambda, n)$ with $\lambda\in X_n$ admits a (unique by [8]) G-module structure (cf. [2]). This bound was improved to $p\geqq 2h-2$ in [10], 4, 5. In both cases $Q(\lambda, n)$ was constructed as a direct summand of $M(\lambda^0)\otimes St_n$. It is however clear that we can replace $M(\lambda^0)$ by $V(\lambda^0)$ or any G-module such that all weights μ satisfy $\mu\leqq\lambda^0$ and such that the λ^0 weight space has dimension 1 (see [5]).

It is proved in [10], 5.9 that $Q(\lambda, n)$ admits a Weyl filtration, i.e. a G-filtration with subquotients of the form $V(\mu)$ with $\mu\in X^+$. All μ occuring satisfy $(\mu, \alpha_0^\vee)<2p^n(h-1)$ and for such μ the Weyl module $V(\mu)$ occurs $[V(\mu): M(\lambda)]$ times as a subquotient in such a filtration. The existence of the filtration can be proved more easily than in [10], 5.6 using [14], Theorem B and Proposition 3.4 together with the realization of $Q(\lambda, n)$ as a direct summand of $V(\lambda^0)\otimes St_n$.

More generally one has

(2.1) **Proposition.** *Assume $p\geqq 2h-2$. For all $\lambda\in X_n$ and $\nu\in X^+$ the tensor product $Q(\lambda, n)\otimes V(\nu)^{(p^n)}$ has highest weight $\lambda^0-\delta+p^n(\nu+\delta)$. It is a direct G-summand of $V(\lambda^0)\otimes St_n\otimes V(\nu)^{(p^n)}$ and admits a Weyl filtration which starts with a submodule isomorphic to $V(\lambda^0-\delta+p^n(\nu+\delta))$.*

Proof. The previous remarks show that $Q(\lambda, n)\otimes V(\nu)^{(p^n)}$ is a direct summand of $V(\lambda^0)\otimes St_n\otimes V(\nu)^{(p^n)}$ which is isomorphic to $V(\lambda^0)\otimes V(p^n(\nu+\delta)-\delta)$ by [10], 5.5. Again using [14] Theorem B and Proposition 3.4 gives the Weyl filtration. (It was first proved to exist in [10], 5.5.) For the statement about the highest weight see e.g. [10], 6.2. Whenever we have a Weyl filtration of a G-module V then we can take any maximal weight μ of V and get a submodule $V(\mu)$ at the beginning of the filtration, cf. [10], 5.2(2).

Let us use the category of $\mathcal{U}_n-T$-modules (the definition in [8] or [10]). For each $\lambda\in X$ there are a universal highest weight module $\hat{Z}(\lambda, n)=\mathcal{U}_n\otimes_{\mathfrak{b}_n} K_\lambda$ of weight λ, its unique irreducible quotient $\hat{M}(\lambda, n)$ and the projective cover $\hat{Q}(\lambda, n)$ of $\hat{M}(\lambda, n)$. For $\lambda\in X_n$ there are isomorphisms of $\mathcal{U}_n$-modules $\hat{M}(\lambda, n)\simeq M(\lambda, n)$, $\hat{Z}(\lambda, n)\simeq Z(\lambda, n)$, $\hat{Q}(\lambda, n)\simeq Q(\lambda, n)$. For all $\lambda, \nu\in X$ there are isomorphisms of $\mathcal{U}_n-T$-modules $\hat{M}(\lambda+p^n\nu, n)\simeq\hat{M}(\lambda, n)\otimes p^n\nu$, $\hat{Z}(\lambda+p^n\nu, n)\simeq\hat{Z}(\lambda, n)\otimes p^n\nu$, $\hat{Q}(\lambda+p^n\nu, n)\simeq\hat{Q}(\lambda, n)\otimes p^n\nu$. For $\lambda\in X_n$ the G-module $M(\lambda)$ is isomorphic to $M(\lambda, n)$ as a $\mathcal{U}_n-T$-module, if $Q(\lambda, n)$ admits a G-module structure, this G-structure induces the $\mathcal{U}_n-T$-module structure of $\hat{Q}(\lambda, n)$.

Let $e(\mu)$, $\mu\in X$ be the canonical basis of $\mathbb{Z}[X]$. Set for any $\mu\in X$

$$S(\mu)=\sum_{\nu\in W_\mu} e(\nu)=\sum_{\sigma\in W}\frac{1}{|\mathrm{Stab}_W\mu|}e(\sigma\mu).$$

By [11], 2.5(2) we have for all $\lambda\in X_n$

$$\mathrm{ch}\,\hat{Q}(\lambda, n)=\sum_{\mu\in X^+}[\hat{Z}(\mu-\delta, n): \hat{M}(\lambda-p^n\delta, n)]\,S(\mu)\,\mathrm{ch}\,St_n,$$

hence for all $v \in X^+$

$$\mathrm{ch}\,\hat{Q}(\lambda, n) \otimes V(v)^{(p^n)} = \sum_{\mu \in X^+} [\hat{Z}(\mu + (p^n - 1)\delta, n) : \hat{M}(\lambda, n)]\, S(\mu)\, \mathrm{ch}(St_n \otimes V(v)^{(p^n)}).$$

As $St_n \otimes V(v)^{(p^n)} \simeq V(p^n(v+\delta) - \delta)$ by [10], 5.5 we get

(2.2) **Proposition.** *For all $\lambda \in X_n$, $v \in X^+$ we have*

$$\mathrm{ch}\,\hat{Q}(\lambda, n) \otimes V(v)^{(p^n)} = \sum_{\mu \in X^+} b_\mu^\lambda \sum_{\sigma \in W} \frac{1}{|\mathrm{Stab}_W \mu|}\, \mathrm{ch}(\sigma\mu + p^n(v+\delta) - \delta)$$

where $b_\mu^\lambda = [\hat{Z}(\mu + (p^n - 1)\delta, n) : \hat{M}(\lambda, n)]$.

Set $\mathscr{S}_{\lambda,n} = \{\mu \in X^+ \mid \mu \uparrow\uparrow \lambda^0 - \delta\}$. For each $\mu \in \mathscr{S}_{\lambda,n}$ we have $\mu \leqq \lambda^0$, hence μ is a Weight of $V(\lambda^0)$ by [3], §13.2 Lemma B and §13.4 Lemma B. Thus $\mathscr{S}_{\lambda,n}$ is a finite set with $\lambda^0 - \delta$ as its largest element.

(2.3) **Lemma.** *Let $\lambda \in X_n$, $\mu \in X^+$. If $b_\mu^\lambda \neq 0$, then $\mu \in \mathscr{S}_{\lambda,n}$.*

Proof. By definition $0 \neq b_\mu^\lambda = [\hat{Z}(\mu + p^n\delta, n) : \hat{M}(\lambda, n)]$. Now [10], 3.3 implies $\mu + p^n\delta \uparrow \sigma_0 \cdot \lambda + 2p^n\delta = \lambda^0 - \delta + p^n\delta$, hence $\mu \uparrow \lambda^0 - \delta$ by (1.4(1)), hence $\mu \uparrow\uparrow \lambda^0 - \delta$ by (1.8(2)).

(2.4) **Proposition.** *Let $\lambda \in X_n$ and $\mu \in \mathscr{S}_{\lambda,n}$. Then for all $\sigma \in W$ we have*

$$\lambda \uparrow \sigma \cdot \mu + p^n\delta \uparrow\uparrow \lambda^0 - \delta + p^n\delta.$$

Proof. By (1.4(4)) we have $\sigma_0 \cdot \mu \uparrow\uparrow \sigma \cdot \mu \uparrow\uparrow \mu$ where we can go step by step with reflections of the form $S_{\beta,0}$, $\beta \in \Phi^+$ only. Then we get

$$\sigma_0 \cdot \mu + p^n\delta \uparrow\uparrow \sigma \cdot \mu + p^n\delta \uparrow\uparrow \mu + p^n\delta$$

as we are using reflections $S_{\beta,m}$ with $m = p^{n-1}(\delta, \beta^\vee) > 0$ only. From $\mu \uparrow\uparrow \lambda^0 - \delta$ we get $\mu + p^n\delta \uparrow\uparrow \lambda^0 - \delta + p^n\delta$, hence $\sigma \cdot \mu + p^n\delta \uparrow\uparrow \lambda^0 - \delta + p^n\delta$. Using (1.4(2)) we get $\sigma_0 \cdot (\lambda^0 - \delta + p^n\delta) \uparrow \sigma_0 \cdot (\mu + p^n\delta)$, hence $\lambda - 2p^n\delta \uparrow \sigma_0 \cdot \mu - p^n\delta$ and $\lambda \uparrow \sigma_0 \cdot \mu + p^n\delta \uparrow \sigma \cdot \mu + p^n\delta$.

(2.5) **Proposition.** *Let $\lambda \in X_n$ and $\mu \in X^+$ with $\mu \leqq \lambda^0$. If $\lambda \uparrow \sigma_0 \cdot \mu + p^n\delta$ or $\mu + p^n\delta \uparrow\uparrow \lambda^0 - \delta + p^n\delta$, then $\mu \in \mathscr{S}_{\lambda,n}$.*

Proof. Certainly $\mu + p^n\delta \uparrow\uparrow \lambda^0 - \delta + p^n\delta$ implies $\mu \uparrow \lambda^0 - \delta$, hence $\mu \in \mathscr{S}_{\lambda,n}$ using (1.8(2)). If $\lambda \uparrow \sigma_0 \cdot \mu + p^n\delta$, apply σ_0 to both sides and then argue as before.

(2.6) **Proposition.** *Let $\lambda \in X_n$, $\mu \in \mathscr{S}_{\lambda,n}$, $\sigma \in W$ and $v \in X^+$. There is $\tau \in W$ with $\tau \cdot (\sigma \cdot \mu + p^n(v+\delta)) \in X^+ - \delta$, and there are a unique $\mu' \in X^+$ and a $\sigma' \in W$ with $\tau \cdot (\sigma \cdot \mu + p^n(v+\delta)) = \sigma' \cdot \mu' + p^n(v+\delta)$. Then $\mu' \in \mathscr{S}_{\lambda,n}$ and $\mu' \uparrow\uparrow \mu$.*

Proof. The existence of τ, μ', σ' and the uniqueness of μ' are clear. We have only to prove the last statements. Let us use induction over the number of $\beta \in \Phi^+$ with $(\sigma \cdot \mu + p^n(v+\delta) + \delta, \beta^\vee) < 0$. If this number is 0, then $\sigma \cdot \mu + p^n(v + \delta) \in X^+ - \delta$ and $\mu' = \mu$.

If not, then there is a simple root α_i with $(\sigma \cdot \mu + p^n(v+\delta) + \delta, \alpha_i^\vee) < 0$. We can write $\sigma_i \cdot (\sigma \cdot \mu + p^n(v+\delta))$ in the form $\sigma' \cdot \mu' + p^n(v+\delta)$ with $\mu' \in X^+$ and $\sigma' \in W$.

We want to prove $\mu' \in \mathscr{S}_{\lambda,n}$ and $\mu' \uparrow\uparrow \mu$. Then we can apply the induction hypothesis to $\sigma' \cdot \mu' + p^n(\nu+\delta)$ and get the proposition.

We have

$$\begin{aligned}\sigma' \cdot \mu' + p^n(\nu+\delta) &= \sigma_i \cdot (\sigma \cdot \mu + p^n(\nu+\delta)) = \sigma_i\sigma \cdot \mu + p^n \sigma_i(\nu+\delta) \\ &= \sigma_i\sigma \cdot \mu + p^n(\nu+\delta) - p^n(\nu+\delta, \alpha_i^\vee)\alpha_i,\end{aligned}$$

hence $\sigma' \cdot \mu' = \sigma_i\sigma \cdot \mu - p^n(\nu+\delta, \alpha_i^\vee)\alpha_i$. By our choice of α_i we have $(\sigma \cdot \mu + \delta, \alpha_i^\vee) < -p^n(\nu+\delta, \alpha_i^\vee)$, hence $(\sigma_i\sigma \cdot \mu + \delta, \alpha_i^\vee) > p^n(\nu+\delta, \alpha_i^\vee) > 0$. Therefore (1.9) implies $\sigma' \cdot \mu' \uparrow\uparrow \sigma_i\sigma \cdot \mu$ and (1.10(2)) implies $\mu' \uparrow\uparrow \mu$ and therefore also $\mu' \in \mathscr{S}_{\lambda,n}$. This was what we had to show.

(2.7) **Corollary.** *Let $\lambda \in X_n$ and $\nu \in X^+$. Then $\operatorname{ch} \hat{Q}(\lambda, n) \otimes V(\nu)^{(p^n)}$ is a linear combination of the $\operatorname{ch}(\sigma \cdot \mu + p^n(\nu+\delta))$ with $\mu \in \mathscr{S}_{\lambda,n}$ and $\sigma \in W$ such that $\sigma \cdot \mu + p^n(\nu+\delta) \in X^+$. The coefficient of $\operatorname{ch}(\sigma \cdot \mu + p^n(\nu+\delta))$ for such μ, σ is equal to*

$$\sum_{\tau \in W} \det(\tau)[\hat{Z}(\sigma \cdot \mu + p^n(\nu+\delta), n) : \hat{M}(\lambda + p^n\tau \cdot \nu, n)].$$

Proof. By (2.2) and (2.3) we know that $\operatorname{ch} \hat{Q}(\lambda, n) \otimes V(\nu)^{(p^n)}$ is a linear combination of all $\operatorname{ch}(\sigma \cdot \mu + p^n(\nu+\delta))$ with $\mu \in \mathscr{S}_{\lambda,n}$ and $\sigma \in W$. If $\sigma \cdot \mu + p^n(\nu+\delta) \notin X^+$ and $\operatorname{ch}(\sigma \cdot \mu + p^n(\nu+\delta)) \neq 0$, then there is a unique $\tau \in W$ with $\tau \cdot (\sigma \cdot \mu + p^n(\nu+\delta)) \in X^+$. Then $\operatorname{ch}(\sigma \cdot \mu + p^n(\nu+\delta)) = \det(\tau) \operatorname{ch}(\tau \cdot (\sigma \cdot \mu + p^n(\nu+\delta)))$ and $\tau \cdot (\sigma \cdot \mu + p^n(\nu+\delta))$ has by (2.6) the required form. This proves that $\operatorname{ch} \hat{Q}(\lambda, n) \otimes V(\nu)^{(p^n)}$ is a linear combination as resired.

In (2.2) the coefficient of $\operatorname{ch}(\sigma \cdot \mu + p^n(\nu+\delta))$ is equal to

$$\begin{aligned}[\hat{Z}(\mu + p^n\delta, n) : \hat{M}(\lambda, n)] &= [Z(\sigma \cdot \mu + p^n\delta, n) : \hat{M}(\lambda, n)] \\ &= [\hat{Z}(\sigma \cdot \mu + p^n(\nu+\delta), n) : \hat{M}(\lambda + p^n\nu, n)].\end{aligned}$$

If $\sigma \cdot \mu + p^n(\nu+\delta) = \tau^{-1} \cdot (\sigma' \cdot \mu' + p^n(\nu+\delta))$ for $\mu' \in \mathscr{S}_{\lambda,n}$, $\tau, \sigma' \in W$, then this number is by [10], 3.2(2) equal to $[\hat{Z}(\sigma' \cdot \mu' + p^n(\nu+\delta), n) : \hat{M}(\lambda + p^n\tau \cdot \nu, n)]$. From this the claim about the coefficients in the corollary follows easily.

(2.8) **Proposition.** *Let ν be a dominant weight such that $(\nu + 2\delta, \alpha_0^\vee) \leqq p$. For $\lambda \in X_n$, $\mu \in \mathscr{S}_{\lambda,n}$ and $\sigma \in W$ with $\sigma \cdot \mu + p^n(\nu+\delta) \in X^+$ we have*

$$\begin{aligned}&[V(\sigma \cdot \mu + p^n(\nu+\delta)) : M(\lambda + p^n\nu)] \\ &\quad = \sum_{\tau \in W} \det(\tau)[\hat{Z}(\sigma \cdot \mu + p^n(\nu+\delta), n) : \hat{M}(\lambda + p^n\tau \cdot \nu, n)].\end{aligned}$$

Proof. By [10] 3.1(5) we have

$$\begin{aligned}&[V(\sigma \cdot \mu + p^n(\nu+\delta)) : M(\lambda + p^n\nu)] \\ &\quad = \sum_{\nu_1 \in X^+} \sum_{\tau \in W} \det(\tau)[\hat{Z}(\sigma \cdot \mu + p^n(\nu+\delta), n) : \hat{M}(\lambda + p^n\tau \cdot \nu_1, n)][V(\nu_1) : M(\nu)].\end{aligned}$$

We have to show that only $\nu_1 = \nu$ contributes to this sum. If we get something from $\nu_1 \neq \nu$, then $[V(\nu_1) : M(\nu)] \neq 0$, hence $\nu_1 = w \cdot \nu$ for some $w \in W_{a,p}$, $w \neq 1$. As $\nu \in \bar{C}_0$, we get $S_{\alpha_{0,1}} \cdot \nu \uparrow\uparrow \nu_1$, hence

$$p^n(S_{\alpha_{0,1}} \cdot \nu, \alpha_0^\nu) \leqq p^n(\nu_1, \alpha_0^\nu) \leqq (\lambda + p^n \nu_1, \alpha_0^\nu) \leqq (\tau^{-1} \cdot (\sigma \cdot \mu + p^n(\nu + \delta)), \alpha_0^\nu)$$
$$\leqq (\sigma \cdot \mu + p^n(\nu + \delta), \alpha_0^\nu) \leqq (\mu + p^n(\nu + \delta), \alpha_0^\nu)$$
$$\leqq (p^n(\nu + 2\delta) - \delta, \alpha_0^\nu) < p^n(\nu + 2\delta, \alpha_0^\nu).$$

This implies

$$(\nu + 2\delta, \alpha_0^\nu) > (S_{\alpha_{0,1}} \cdot \nu, \alpha_0^\nu) = (\sigma_{\alpha_0}(\nu + \delta) - \delta + p\alpha_0, \alpha_0^\nu) = 2p - (\nu + 2\delta, \alpha_0^\nu)$$

and $2p < 2(\nu + 2\delta, \alpha_0^\nu) \leqq 2p$ a contradiction.

(2.9) *Remarks.* (1) Comparing (2.8) and (2.7) we get a reciprocity for the Weyl filtration of $\hat{Q}(\lambda, n) \otimes V(\nu)^{(p^n)}$ generalizing that in [10], 5.9. (Note that the existence of ν as in (2.8) implies $p \geqq 2h - 2$.)

(2) We call the formula in (2.2) the pattern of a Weyl filtration of $Q(\lambda, n) \otimes V(\nu)^{(p^n)}$, briefly the filtration pattern of $Q(\lambda, n) \otimes V(\nu)^{(p^n)}$. It is obvious that the filtration pattern of $Q(\lambda, n) \otimes V(\nu)^{(p^n)}$ is only the translation of the filtration pattern of $Q(\lambda, n)$ by $p^n \nu$. The filtration pattern of $Q(\lambda, n)$ does not depend on the chosen λ but only on the facet to which λ belongs because of Jantzen's translation principle (cf. [6, 9]). For regular λ there are exactly $\frac{|W|}{f} p^{2(n-1)}$ filtration patterns.

(3) Because of (1) the filtration pattern is dual to Jantzen's generic decomposition pattern for Weyl modules (cf. [6]).

3. The Case $n = 1$

(3.1) **Lemma.** *For any $\beta \in \Phi^-$ there exist $\sigma \in W$ and $\alpha_k \in \Delta$ with $\sigma(\alpha_k) = \beta$ and $\sigma(\alpha_i) \in \Phi^+$ for all $i \neq k$, $1 \leqq i \leqq l$.*

Proof. Let $-\beta = \sum_{i=1}^{l} a_i \alpha_i$, where $a_i \in \mathbb{Z}^+$, $1 \leqq i \leqq l$. We use induction on $ht(-\beta) = \sum_{i=1}^{l} a_i$.

If $ht(-\beta) = 1$, then $\beta = -\alpha_k$ for some simple root α_k and we can take $\sigma = \sigma_k$. Suppose $ht(-\beta) > 1$. There exists $\alpha_j \in \Delta$ with $(\beta, \alpha_j^\nu) < 0$. Then $\beta' = \sigma_j(\beta) = \beta - (\beta, \alpha_j^\nu)\alpha_j \in \Phi^-$ and $ht(-\beta') < ht(-\beta)$. By the induction hypothesis there exist $\sigma' \in W$, $\alpha_k \in \Delta$ such that $\sigma'(\alpha_k) = \beta'$, $\sigma'(\alpha_i) \in \Phi^+$ for all $i \neq k$, $1 \leqq i \leqq l$. We take now $\sigma = \sigma_j \sigma'$, thus $\sigma(\alpha_k) = \sigma_j \sigma(\alpha_k) = \sigma_j(\beta') = \beta$. Consider $\alpha_i \in \Delta - \{\alpha_k\}$. In order to prove that $\sigma(\alpha_i) = \sigma_j \sigma'(\alpha_i) \in \Phi^+$, we have to prove that $\sigma'(\alpha_i) \neq \alpha_j$. If $\sigma'(\alpha_i) = \alpha_j$, then

$$\alpha_k = (\sigma')^{-1}(\beta') = (\sigma')^{-1}(\sigma_j(\beta)) = (\sigma')^{-1}(\beta - (\beta, \alpha_j^\nu)\alpha_j)$$
$$= (\sigma')^{-1}(\beta) - (\beta, \alpha_j^\nu)(\sigma')^{-1}(\alpha_j) = (\sigma')^{-1}(\beta) - (\beta, \alpha_j^\nu)\alpha_i$$

and $(\sigma')^{-1}(\beta) = \alpha_k + (\beta, \alpha_j^\nu)\alpha_j$. As $(\beta, \alpha_j^\nu) < 0$, this is a contradiction to the definition of a basis of a root system. Hence $\sigma'(\alpha_i) \neq \alpha_j$.

We call a weight $\mu' \in X^+$ maximal with respect to a weight $\mu \in X^+$ if $\mu' \uparrow\uparrow \mu$, $\mu \neq \mu'$, and if there is no $\mu'' \in X^+$ with $\mu' \uparrow\uparrow \mu'' \uparrow\uparrow \mu$ and $\mu'' \neq \mu, \mu'$. If $\mu \in C$, $\mu' \in C'$

for $C, C' \in \mathscr{C}$, then μ' is maximal with respect to μ if and only if $\mu' \uparrow\uparrow \mu$ and $d(C') = d(C) - 1$ (by [7], Corollary 2 to Satz 9).

We shall from now on restrict ourselves to the case of p-regular $\lambda \in X_1$, i.e. to λ inside an alcove. Then also $\lambda^0 - \delta$ and all μ with $\mu \in \mathscr{S}_{\lambda,1}$ are p-regular.

(3.2) **Lemma.** *Suppose $\lambda \in X_1$ p-regular and $\mu \in \mathscr{S}_{\lambda,1}$. Let C be the alcove with $\mu \in C$.*

(1) *If $C = C_0$ then $\sigma \cdot \mu + p(\nu + \delta) \in X^+$ for all $\sigma \in W$ and $\nu \in X^+$.*

(2) *Let $C \neq C_0$ and let $\mu' \in \mathscr{S}_{\lambda,1}$ be maximal with respect to μ. Then there exist $\sigma \in W$, $\nu \in X^+$ and a simple reflection σ_k such that $\sigma_k \cdot (\sigma \cdot \mu + p(\nu + \delta)) = \sigma \cdot \mu' + p(\nu + \delta) \in X^+$ and such that for all $\tau \in W$, $\tau \neq 1$, σ_k the weight $\tau \cdot (\sigma \cdot \mu' + p(\nu + \delta))$ is not of the form $\sigma'' \cdot \mu'' + p(\nu + \delta)$ with $\mu'' \in \mathscr{S}_{\lambda,1}$ and $\sigma'' \in W$.*

Proof. (1) As $\mu \in C_0$ we have $0 \leqq (\mu + \delta, \beta^\nu) \leqq (\mu + \delta, \alpha_0^\nu) < p$ for all $\beta \in \Phi^+$, hence $(\sigma \cdot \mu + \delta, \beta^\nu) > -p$ for all $\sigma \in W$ and

$$(\sigma \cdot \mu + p(\nu + \delta) + \delta, \beta^\nu) > p(\delta, \beta^\nu) - p \geqq 0.$$

This shows $\sigma \cdot \mu + p(\nu + \delta) \in X^+$.

(2) As μ' is maximal with respect to μ there are $\beta \in \Phi^+$ and $m \in \mathbb{Z}^+$ with $\mu' = S_{\beta,m} \cdot \mu$. Furthermore there has to be $a_\beta \in \mathbb{Z}$ with $0 < a_\beta < p$ and $(\mu + \delta, \beta^\nu) = mp + a_\beta$, hence $(\mu' + \delta, \beta^\nu) = mp - a_\beta$ and $\mu' = \mu - a_\beta \beta$. By (3.1) there exist $\sigma \in W$ and $\alpha_k \in \Delta$ such that $\sigma(\beta) = -\alpha_k$ and $\sigma^{-1}(\alpha_i) \in \Phi^+$ for all $i \neq k$, $1 \leqq i \leqq l$. Then $(\sigma \cdot \mu + \delta, \alpha_k^\nu) = -(\mu + \delta, \beta^\nu) = -(mp + a_\beta)$ and $(\sigma \cdot \mu' + \delta, \alpha_k^\nu) = -mp + a_\beta$ whereas $(\sigma \cdot \mu + \delta, \alpha_i^\nu) \geqq 0$ and $(\sigma \cdot \mu' + \delta, \alpha_i^\nu) \geqq 0$ for $i \neq k$, $1 \leqq i \leqq l$. Set $\nu = (m-1)\omega_{\alpha_k} + \sum_{i \neq k} m_i \omega_{\alpha_i}$ with suitable $m_i \in \mathbb{Z}^+$ to be determined later. Then $(\sigma \cdot \mu + p(\nu + \delta) + \delta, \alpha_k^\nu) = -a_\beta$ and

$$\begin{aligned} \sigma_k \cdot (\sigma \cdot \mu + p(\nu + \delta)) &= \sigma \cdot \mu + p(\nu + \delta) - (\sigma \cdot \mu + p(\nu + \delta) + \delta, \alpha_k^\nu)\alpha_k \\ &= \sigma \cdot \mu + p(\nu + \delta) + a_\beta \alpha_k = \sigma \cdot (\mu - a_\beta \beta) + p(\nu + \delta) \\ &= \sigma \cdot \mu' + p(\nu + \delta). \end{aligned}$$

For all $i \neq k$ we have $(\sigma \cdot \mu' + p(\nu + \delta) + \delta, \alpha_i^\nu) \geqq (\sigma \cdot \mu' + \delta, \alpha_i^\nu) \geqq 0$, hence $\sigma \cdot \mu' + p(\nu + \delta) \in X^+$. The set $\{\sigma'' \cdot \mu'' \mid \mu'' \in \mathscr{S}_{\lambda,1}, \sigma'' \in W\}$ is finite. Therefore we can choose the m_i in the definition of ν such that $(\sigma'' \cdot \mu'' + p(\nu + \delta) + \delta, \gamma^\nu) > 0$ for all $\gamma \in \Phi^+$, $\gamma \neq \alpha_k$ and all these $\sigma'' \cdot \mu''$. Then either $\sigma'' \cdot \mu'' + p(\nu + \delta) \in X^+$ or $\sigma_k \cdot (\sigma'' \cdot \mu'' + p(\nu + \delta)) \in X^+$. From this the last claim of the lemma follows.

(3.3) **Theorem.** *Suppose $p \geqq 2h - 2$. Let $\lambda \in X_1$ be p-regular. Then we have*

(1) $b^\lambda_{\lambda^0 - \delta} = 1$.

(2) *If $\mu, \mu' \in \mathscr{S}_{\lambda,1}$ with $\mu' \uparrow\uparrow \mu$, then $b^\lambda_\mu \leqq b^\lambda_{\mu'}$.*

Proof. (1) By definition

$$b^\lambda_{\lambda^0 - \delta} = [\hat{Z}(\lambda^0 + (p^n - 1)\delta, 1) : \hat{M}(\lambda, 1)] = [\hat{Z}(\sigma_0 \cdot \lambda + 2p^n \delta, 1) : \hat{M}(\lambda, 1)].$$

This number is equal to 1 by [10], Bemerkung 1 in 3.3.

(2) We may assume that μ' is maximal with respect to μ. Choose ν, σ as in (3.2(2)). We can express $\mathrm{ch}\, Q(\lambda, 1) \otimes V(\nu)^{(p)}$ as linear combination of the $\mathrm{ch}(\mu_1)$

with $\mu_1 \in X^+$. The coefficient of $\mathrm{ch}(\sigma \cdot \mu' + p(\nu+\delta))$ is by (3.2(2)) and by (2.7) equal to $b^\lambda_{\mu'} - b^\lambda_\mu$. As $p \geqq 2h-2$ we know by (2.1) that $Q(\lambda, 1) \equiv V(\nu)^{(p)}$ has a Weyl filtration. Therefore the coefficient of each $\mathrm{ch}(\mu_1)$ has to be non-negative. This implies $b^\lambda_{\mu'} - b^\lambda_\mu \geqq 0$.

Remark. Observe that (1) holds not only for $n=1$ but for arbitrary n.

(3.4) **Corollary.** *Suppose $p \geqq 2h-2$. Let $\lambda \in X_1$ be p-regular. Then $b^\lambda_\mu \geqq 1$ for all $\mu \in \mathscr{S}_{\lambda,1}$.*

This is obvious as $\mu \uparrow\uparrow \lambda^0 - \delta$.

(3.5) **Corollary.** *Suppose $p \geqq 2h-2$. Let $\lambda \in X_1$ be p-regular. Then*

$$\{\lambda' \in X \mid [\hat{Z}(\lambda', 1) : \hat{M}(\lambda, 1)] \neq 0\} = \{\sigma \cdot \mu + p\delta \mid \mu \in \mathscr{S}_{\lambda,1}, \sigma \in W\}.$$

This is now clear by [10], 3.2, by the definition of b^λ_μ and by (2.3).

(3.6) *Remarks.* (1) For $n>1$ Corollary 3.4 does not hold in general. For G of type A_2, for $n=2$ and $p=5>2h-2=4$ one has

$$\mathscr{S}_{\lambda,2} = \{(1.12), (3.8), (6.2), (2.7), (3.4), (1.2)\} \quad \text{for } \lambda = (12.23).$$

One can show $b^\lambda_{(3.4)} = b^\lambda_{(1.2)} = 0$.

(2) Let $\lambda = \lambda_0 + p\lambda_1 + \ldots + p^{n-1}\lambda_{n-1}$ be the p-adic expansion of some $\lambda \in X_n$. Then we have an isomorphism of G-modules

$$Q(\lambda, n) \simeq Q(\lambda_0, 1) \otimes Q(\lambda_1, 1)^{(p)} \otimes \ldots \otimes Q(\lambda_{n-1}, 1)^{(p^{n-1})}.$$

We can obtain the filtration pattern of $Q(\lambda, n)$ from those of the $Q(\lambda_i, 1)$ by decomposing tensor products.

(3.7) **Proposition.** *Let $\lambda \in X_n$ be p-regular, let $C \in \mathscr{C}$ be the alcove with $\lambda \in C$. Let $s \in W_{a,p}$ be a reflection with respect to a facet of the alcove C such that $C \underset{\bar{A}}{\leqq} s \cdot C$. Let $\mu \in \mathscr{S}_{\lambda,n}$. Then there is $w \in W_{a,p}$ with $\mu + p^n\delta = w \cdot \lambda$, set $\mu' = ws \cdot \lambda - p^n\delta$. Then we have that if $\mu' \in X^+$, then $b^\lambda_{\mu'} = b^\lambda_\mu$.*

Proof. We have to show

$$[\hat{Z}(\mu + p^n\delta, n) : \hat{M}(\lambda, n)] = [\hat{Z}(\mu' + p^n\delta, n) : \hat{M}(\lambda, n)],$$

i.e.

$$[\hat{Z}(w \cdot \lambda, n) : \hat{M}(\lambda, n)] = [\hat{Z}(ws \cdot \lambda, n) : \hat{M}(\lambda, n)].$$

Applying the translations functors for the category of $\mathscr{U}_n - T$-modules (cf. [10] 5.2) this follows in the same way as the corresponding statement for Weyl modules in, say [9], § 3, Theorem (b).

(3.8) **Proposition.** *Let $\lambda \in X_n$ be p-regular. Suppose $\mu \in \mathscr{S}_{\lambda,n}$ is maximal with respect to $\lambda^0 - \delta$. Then $b^\lambda_\mu = 1$.*

Proof. We have to show $[\hat{Z}(\mu + p^n\delta, n) : \hat{M}(\lambda, n) = 1$ or, equivalently, by [10], 3.2, $[\hat{Z}(\sigma_0 \cdot \mu + p^n\delta, n) : \hat{M}(\lambda, n)] = 1$ or, equivalently, for some $\nu \in X$

$$[\hat{Z}(\sigma_0 \cdot \mu + p^n(\nu+\delta), n) : \hat{M}(\lambda + p^n\nu, n)] = 1.$$

As $\mu \uparrow\uparrow \lambda^0 - \delta$ and $\mu \neq \lambda^0 - \delta$ is maximal with this property we have also $\sigma_0 \cdot (\lambda^0 - \delta) \uparrow \sigma_0 \cdot \mu$ and maximality of $\sigma_0 \cdot (\lambda^0 - \delta) = \lambda - p^n \delta$ for this property. Hence we see for all $v \in X$ that $\lambda + p^n v \uparrow \sigma_0 \cdot \mu + p^n(v+\delta)$, that $\lambda + p^n v \neq \sigma_0 \cdot \mu + p^n(v+\delta)$ and that $\lambda + p^n v$ is maximal with this property. If we choose $v \in X^+$ sufficiently large such that $[\hat{Z}(\sigma_0 \cdot \mu + p^n(v+\delta)), n) : \hat{M}(\lambda', n)] \neq 0$ implies $\lambda' \in X^+$, then $\sigma_0 \cdot \mu + p^n(v+\delta) \in X^+$ and $[V(\sigma_0 \cdot \mu + p^n(v+\delta)) : M(\lambda + p^n v)] = 1$ by [7], Satz 10. Using [10], 3.1 (5) and Satz 3.3 one gets the desired equality.

Remark. The following results about $\operatorname{ch} Q(\lambda, n)$ are well known, and can now easily be proved. Set $C^0 = \sigma_0 \cdot C_0 + p\delta$. Then

(1) If $\lambda \in C^0 \cap X_1$, then $\operatorname{ch} Q(\lambda, 1) = \sum_{\sigma \in W} \operatorname{ch}(\sigma \cdot (\lambda^0 - \delta) + p\delta)$.

(2) *If* $\lambda \in S_{\alpha_0, h-2} \cdot C^0 \cap X_1$, then

$$\operatorname{ch} Q(\lambda, 1) = \sum_{\sigma \in W_1} (\operatorname{ch}(\sigma \cdot (\lambda^0 - \delta) + p\delta) + \operatorname{ch}(\sigma \cdot \mu + p\delta))$$

where μ is the unique element in $\mathscr{S}_{\lambda,1}$ with $\mu \neq \lambda^0 - \delta$. Here $W_1 = \{\sigma \in W \mid \sigma(\alpha_0) \notin -\Delta\}$.

References

1. Andersen, H.H.: The strong linkage principle. J. Reine Angew. Math. **315**, 53–59 (1980)
2. Ballard, J.W.: Injective modules for restricted enveloping algebras. Math. Z. **163**, 57–63 (1978)
3. Humphreys, J.E.: Introduction to Lie algebras and representation theory, GTM 9. Berlin-Heidelberg-New York: Springer 1972
4. Humphreys, J.E.: Linear algebraic groups, GTM 21. Berlin-Heidelberg-New York: Springer 1975
5. Humphreys, J.E.: Ordinary and modular representation of Chevalley groups. Lecture Notes in Math. **528**. Berlin-Heidelberg-New York: Springer 1976
6. Jantzen, J.C.: Über das Dekompositionsverhalten gewisser modularer Darstellungen halbeinfacher Gruppen und ihrer Lie-Algebren. J. Algebra **49**, 441–469 (1977)
7. Jantzen, J.C.: Darstellungen halbeinfacher Gruppen und Kontravariante Formen. J. Reine Angew. Math. **290**, 117–141 (1977)
8. Jantzen, J.C.: Über Darstellungen höherer Frobenius-Kerne halbeinfacher algebraischer Gruppen. Math. Z. **164**, 271–292 (1979)
9. Jantzen, J.C.: Weyl modules for group of Lie type, in: Finite simple groups II, pp. 291–300. London: Academic Press 1980
10. Jantzen, J.C.: Darstellungen halbeinfacher Gruppen und ihrer Frobenius-Kerne. J. Reine Angew. Math. **317**, 157–199 (1980)
11. Jantzen, J.C.: Zur Reduction modulo p unipotenter Charaktere endlicher Chevalley-Gruppen. Math. Z. **181**, 97–128 (1982)
12. Lusztig, G.: Hecke algebras and Jantzen's generic decomposition patterns. Adv. Math. **37**, 121–164 (1980)
13. Verma, D.N.: The role of the affine Weyl groups in the representation theory of algebraic Chevalley groups and their Lie algebras, in I. M. Gelfand (ed.), Lie group and their representations, pp. 653–705. New York: Halsted 1975
14. Wang, J.-p.: Sheaf cohomology on G/B and tensor products of Weyl modules. J. Algebra **77**, 162–185 (1982)

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