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# Unfolding Bifurcations of an Elliptic Boundary Value Problem

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The object of this note is to show that if one considers the two parameter family of elliptic equations on a domain $\Omega \subset \mathbb{R}^n$ with say $C^\infty$ boundary $\partial\Omega$, given by

$$\begin{aligned} \Delta u + \lambda f(u) &= \mu g \quad \text{in } \Omega \\ u &= \mu \phi \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

$(\lambda, \mu) \in \mathbb{R}^2$, $f$, $g$: $\mathbb{R} \to \mathbb{R}$ smooth, $f(0)=0$, and $\phi \not\equiv 0$ smooth, then under some mild restrictions on $f$ a bifurcation around the solution $u \equiv 0$ near $(\lambda_0, \mu_0)$, where $\lambda_0$ is a simple eigenvalue of the linearized equation

$$\begin{aligned} \Delta w + \lambda_0 f'(0) w &= 0 \quad \text{in } \Omega \\ w &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{2}$$

and $\mu_0 = 0$, can be described by the cusp-catastrophe of Thom.

We begin by developing some analytical tools necessary to prove the main theoretical result. For this we follow [7].

Let $G$ be a Banach space with a continuous symmetric bilinear pairing $\langle\ ,\ \rangle$ with the property that $\langle w, w\rangle \geqq 0$ with equality holding only when $w=0$. Let $U \subset G$ be open with $E$: $U \to \mathbb{R}$ a $C^r$, $r \geqq 1$, real valued map. Then $E$ is called $C^r$-*acceptable* with respect to $\langle\ ,\ \rangle$ if for each $p \in U$ there is a $C^{r-1}$ vector field $X$: $U \to G$ such that the differential of $E$, $dE$ satisfies

$$dE(p)[h] = \langle X(p), h\rangle \tag{3}$$

for all $h \in G$ and $p \in U$. $X$ is the gradient of $E$ with respect to $\langle\ ,\ \rangle$. If $X$ exists it is necessarily unique. The following notion will be crucial.

**Definition.** Let $E$: $U \to \mathbb{R}$ be a $C^r$-acceptable map. A critical point $q \in U$ is *non-degenerate* if the Frechét derivative of the *associated vector field* $X$ at $q$, say $DX(q)$: $G \to G$ is an isomorphism.

Now let $E_{\lambda,\mu}$ be a two parameter family of maps ($(\lambda,\mu)\in\mathbb{R}^2$) with $(\lambda,\mu,p)\to E_{\lambda,\mu}(p)$ $C^r$, $r>1$. If each $E_{\lambda,\mu}$ is $C^r$-acceptable and $(\lambda,\mu,p)\to X_{\lambda,\mu}(p)$ is $C^{r-1}$, we say that $E_{\lambda,\mu}$ is a $C^r$-acceptable family.

The following normal form theorem from [1] is crucial for us:

**Theorem 1.** *Suppose that $E_{\lambda,\mu}$: $U\to\mathbb{R}$ is a $C^\infty$-acceptable family with $q\in U$ a critical point of $E_{\lambda_0,\mu_0}$. Assume further that the kernel $J$ of the Hessian $D^2E_{\lambda_0,\mu_0}(q)$: $G\times G\to\mathbb{R}$ is a one dimensional subspace spanned by $e\in G$. Let $J^\perp$ denote a complement of $J$ in $G$; $G=J\oplus J^\perp$. Let $E$ denote $E_{\lambda_0,\mu_0}$ and assume further that the following partial derivatives all evaluated at $(\lambda_0,\mu_0,q)$ satisfy the conditions*

(i) $$\frac{\partial^3 E}{\partial e^3}=0;$$

(ii) $$\frac{\partial^4 E}{\partial e^4}\neq 0;$$

(iii) $$\frac{\partial^3 E_{\lambda,\mu}}{\partial\lambda\,\partial e^2}\neq 0;$$

(iv) $$\frac{\partial^2 E_{\lambda,\mu}}{\partial\mu\,\partial e}\neq 0.$$

*After a suitable translation we may assume that $q=0$. Then there are neighborhoods $W$ of $q$ in $U$ and $V$ of $(\lambda_0,\mu_0)$ in $\mathbb{R}^2$ and an origin (in $G$) preserving diffeomorphism*

$$\Psi\colon\ V\times W\to\mathbb{R}^2\times G=\mathbb{R}^2\times J\times J^\perp$$

*such that $E(\lambda,\mu,p)=E_{\lambda,\mu}(p)$ satisfies*

(a) $E\circ\Psi(\lambda,\mu,t,p)=\frac{1}{4}t^4-\zeta t^2+\gamma t+\frac{1}{2}D^2E_{\lambda,\mu}(q)[p,p]+E_{\lambda,\mu}(q)$,

*where $\zeta$ and $\gamma$ depend linearily on $\lambda$ and $\mu$ and where $D^2E_{\lambda,\mu}(q)[p,p]$ denotes the second derivative of $E_{\lambda,\mu}$ along $J^\perp$ evaluated at $q$. Thus, for fixed $(\lambda,\mu)$*

$$p\mapsto D^2E_{\lambda,\mu}(q)[p,p]$$

*is a quadratic form on $G$, which is "non-degenerate" in the weak sense that if for all $h$, $D^2E_{\lambda,\mu}(q)[p,h]=0$, then $p=0$.*

(b) *If* $\dfrac{\partial^2 E_{\lambda,\mu}}{\partial\lambda\,\partial e}=0$, $\dfrac{\partial^4 E}{\partial e^4}>0$ *and* $\dfrac{\partial^3 E_{\lambda,\mu}}{\partial\lambda\,\partial e^2}<0$,

*then in the above normal form one can take $\zeta=\lambda$ and $\gamma=\mu$, and the diffeomorphism $\Psi=(\Psi_1,\Psi_2,\Psi_3,\Psi_4)$ so that at $(\lambda_0,\mu_0,q)$*

$$\frac{\partial\Psi_2}{\partial\lambda}>0.$$

*These last conditions are designed to locate the axis and direction of the cusp and hence the direction of bifurcation. In this last coordinate representation the set of critical points (locally, of course) is the set $(\lambda,\mu,t,0)$ in $\mathbb{R}^2\times J\times J^\perp$ with*

$$t^3-\lambda t+\mu=0,$$

*i.e. the cusp of Thom.*

We now apply these ideas to the study of bifurcation of solutions to (1). Let $s>\frac{n}{2}+2$ and introduce the space

$$\mathscr{H}_\mu=\{u\in H^s(\Omega,\mathbb{R}): u=\mu\phi \text{ on } \partial\Omega\}.$$

This is an affine subspace of the Sobolev space $H^s(\Omega,\mathbb{R})$ of $s$-times differentiables functions with (distributional) derivatives which are square integrable. Let

$$\mathscr{H}=\bigcup_{\mu\in\mathbb{R}}\mathscr{H}_\mu.$$

Then $\mathscr{H}$ has the structure of a $C^\infty$ trivial vector bundle over $\mathbb{R}$ with trivialization

$$\mathscr{H}_0\times\mathbb{R}\to\mathscr{H} \tag{4}$$

given by $(v,\mu)\to v+\mu\psi$, where $\psi$ is the unique solution of ($\phi\not\equiv 0$)

$$\begin{aligned}\Delta\psi&=0 \quad \text{in } \Omega\\ \psi&=\phi \quad \text{on } \partial\Omega\end{aligned} \tag{5}$$

Consider the energy functional $E_{\lambda,\mu}\colon \mathscr{H}\to\mathbb{R}$ given by

$$E_{\lambda,\mu}(u)=\tfrac{1}{2}\int_\Omega \nabla u\cdot\nabla u\,dx-\lambda\int_\Omega F(u)\,dx+\mu\int_\Omega g(x)u\,dx, \tag{6}$$

where $F\colon\mathbb{R}\to\mathbb{R}$ is $C^\infty$ and $F'(s)=f(s)$.

Note that the parameter $\mu$ in (6) also enters in the space $\mathscr{H}$. This is not uncommon in complicated variational problems (e.g. see [1, 7]).

By using the trivialization (4) and Eq. (5) we may view $E_{\lambda,\mu}$ as a function on $\mathscr{H}_0$ given by

$$\begin{aligned}E_{\lambda,\mu}(v)=\tfrac{1}{2}\int_\Omega \nabla(v+\mu\psi)\cdot\nabla(v+\mu\psi)\,dx-\lambda\int_\Omega F(v+\mu\psi)\,dx\\ +\mu\int_\Omega g(x)(v+\mu\psi)\,dx.\end{aligned} \tag{7}$$

The Euler-Lagrange operator associated to (7) at a critical point $v$ is given by (for fixed $\lambda$ and $\mu$)

$$v\mapsto -\Delta v-\lambda f(v+\mu\psi)+\mu g.$$

From the Sobolev embedding theorem it follows that $H^s(\Omega)$, $s>n/2+2$, may be viewed as a subspace of the $C^2$ functions on $\Omega$. One consequence of this is that $(\lambda,\mu,v)\to E_{\lambda,\mu}(v)$ is $C^\infty$ as a functional on $\mathbb{R}^2\times H^s(\Omega)$. If, however, one worked with the space $H^1(\Omega)$, which might seem natural, we would not in general obtain that $E$ is $C^\infty$. But with $H^s(\Omega)$, with $s>n/2+2$, all the critical points of $E_{\lambda,\mu}$ for fixed $(\lambda,\mu)$ must be degenerate in the classical sense that the Hessian induces an isomorphism between the space $\mathscr{H}_0$ and its dual. Moreover the generalizations of classical singularity theory to Banach spaces as carried out

by Magnus [4] and Chillingworth [2] do not apply to these problems (see for example the introduction to [7]).

The second author developed an approach to singularity theory on Banach manifolds which would apply to problems in the classical calculus of variations. This was first presented in abstract form in [6], 1976 where it was applied to normal forms for the energy whose critical points are geodesics on a Riemannian manifold. Somewhat later less abstract versions were given in [1] and [7], where applications to minimal surfaces were given. Recent generalizations of this approach have been given by Golubitsky and Marsden [3].

Our main result is another application of this approach.

**Theorem 2.** *Let $f, g\colon \mathbb{R}\to\mathbb{R}$ and $\phi$ be $C^\infty$ and let $\lambda_0$ be a simple eigenvalue of* (2) *with eigenfunction $e$ (necessarily $C^\infty$). We make the following assumptions on $e$, $f$, $g$ and $\psi$, where $\psi$ is the unique function given in* (5):

(H1) $$f(0)=0, \quad f'(0)=1, \quad f''(0)=0, \quad f'''(0)\neq 0$$

(H2) $$\int_\Omega (\lambda_0\psi - g)e\,dx \neq 0.$$

*We then may conclude that the bifurcation of solutions to* (1) *about $u\equiv 0$ for $\lambda$ near $\lambda_0$ and $\mu$ near $\mu_0=0$ can be described by the cusp in the sense of Theorem* 1.

*Proof.* Solutions of (1) are critical points of $E_{\lambda,\mu}$. To apply Theorem 1 one first must show that $E_{\lambda,\mu}$ is a $C^\infty$-acceptable family and then verify conditions (i) through (iv). Define $\langle\ ,\ \rangle\colon \mathscr{H}_0\times\mathscr{H}_0\to\mathbb{R}$ by

$$\langle v, w\rangle = \int_\Omega \nabla v\cdot\nabla w\,dx.$$

Define $X_{\lambda,\mu}\colon \mathscr{H}_0\to\mathscr{H}_0$ as follows. Let $v\in\mathscr{H}_0$ and define $X_{\lambda,\mu}$ by the partial differential equation

$$X_{\lambda,\mu}(v)\colon \Omega\to\mathbb{R}, \quad X_{\lambda,\mu}(v)=0 \quad \text{on } \partial\Omega$$

and

$$-\Delta X_{\lambda,\mu}(v) = -\Delta v - \lambda f(v+\mu\psi)+\mu g \quad \text{in } \Omega.$$

One readily checks by elliptic regularity theory that $X_{\lambda,\mu}(v)\in\mathscr{H}_0$ and that

$$(\lambda,\mu,v)\mapsto X_{\lambda;\mu}(v)$$

is $C^\infty$. Moreover it is trivial to see that

$$dE_{\lambda,\mu}(v)[h] = \langle h, X_{\lambda,\mu}(v)\rangle.$$

Consequently it remains to compute the derivatives (i) through (iv) as given in Theorem 1. Since $u\equiv 0$ satisfies (1) for $\mu=0$ it is a critical point of $E_{\lambda_0,\mu_0}$, which for notational convenience we now simply denote by $E$.

We have that

$$\begin{aligned} D^2E(0)[h,k] &= \int_\Omega \nabla h\nabla k\,dx - \lambda_0\int_\Omega hk\,dx \\ &= \int_\Omega (-\Delta h - \lambda_0 h)k\,dx. \end{aligned}$$

Since $\lambda_0$ is a simple eigenvalue of (2), $D^2E(0)$ has a kernel spanned by $e \in C^\infty(\Omega, \mathbb{R})$. Finally, we obtain by simple computation at the point $q = (\lambda_0, 0, 0) \in \mathbb{R}^2 \times \mathscr{H}_0$ using $(H_1)$, $(H_2)$:

(i') $$\frac{\partial^3 E}{\partial e^3} = -\lambda_0 \int_\Omega f''(0) e^3 \, dx = 0;$$

(ii') $$\frac{\partial^4 E}{\partial e^4} = -\lambda_0 \int_\Omega f'''(0) e^4 \, dx \neq 0;$$

(iii') $$\frac{\partial^3 E_{\lambda,\mu}}{\partial \lambda \, \partial e^2} = -\int_\Omega f'(0) e^2 \, dx \neq 0;$$

(iv') $$\frac{\partial^2 E_{\lambda,\mu}}{\partial \mu \, \partial e} = \int_\Omega \nabla e \nabla \psi \, dx - \lambda_0 \int_\Omega f'(0) e \psi \, dx + \int_\Omega g e \, dx \neq 0,$$

where the first term is zero because of (5) and (2) (integrate by parts). This concludes the proof.

*Remarks.* (a) The assumption of $C^\infty$ on $f$ is not necessary. One must only choose $f$ to be somewhat smoother than that required by the standard normal form theorem for two parameter families of functions of one variable.

(b) In [5] a weak form of Theorem 2 was obtained for $g \equiv 0$ using continuation techniques under less restrictive assumptions on $f$. More precisely, there it was shown that the sections of constant $\lambda$ of solutions to (1) provide $\mu$-continua of solutions, which fit together in a "cusp-like" picture.

(c) Assumption (H1) appears to be quite natural for a singularity approach. We want to comment on assumption (H2). Consider the problem

(8) $$\begin{aligned} u'' + \lambda f(u) &= 0 \\ u(0) = \mu &= u(\pi). \end{aligned}$$

and assume that $f$ satisfies (H1). It is well known, that $\lambda_n = n^2$, $n = 1, 2, \ldots$ are simple eigenvalues of

(9) $$\begin{aligned} u'' + \lambda u &= 0 \\ u(0) = 0 &= u(\pi) \end{aligned}$$

with eigenfunctions $e_n(x) = \sin(nx)$ and that each $\lambda_n$ is a point of bifurcation to solutions of (8) near $u \equiv 0$ and $\mu_0 = 0$.

Note that

$$\int_0^1 e_n \, dx \begin{cases} \neq 0 & \text{for } n \text{ odd} \\ = 0 & \text{for } n \text{ even.} \end{cases}$$

Thus, according to Theorem 2 solutions of (8) near $u \equiv 0$, $\mu_0 = 0$ and $\lambda_n$, $n$ odd, form a cusp. The question is, what is the picture near $u \equiv 0$, $\mu_0 = 0$ and $\lambda_n$, $n$ even? This is answered in [5] where it is shown by continuation techniques, that nontrivial solutions locally look like a "paraboloid", i.e. not a cusp. In this sense assumption (H2) seems to be a necessary assumption.

(d) Theorem 2 can be generalized in various directions, e.g. one can formulate a theorem for systems of non-linear elliptic variational problems, which

we omit here. We want to present here, however, a generalization which appears to be noteworthy:

$$E_{\lambda,\mu}(v)=\int_{\Omega} H(\nabla v)\,dx+\lambda\int_{\Omega} F(v+\mu 1)\,dx, \tag{10}$$

where $E_{\lambda,\mu}$: $\mathscr{H}_0\to\mathbb{R}$. Assume that $F$: $\mathbb{R}\to\mathbb{R}$ and $H$: $\mathbb{R}^2\to\mathbb{R}$ are $C^\infty$ and $D^2H(p)$: $\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}$ is positive definite for all $p\in\mathbb{R}^2$. Moreover assume that $DH(0)=0$, $D^3H(0)=0$ and $D^4H(0)=0$. Write $H=H(p_1,p_2)$ and

$$A=\frac{\partial^2 H}{\partial p_1^2}(0),\qquad B=\frac{\partial^2 H}{\partial p_1\,\partial p_2}(0)\quad\text{and}\quad C=\frac{\partial^2 H}{\partial p_2^2}(0).$$

**Theorem 3.** *If $\lambda_0$ is a simple eigenvalue of the elliptic equation*

$$A\frac{\partial^2 h}{\partial p_1^2}+2B\frac{\partial^2 h}{\partial p_1\,\partial p_2}+C\frac{\partial^2 h}{\partial p_2}+\lambda h=0\quad \text{in } \Omega$$
$$h=0\quad \text{on } \partial\Omega$$

*with eigenfunction $e$ satisfying $\int_{\Omega} e\,dx\neq 0$. Furthermore assume that $f(s)=F'(s)$ satisfies* (H1). *Then the two parameter bifurcation from the critical point $v\equiv 0$ near $\mu_0=0$ and $\lambda_0$ is described by the cusp catastrophe.*

The proof is strictly analogous to the proof above.

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