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Actions of Compact Lie Groups on Homogeneous Spaces

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0. Introduction

Let G be a compact connected Lie group, $H \subset G$ a closed connected subgroup, then G acts on the homogeneous space $X = G/H$ in a natural way by left translations. We call this action of G on X the natural action or the linear action. One might ask, given a G and a homogeneous space G/H, if, beside the linear action, there exist essentially different actions of G on G/H. This is a classical problem in transformation group theory and the present situation seems to be far from a complete and satisfying solution of it. If $H = \{e\}$ is the trivial subgroup we are in the special case of actions of G on itself. In this case we know of at least two actions, namely the natural action and the adjoint action by conjugation, and in some cases (see e.g. [4], p. 151) it has been proved that these are indeed all actions up to an equivariant equivalence of G on itself. Nevertheless even this question remains totally open.

In the more general case $H \neq \{e\}$ some new results have been obtained, so for example for the actions of $SU(n)$ on its corresponding Stiefel manifolds, by W.Y. Hsiang and P. Tomter in [5], where they show, that up to conjugacy, any differentiable action of $SU(n)$ on $W_{n,k} = SU(n)/SU(k)$, $k > \frac{1}{2}n$, is the natural action.

The case $X = G/NT$, G a compact Lie group, NT the normalizor of the maximal torus T of G, has been completely settled by the author, in fact with the same answer. (See [3], Satz 3.1, ii.)

In this paper we are concerned with the case of the flag manifolds G/T. (This case can not be treated by the lifting methods in [3].) Here our main theorem is:

Theorem 1. *Let X be a simply connected compact closed oriented manifold with $H^*(X; \mathbb{R}) \cong H^*(G/T; \mathbb{R})$, $\mathbb{R}$ the reals, G a simple compact Lie group, $T \subset G$ its maximal torus. If G acts on X with finitely many orbit types, then $X = G/T$ and every G-action on X is conjugate to the natural action.*

As a corollary we obtain:

Theorem 1'. *Let G be a compact simple Lie group, $T \subset G$ a fixed maximal torus, let G act on $X = G/T$ with finitely many orbit types. Then the action is conjugate to the natural action.*

One might conjecture that similar results hold at least in all cases G/H, where H is a closed subgroup of maximal rank, see e.g. [4], p. 144.

I conclude these remarks with an outline of the proof. If $H \subset G$ is a closed subgroup of maximal rank, then $H^*(B_H)$ is a free finitely generated module over $H^*(B_G)$ if the coefficient field of the cohomology has characteristic zero. Let Γ be a compact Lie group and let $\rho: \Gamma \to G$ be a homomorphism. Then this gives an action of Γ on G and all its homogeneous spaces G/H by left translations: $\varphi_\rho: \Gamma \times G/H \to G/H$, $\varphi_\rho(\gamma, gH) = \rho(\gamma)\, gH$. In the case $\operatorname{rk} G = \operatorname{rk} H$ it is now easy to compute the equivariant cohomology $H^*_\Gamma(G/H) = H^*_\Gamma(G/H; k)$, $\operatorname{char} k = 0$. We look on the obvious pullback diagram

$$\begin{array}{ccc} E_\Gamma \times_\Gamma X & \longrightarrow & B_H \\ \downarrow & & \downarrow \\ B_\Gamma & \longrightarrow & B_G \end{array}$$

By the methods of the Eilenberg-Moore spectral sequence we obtain for $H^*_\Gamma(X) = H^*(E_\Gamma \times_\Gamma X)$

$$H^*_\Gamma(X) \cong R_\Gamma \otimes_{R_G} R_H.$$

(See e.g. [4], p. 38.) Here is $R_K = H^*(B_K)$ for a group K. In particular, if G acts on G/H by left translations, we obtain

$$H^*_G(X) \cong R_H$$

or, if $T \subset G$ is a maximal torus

$$H^*_T(X) \cong R_T \otimes_{R_G} R_H.$$

Conversely, it can be shown: If X equals G/H, $\operatorname{rk} G = \operatorname{rk} H$, and G acts on X such that $H^*_G(X) \cong R_H$, then the action is linear. So, our main problem in proving the above theorems, is to show that a G-action on G/H has a "linear" equivariant cohomology. In the case G/T, G simple, this can be done in a surprisingly simple way using solely elementary facts from commutative algebra. For example it is shown, that any graded endomorphism of the k-algebra $H^*(G/T)$ is either trivial or an isomorphism, a result which in an other context has also been proved by A. Liulevicius.

1. Localization and Other Preliminary Facts

We begin with some commutative algebra. To fix notations let k be a field of characteristic zero. Denote by $k[x_1, \ldots, x_n]$ the graded commutative polynomial algebra in the indeterminants $x_1, \ldots, x_n$ of positive degrees. Note

$\deg x_i \equiv 0(2)$, $i=1, \ldots, n$. If B is a positively graded k-algebra let $B_+ \subset B$ denote the augmentation ideal.

Theorem 1.1. *Let $R=k[t_1, \ldots, t_r]$, $P=k[x_1, \ldots, x_n]$ and $I_0 \subset P_+$ the ideal generated by the homogeneous elements $f_1, \ldots, f_m$. Set $A_0=P/I_0$ and assume $\dim_k A_0 < \infty$. If A is a homogeneous R-algebra, free as an R-module, such that $A \otimes_R k \cong A_0$ in the graded sense, then $A \cong P_R/I$, where $P_R=R \otimes_k P$ and $I \subset P_R$ is an ideal generated by homogeneous elements of the form $1 \otimes f_j - r_j$, $j=1, \ldots, m$, where the r_j are elements of $R_+ P_R$. In particular A is a finitely generated R-module.*

Proof. From $A \otimes_R k = A_0$ it follows $\mathrm{rk}_R A = \dim_k A_0$, i.e., A is finitely generated.

Let $p: A \to A_0$ denote the canonical projection, let $\bar{X}_1, \ldots, \bar{X}_n \in A$ be homogeneous elements with $p(\bar{X}_i)=\bar{x}_i$, where $\bar{x}_i$ is the remainder of $x_i \in P$ modulo I_0. Then A_0 is generated as a k-vector space by 1 and some polynomials $p_j(\bar{x}_1, \ldots, \bar{x}_n)$, $j=1, \ldots, N-1$, $N=\dim_k A_0$. So, it follows by Nakayama, that A as a R-module is generated by 1 and the polynomials $p_j(\bar{X}_1, \ldots, \bar{X}_n)$, $j=1, \ldots, N-1$. We write $P_R=R[X_1, \ldots, X_n]$, $X_i=1 \otimes x_i$. So the homomorphism $P_R \to A$, $X_i \to \bar{X}_i$ is surjective, i.e., we can write $A=P_R/I$, where I is a certain ideal. Let $F_j=1 \otimes f_j \in P_R$ and $\bar{F}_j$ its image in A, then $p(\bar{F}_j)=0 \in A_0$. This shows $\bar{F}_j \in R_+ A$. Let $r_j \in R_+ P_R$ such that $\bar{F}_j - \bar{r}_j \in 0$ in A. Then $F_j - r_j \in I$, $j=1, \ldots, m$. Let $I' \subset I$ be the ideal generated by the $F_j - r_j$, $j=1, \ldots, m$. Let $\varphi: P_R/I' \to P_R/I$ denote the obvious quotient map and let $K=\ker \varphi$. Then by the freeness of A the exact sequence

$$0 \to K \to P_R/I' \to P_R/I \to 0$$

induces an exact sequence of k-vector spaces. In particular the induced homomorphism $\bar{\varphi}: (P_R/I) \otimes_R k \to A_0$ becomes an isomorphism. In other words $\bar{K}=K \otimes_R k=0$. If we have chosen the r_j homogeneous such that I' is homogeneously generated, the above sequence is an exact sequence of graded modules over the graded polynomial ring R. So $\bar{K}=0$ implies $K=0$ (see e.g. Hilton-Stammbach, p. 252, Prop. 7.2). This shows φ to be an isomorphism, *i.e.*, $I=I'$. □

Let now X be a topological space, let G be a group having a universal fibration $E_G \to B_G$, let G act on X as a group of homeomorphisms. We then consider the Borel-fibration

$$X \to E_G \times_G X \to B_G.$$

We set $X_G=E_G \times_G X$ and $H_G^*(X) \cong H^*(X_G)$, where we use Cech-cohomology with compact supports in the coefficient field k. We denote the projection of the above fibre bundle by π. If $R_G=H^*(B_G)$, then $H_G^*(X)$ is a R_G-algebra via π^* and the cup-product. Obviously, the above construction is functorial with respect to equivariant maps. The functor obtained in this way is called the ordinary equivariant cohomology with coefficients in the constant sheaf $\underline{k}$ given by the ground field. In the following we assume always, that G is acting with finitely many orbit types.

Let X be a space such that $H^*(X)$ is multiplicatively generated by finitely many elements of even degree, i.e.,

$$H^*(X) \cong k[x_1, \ldots, x_n]/I_0,$$
$$\deg x_i \equiv 0 \pmod 2,$$
$$I_0 = (f_1, \ldots, f_m), \quad f_j \text{ homogeneous.}$$

Let G be a compact Lie group acting on X, then the Serre-spectral sequence of $X \to X_G \to B_G$ degenerates at the 2-term. This shows $H_G^*(X)$ to be a free graded finitely generated R_G-module. Let $R_G^+ \subset R_G$ the maximal ideal generated by the elements of strictly positive degree, then $R_G/R_G^+ = k$. The homomorphism $i^*: H_G^*(X) \to H^*(X)$ induced by the inclusion $X \subset X_G$ is then given by the projection map

$$H_G^*(X) \to H_G^*(X)/R_G^+ H_G^*(X) \cong H^*(X).$$

(See e.g. [1], Th. 14.2) Therefore by Theorem 1.1 we get

$$H_G^*(X) = R_G[X_1, \ldots, X_n]/I,$$

where the defining ideal I is generated by elements of the form $F_j - r_j$, $r_j \in R_G^+ \cdot R_G[X_1, \ldots, X_n]$, $F_j = 1 \otimes f_j$, $j = 1, \ldots, m$. Obviously, the polynomials r_j depend from the action. Therefore we have to investigate what kind of polynomials r_j can actually occur from a G-action. A partial answer to this question is given by the localization theorem in ordinary equivariant cohomology theory. Let $S \subset R_G$ a multiplicatively closed subset, let $X^S \subset X$ be the closed G-equivariant subspace defined as follows: If $x \in X$ is a point, $G_x \subset G$ the closed isotropy subgroup of x, let $P_x \subset R_G$ be the kernel of the natural restriction homomorphism $R_G \to R_{G_x}$. Then X^S is the subset of all points $y \in X$, s.t., $P_y \cap S = \varnothing$.

Theorem 1.2 (Localization theorem) [4]. *Let X be a compact or paracompact Hausdorff space of finite k-cohomology dimension. Let the compact Lie group G act on X with finitely many orbit types. If $S \subset R_G$ is a multiplicatively closed subset, then the inclusion $X^S \subset X$ induces an isomorphism of graded $S^{-1}R_G$-algebras $S^{-1}H_G^*(X) \cong S^{-1}H_G^*(X^S)$.*

As an example, we consider the special case $S = R_G - \{0\}$, G a torus. Then X^S is the fixed set $F = F(G; X)$ and $S^{-1}H_G^*(X) \cong S^{-1}R_G \otimes_k H^*(F)$.

Theorem 1.3 (Special case of Th. IV.1 in [4]). *Let G be a torus and X a compact space or a paracompact space with finite k-cohomology dimension such that $H^*(X)$ is generated as a k-algebra by elements of even degree $X_1, \ldots, X_n$. Let $P_G = R_G[X_1, \ldots, X_n]$ and $H_G^*(X) = P_G/I$. If $S = R_G - \{0\}$, we write $J = S^{-1}I$ as an ideal in $S^{-1}P_G$.*

(i) *If $J = \bigcap_{i \in \Delta} J_i$ is the reduced primary decomposition of J, then there is a natural correspondence between the sets $\{J_i\}_{i \in \Delta}$ and $\pi_0 F(T; X)$ of connected components F_i of the fixed set such that $S^{-1}R_G \otimes_k H^*(F_i) \cong S^{-1}P_G/J_i$ and the localization isomorphism is given by the canonical homomorphism*

$$S^{-1}P_G/J \to \prod_{i \in \Delta} S^{-1}P_G/J_i.$$

(ii) *The set of associated prime ideals of J is given by a family $\{\underline{m}_i\}_{i\in\Delta}$ of maximal ideals $\underline{m}_i=(X_1-\omega_{i1},\ldots,X_n-\omega_{in})$, where the ω_{ij} are homogeneous elements of R_G with $\deg\omega_{ij}=\deg X_j$.*

As a simple consequence it follows:

Lemma 1.1. *Under the above assumptions the set of associated prime ideals $\mathrm{Ass}_{P_G} H_G^*(X)$ is in a natural way bijective with the set of connected components $\pi_0 F(G;X)$. Moreover any $P\in\mathrm{Ass}_{P_G} H_G^*(X)$ is of the form*

$$P_i=(X_1-\omega_{i1},\ldots,X_n-\omega_{in}),$$

i.e., $H_G^*(X)/P_i H_G^*(X)\cong R_G$.

Proof. The first statement follows from (i) of the above theorem. The second is a consequence of Bourbaki, Alg. comm., Ch IV, 1.2, Prop. 5.

2. A Special Case

As a very important special case we are considering spaces X, such that $H^*(X)$ as a graded k-algebra is generated by finitely many elements $x_1,\ldots,x_n$ of degree two. Then $H_G^*(X)$ is a free R_G-module and the long exact sequence of the couple (X_G, F_G) degenerates into the short exact sequence

$$(*)\qquad 0\to H_G^*(X)\to H_G^*(F)\to H_G^{\mathrm{odd}}(X,F)\to 0.$$

(By the localization theorem $H_G^*(X,F)$ is a torsion module over R_G. By the freeness of $H_G^*(X)$ the second arrow must be injective.)

Theorem 2.1. *Let G be a torus and X a finitistic space such that $H^*(X)$ is generated as a k-algebra only by elements of degree two. Let $H^*(X)=P/I_0$, $H_G^*(X)=P_G/I$, then we have:*

(i) *The set of associated prime ideals $\mathrm{Ass}_{P_G} H_G^*(X)=\{P_i\}_{i\in\Delta}$, is in a natural correspondence with the connected components of the fixed set; moreover any P_i is of the form $P_i=(X_1-\omega_{i1},\ldots,X_n-\omega_{in})$, $\omega_{ij}\in R_G$, where the homomorphism $H_G^*(X)\to H_G^*(\{s_i\}\times B_G)$, $s_i\in F_i$, is given by $X_j\to\omega_{ij}$.*

(ii) *Let $I=\bigcap_{i\in\Delta} I_i$ the corresponding reduced primary decomposition, then the sequence $(*)$ is equivalent to the canonical sequence*

$$0\to P_G/I\to\prod_{i\in\Delta} P_G/I_i\to\mathrm{coker}\to 0.$$

Proof. The first statement is an immediate consequence of Lemma 1.1. For the second statement we observe that the homomorphism $r: H_G^*(X)\to H_G^*(F')$ induced by the inclusion of a connected fixed space component $F'\subset X$ is given by $X_j\to Y_j+\omega_j$ with $Y_j\in H^2(F')$, $\omega_j\in H^2(B_G)$. By the localization theorem the homomorphism $S^{-1}r: S^{-1}H_G^*(X)\to S^{-1}H_G^*(F')$ is an epimorphism. From the isomorphism $S^{-1}H_G(F')\cong S^{-1}R_G\otimes_k H^*(F')$ it follows that $Y_1,\ldots,Y_n\in H^2(F')$ is a set of generators of the k-algebra $H^*(F')$. This shows $\bar{r}=r\otimes 1: H_G^*(X)$

$\otimes_{R_G} k \to H_G^*(F') \otimes_{R_G} k \cong H^*(F')$ to be an epimorphism. Let $C = \operatorname{coker} r$, then $\bar{C} = C \otimes_{R_G} k = 0$. Because C is finitely generated, the R_G-module C must be the zero module. So, the map r is an epimorphism too. We conclude that the restriction map $i_i^*: H_G^*(X) \to H_G^*(F_i)$ is an epimorphism for all i. This shows i_i^* to be a projection map $P_G/I \to P_G/I_i$ for an ideal $I_i \subset P_G$ with $I \subset I_i$. Let $\{I_i\}_{i \in \Delta}$ be the family of ideals obtained in this way. We define $I' = \bigcap_{i \in \Delta} I_i$. Then we consider the following commutative diagram:

$$\begin{array}{ccc} H_G^*(X) & \xrightarrow{i^*} & \prod_{i \in \Delta} H_G^*(F_i) \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \cong} \\ P_G/I' & \xrightarrow[\mu]{} & \prod_{i \in \Delta} P_G/I_i \end{array}$$

Here, the first horizontal arrow is induced by the inclusion of the fixed set, whereas μ is the obvious canonical map. The map p is the projection induced by $I \subset I'$. By the injectivity of i^* it follows that $I' \subset I$, i.e., $I = I'$. So, p is an isomorphism and we are done.

As a corollary we obtain:

Corollary. *Let X be a compact closed oriented manifold, $H^*(X) = P/I_0$, $P = k[x_1, \ldots, x_n]$, $\deg x_i = 2$, let G a torus acting continuously and nontrivially and with finitely many orbit types on X. Then there exists a connected component F of the fixed set $F(T; X)$ such that the restriction homomorphism $H_G^*(X) \to H_G^*(\{q\}) \cong R_G$, $q \in F$, is nontrivial, i.e., the prime ideal P of the component F is given as $P = (X_1 - \omega_1, \ldots, X_n - \omega_n)$, $\omega_i \neq 0$, for at least one index i, $1 \leqq i \leqq n$.*

Proof. If all the associated prime ideals P of the P_G-module $H_G^*(X)$ are trivial, then $\operatorname{Ass}_{P_G} H_G^*(X) = \{P\}$, $P = (X_1, \ldots, X_n)$. Then, by the theorem, we have $H_G^*(X) = P_G/I$, $\operatorname{Rad}(I) = P$. Therefore $H_G^*(X) \cong R_G \otimes_k H^*(F)$, F being the single connected component. It follows $H^*(X) \cong H^*(F)$. This implies $\dim F \geqq \dim_c F = \dim_c X$, where $\dim_c(-)$ denotes the k-cohomology dimension. Now, because X is a closed compact oriented manifold, it follows $\dim_c X = \dim X$. This shows $\dim F \geqq \dim X$, i.e., the action is trivial, which is a contradiction.

3. Equivariant Cohomology of Linear Actions

Let G be a compact Lie group, $H \subset G$ a closed subgroup, let Γ be a compact connected Lie group and let $\rho: \Gamma \to G$ be a homomorphism. Then ρ induces an action of Γ on $X = G/H$ in the following way:

$$\begin{aligned} \varphi_\rho: \Gamma \times G/H &\to G/H \\ (\gamma, gH) &\to \rho(\gamma)\, gH. \end{aligned}$$

In the following such actions will be called linear or homomorphic. If ρ is injective, then the induced action φ_ρ is effective. We assume, that ρ is always

injective, so Γ can be considered as a subgroup of G. If $H \subset G$ is a closed connected subgroup of maximal rank, it is not very difficult to compute the equivariant cohomology $H^*_\rho(G/H) = H^*_\Gamma(G/H)$, $\operatorname{char} k = 0$, of a linear action. Let $i: H \subset G$ denote the inclusion homomorphism. We consider the obvious fiber square

$$\begin{array}{ccc} E_\Gamma \times_\Gamma G/H & \longrightarrow & B_H \\ \downarrow & & \downarrow{\scriptstyle Bi} \\ B_\Gamma & \xrightarrow[B\rho]{} & B_G \end{array}$$

Let $R_G = H^*(B_G)$, $R_H = H^*(B_H)$ and $R_\Gamma = H^*(B_\Gamma)$, then R_H and R_Γ are R_G-modules by the induced mappings $(Bi)^*$ and $(B\rho)^*$. In particular, R_H is a free R_G-module. The E_2-term of the corresponding Eilenberg-Moore spectral sequence is then given by:

$$\begin{aligned} E_2 &\cong \operatorname{Tor}_{R_G}(R_\Gamma, R_H) \\ &\cong R_\Gamma \otimes_{R_G} R_H. \\ E_2^{*,q} &= 0, \quad q > 0. \end{aligned}$$

Therefore the spectral sequence degenerates at the 2-term and we get:

$$H^*_\Gamma(G/H) \cong R_\Gamma \otimes_{R_G} R_H,$$

where the isomorphism is induced by the natural map $R_\Gamma \otimes_{R_G} R_H \to H^*_\Gamma(G/H)$.

Lemma 3.1. *Let G be a compact connected Lie group and $H \subset G$ a closed connected subgroup of maximal rank, let $X = G/H$, let $\varphi_\rho: \Gamma \times X \to X$ be a linear action of a compact connected Lie group Γ, then the canonical homomorphism $R_\Gamma \otimes_{R_G} R_H \to H^*_\Gamma(X)$ is an isomorphism of graded R_Γ-algebras.*

As an example take $H = T$, a fixed maximal torus, let $\rho: \Gamma \to T$ be an isomorphism, then $H^*_\Gamma(G/T) \cong R_T \otimes_{R_G} R_T$.

In the following we want to investigate in greater detail the structure of $\operatorname{Spec} R_T \otimes_{R_G} R_T$. Let $\underline{g}$ be the Lie algebra of G, $\underline{h} \subset \underline{g}$ the Lie algebra of T (both real). Let $\mathcal{O}_{\underline{h}} = k[\underline{h}]$ denote the ring of k-valued regular functions on $\underline{h}$. Then the Weyl group $W = W_G = NT/T$ acts in a natural way on $\underline{h}$ and $\mathcal{O}_{\underline{h}}$. Let $\mathcal{O}_{\underline{h}}^W \subset \mathcal{O}_{\underline{h}}$ be the corresponding invariant ring. Because W acts on $\underline{h}$ as a real reflection group, the ring $\mathcal{O}_{\underline{h}}^W$ is a polynomial ring [2]. It is a well known fact in fiber bundle theory that the natural homomorphism $\pi^*: R_G \to R_T$ induced by the projection map $\pi: B_T \to B_G$ can be identified with the inclusion $\mathcal{O}_{\underline{h}}^W \subset \mathcal{O}_{\underline{h}}$ of invariants [1]. The push-out diagram

$$\begin{array}{ccc} R_T & \longrightarrow & R_T \otimes_{R_G} R_T \\ \uparrow & & \uparrow \\ R_G & \longrightarrow & R_T \end{array}$$

corresponds to the pullback-diagram

$$\begin{array}{ccc} \underline{h} \times_{\underline{h}/W} \underline{h} & \longrightarrow & \underline{h} \\ \downarrow & & \downarrow p \\ \underline{h} & \xrightarrow[p]{} & \underline{h}/W, \end{array}$$

where $p: \underline{h} \to \underline{h}/W$ is the obvious quotient morphism. Because of the $(\underline{h}/W)$-automorphisms induced by the elements $\sigma \in W$,

$$\begin{array}{ccc} \underline{h} & \xrightarrow{\sigma} & \underline{h} \\ & \searrow \quad \swarrow & \\ & \underline{h}/W, & \end{array}$$

we obtain $|W|$-many sections $s_\sigma: \underline{h} \to \underline{h} \times_{\underline{h}/W} \underline{h}$, $\sigma \in W$, i.e., there exist $|W|$-many different surjective $\mathcal{O}_{\underline{h}}$-homomorphisms

$$\varphi_\sigma: \mathcal{O}_{\underline{h}} \otimes_{\mathcal{O}_{\underline{h}/W}} \mathcal{O}_{\underline{h}} \to \mathcal{O}_{\underline{h}}$$

or

$$\varphi_\sigma: R_T \otimes_{R_G} R_T \to R_T.$$

Let

$$R_T^l = k[t_1, \ldots, t_n],$$
$$R_T^r = k[x_1, \ldots, x_n],$$

then we have

$$R_T \otimes_{R_G} R_T = \frac{R_T^l \otimes_k R_T^r}{I} \cong \frac{R_T[x_1, \ldots, x_n]}{I},$$

where the ideal I is generated by the elements $g_i(x) - g_i(t)$, all i. Here $g_1, \ldots, g_n$ is a system of algebraically independent homogeneous invariants in R_T, such that $R_G = k[g_1, \ldots, g_n]$. So we see that the kernel of φ_σ as an ideal is generated by the "diagonal" elements $x_i - t_i^\sigma$, $\sigma \in W$. Let $P^\sigma = (x_1 - t_1^\sigma, \ldots, x_n - t_n^\sigma) \subset R_T^l \otimes_k R_T^r$, then P^σ is an associated prime ideal of I. Conversely, the following lemma shows, that any associated prime ideal of I is given by a P^σ.

Lemma 3.2. *Let* $P^\sigma = (x_1 - t_1^\sigma, \ldots, x_n - t_n^\sigma) \subset R_T[x]$, $\sigma \in W$, *then* $I = \bigcap_{\sigma \in W} P^\sigma$, *in particular, the action of* W *on* R_T *induces a transitive and free action on the set of associated prime ideals* $\mathrm{Ass}_{R_T[x]}(R_T[x]/I)$.

Proof. Let $A = R_T[x]/I$, let $I' = \bigcap_{\sigma \in W} P^\sigma$ and $B = R_T[x]/I'$. Obviously, we have $I \subset P^\sigma$, all $\sigma \in W$, i.e., $I \subset I'$. The inclusion $I \subset I'$ induces an epimorphism $\varphi: A \to B$ of graded R_T-modules. Now, A is a free graded R_T-module, therefore $\dim_k A \otimes_{R_T} k = |W|$, implies $\mathrm{rk}_{R_T} A = |W|$. So B is a finitely generated R_T-module. Let $S = R_T - \{0\}$, let $Q = S^{-1} R_T$, then $\dim_Q S^{-1} B = |W|$. By the semicontinuity of the dimension function, the generic dimension of B is less or equal to the closed dimension, i.e.,

$$\dim_Q S^{-1} B \leqq \dim_k B \otimes_{R_T} k.$$

Moreover, the existence of the surjection $\varphi: A \to B$ implies

$$\dim_k B \otimes_{R_T} k \leqq \dim_k A \otimes_{R_T} k.$$

So we conclude

$$\dim_k B \otimes_{R_T} k = \dim_Q S^{-1} B = |W|.$$

This implies B to be a free R_T-module with $\mathrm{rk}_{R_T} B = \mathrm{rk}_{R_T} A$, i.e., φ is an isomorphism, and so it follows $I = I'$. □

4. On a Rigidity Property of Flag Manifolds

Let k be a field of characteristic zero, let $\mathscr{A}_k$ the category whose objects are the graded commutative k-algebras A with $A = P/J$, $P = k[x_1, \ldots, x_n]$, $\deg x_i = 2$, $J = (g_1, \ldots, g_m)$, g_j homogeneous with respect to the above grading and $\deg g_1 < \deg g_2 \leqq \ldots \leqq \deg g_m$, $\deg g_1 = 4$. Moreover, let g_1 be the quadratic form of a definite symmetric bilinear form. As morphisms we take the gradation preserving homomorphisms. So, in the defining ideal J there exists precisely one definite quadratic form $q = g_1$.

Lemma 4.1. *Let $\varphi: A \to B$ be a morphism in $\mathscr{A}_k$, then either $\mathrm{Im}\,\varphi = B^0 \subset B$ or φ is an epimorphism.*

Proof. Let $A = P/J^A$, $B = S/J^B$, $P = k[x_1, \ldots, x_n]$, $S = k[y_1, \ldots, y_m]$, $J^A \subset P$, $J^B \subset S$, the defining ideals. We can assume φ to be induced by a gradation preserving homomorphism Φ, such that the diagram

$$\begin{array}{ccc} J^A & \xrightarrow{\Phi|_{J^A}} & J^B \\ \downarrow & & \downarrow \\ P & \xrightarrow{\Phi} & S \\ \downarrow & & \downarrow \\ A & \xrightarrow{\varphi} & B \end{array}$$

is commutative. Let $g^A \subset J^A$ and $g^B \subset J^B$ be the corresponding quadratic forms. Then we have by degree reasons: $\Phi(g^A) = a g^B$, $a \in k$. Now, we have

$$\Phi(g^A(x_1, \ldots, x_n)) = g^A(\Phi(x_1), \ldots, \Phi(x_n)) = a g^B(y_1, \ldots, y_m).$$

If $a = 0$, then it follows $g^A(\Phi(x_1), \ldots, \Phi(x_n)) = 0$, and therefore by definiteness $\Phi(x_1) = \Phi(x_2) = \ldots = \Phi(x_n) = 0$, i.e., $\mathrm{Im}\,\varphi = B^0$.

Let $a \neq 0$, then by degree reasons again, we have $\Phi(x_i) = \omega_i$, $\omega_i = \sum_{j=1}^{m} \omega_{ij} y_j$, $\omega_{ij} \in k$. Therefore, we have $g^A(\omega_1, \ldots, \omega_n) = a g^B(y_1, \ldots, y_m)$.

Let $\omega: k^m \to k^n$ the linear map given by the (m, n)-matrix $\omega = (\omega_{ij})$. Then, for a $v \in k^m$, we have $g^B(v) = \frac{1}{a} (g^A \omega)(v)$. Let $\mathrm{rk}\,\omega < m$, then there exists a nonvanish-

ing v such that $\omega(v)=0$. So $g^B(v)=0$ in contradiction to the definiteness of g^B. This shows $\operatorname{rk}\omega=m$, i.e., the elements ω_i, $i=1,\ldots,n$, span the subspace B^2. So Φ and φ must be epimorphisms.

Lemma 4.2. *Let G be a compact simple connected Lie group, let $T\subset G$ a maximal torus. Then $H^*(G/T;\mathbb{R})$ is an object in the category $\mathscr{A}_{\mathbb{R}}$.*

Proof. We have $H^*(G/T;\mathbb{R})\cong H^*(B_T;\mathbb{R})/(\pi^* H^*_+(B_G;\mathbb{R}))$, $H^*(B_G;\mathbb{R})\cong\mathbb{R}[g_1,\ldots,g_n]$, g_j homogeneous, $n=\operatorname{rk}G$. The degrees $d_j=\deg g_j$ are given by the exponents e_j of the Weylgroup $W_G=NT/T$ as follows: $d_j=2e_j+2$, $j=1,\ldots,n$. See e.g. [6], p. 112. Therefore it exists precisely one generator, say g_1, with $d_1=4$, in other terms, g_1 is a quadratic form. By degree reasons any WG-invariant quadratic polynomial $q\in H^*(B_T;\mathbb{R})$ must be proportional to g_1. Let q the restriction to the Lie algebra LT of T of the quadratic polynomial which stems from the Killing form κ: $LG\times LG\to\mathbb{R}$. By the compactness of G the form q must be negatively definite and WG-invariant. So, g_1 is also definite.

Theorem 4.1. *Let X be a compact closed oriented manifold, such that $H^*(X;\mathbb{R})\in|\mathscr{A}_{\mathbb{R}}|$. Let G be a compact connected simple Lie group acting on X with finitely many orbit types. Then there exists a surjective graded R_G-algebra homomorphism $\varphi: H^*_G(X)\to R_T$ and therefore a surjective graded $\mathbb{R}$-algebra homomorphism $\bar{\varphi}: H^*(X)\to H^*(G/T)$.*

Corollary. *Let X be a compact closed oriented manifold such that $H^*(X;\mathbb{R})\in|\mathscr{A}_{\mathbb{R}}|$. Let G be a compact connected simple Lie group acting on X with finitely many orbit types. Then we have the following inequality of Euler characteristics:*

$$\chi(X)\geqq\chi(G/T).$$

Proof. The commutative diagram (see e.g. [4], p. 38)

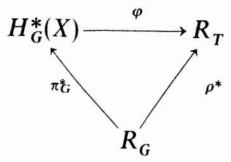
$$\begin{array}{ccc} X_T & \xrightarrow{\tilde{\rho}} & X_G \\ {\scriptstyle\pi_T}\downarrow & & \downarrow{\scriptstyle\pi_G} \\ B_T & \xrightarrow[\rho]{} & B_G \end{array}$$

induces a commutative diagram

$$\begin{array}{ccc} H^*_G(X) & \xrightarrow{\tilde{\rho}^*} & H^*_T(X) \\ {\scriptstyle\pi_G^*}\uparrow & & \uparrow{\scriptstyle\pi_T^*} \\ R_G & \xrightarrow[\rho^*]{} & R_T \end{array}$$

Here we have the isomorphism $H^*_T(X)\cong H^*_G(X)\otimes_{R_G}R_T$, which follows from [4], p. 38, proposition 1 (ii). First, we remark, that the choice of a fixed point $q\in F(T,X)$ induces a section s in the fibration $X_T\to B_T$ in the following way:
If $b\in B_T$, the choose an element $e\in E_T$ which projects to b. Then set $s(b)=[e,q]\in E_T\times_T X$. It follows $\pi_T\circ s=id_{B_T}$, i.e., s is a section. In particular, we

have $s^* \circ \pi_T^* = id_{R_T}$. Define φ to be the composition $\varphi = s^* \circ \tilde{\rho}^*$. We then have a commutative triangle



as can be shown by a simple diagram chasing argument. In other terms, φ is a graded R_G-algebra homomorphism. We want to describe the homomorphism φ a little bit more closely: Let

$$H_G^*(X) = \frac{R_G[X_1, \ldots, X_n]}{I}, \quad \text{(Th. 1.1, ii)}$$

let $J \subset R_T[X_1, \ldots, X_n]$ be the ideal generated by the elements of I, using the canonical inclusion $R_G[X_1, \ldots, X_n] \subset R_T[X_1, \ldots, X_n]$, then we have

$$H_T^*(X) = \frac{R_T[X_1, \ldots, X_n]}{J}.$$

Now, we write

$$R_T = \frac{R_G[t_1, \ldots, t_n]}{K},$$

where the ideal K is generated by $g_k - g_k(t_1, \ldots, t_n)$, $k = 1, \ldots, n$. Here we put $R_G = \mathbb{R}[g_1, \ldots, g_n]$. Now, s^* is given by $X_j \to w_j$, $j = 1, \ldots, n$, where the w_j are linear polynomials in the t's. So, it follows $\varphi(X_j) = w_j$, $j = 1, \ldots, n$. By Th. 2.1, Cor., the point q can be chosen in such a way, that at least one of the w_j does not vanish. Therefore φ induces nontrivial graded homomorphism of $\mathbb{R}$-algebras:

$$\bar{\varphi}\colon H^*(X) \cong H_G^*(X) \otimes_{R_G} \mathbb{R} \to R_T \otimes_{R_G} \mathbb{R} \cong H^*(G/T).$$

By the Lemmata 4.1 and 4.2 the homomorphism $\bar{\varphi}$ is epi. Because we are considering graded modules over the graded polynomial ring R_G, the map φ must be an epimorphism too.

As a second corollary of Th. 4.1 we obtain:

Lemma 4.3. *Let X be compact closed oriented manifold with $H^*(X; \mathbb{R}) \in |\mathscr{A}_{\mathbb{R}}|$ and $\chi(X) = |WG|$. Let the compact connected simple Lie group G set on X with finitely many orbit types. Then $H_G^*(X)$ is isomorphic to R_T as a graded R_G-algebra.*

Now, we come to the proof of our main theorems:

Theorem 1. *Let X be a simply connected closed oriented compact manifold with $H^*(X; \mathbb{R}) \cong H^*(G/T; \mathbb{R})$, G a simple compact Lie group, $T \subset G$ its maximal torus. If G acts on X with finitely many orbit types, then $X = G/T$ and every G-action on X is conjugate to the natural action.*

As a corollary we get:

Theorem 1'. *Let G be a compact simple Lie group, $T \subset G$ a fixed maximal torus, let G act on $X = G/T$ with finitely many orbit types. Then the action is conjugate to the natural action.*

Proof of Theorem 1. By the lemma 4.3 the R_G-algebra $H_G^*(X)$ is isomorphic to R_T. It follows that $H_T^*(X)$ is isomorphic to $R_T \otimes_{R_G} R_T$. The lemma 3.2 says that the Weyl group WG acts freely and transitively on the set of associated prime ideals of the $R_T \otimes_{\mathbb{R}} R_T$-module $H_T^*(X)$. This implies that the action of WG on the set $\pi_0 F(T; X)$ of connected components of the fixed set $F(T; X)$ is free and transitive. Let F be a connected component of $F(T; X)$ and let $x \in F$. Denote by G_x the isotropy group of x, let $G_x^0 \lhd G_x$ its one-connected component. Then it follows that the Weyl group WG_x^0 of G_x^0 is trivial, i.e., $G_x^0 = T$. This shows $\dim G(x) = \dim G/T = \dim X$, i.e., $G(x) = X$. Because X is simply connected we have $G_x = G_x^0 = T$. □

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