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Surfaces in Three-Manifolds and Non-Singular Equations in Groups

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I. Group Theory

1. Definitions

1.1. Group notation is multiplicative. The *product* of two subsets A, B of a group G is denoted $A \cdot B$.

The *abelianization* of a group G is denoted G^{ab} or $H_1(G)$. Let Z_p denote the additive cyclic group of order p; sometimes it will be considered the field of order p. $H_1(G; Z_p) = G^{ab} \otimes Z_p$.

The *conjugacy class* of an element a in a group G is

$$[a]^G = \{b \in G \mid \text{there exists } g \in G \text{ such that } gag^{-1} = b\}.$$

The *commutator* $[a,b] = aba^{-1}b^{-1}$.

If A, B are subgroups of G, then $[A, B]$ is the subgroup generated by all commutators $[a, b]$ for $a \in A$, $b \in B$. If C is also a subgroup, $C^n[A,B]$ is the subgroup generated by all elements of the form $c^n[a,b]$ for $a \in A$, $b \in B$, $c \in C$.

1.2. For A a *normal* subgroup of B, we write $A \lhd B$. A subgroup A of B is *subnormal* in B if there is a finite chain of subgroups, each normal in the next, connecting A to B:

$$A = C_1 \lhd C_2 \lhd \ldots \lhd C_n = B.$$

If X is a subset of a group G, then $\langle X \rangle_G$ denotes the *normal closure* of X in G, the smallest normal subgroup of G containing X.

1.3. If $A \subset B$ are groups, we say that A is *normal-convex* in B when for each $R \lhd A$, $\langle R \rangle_B \cap A = R$. Equivalently, for every subset X of A,

$$\langle X \rangle_B \cap A = \langle X \rangle_A$$

or,

$$A/\langle X \rangle_A \to B/\langle X \rangle_B \quad \text{is injective.}$$

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It is easy to show the following fact:
1.3.1. If A is a subgroup of both B and C, and there exists a homomorphism $\phi: B \to C$ extending the identity on A, and if A is normal-convex in C, then A is normal-convex in B.
Two special cases of this are:

1.3.2. If A is a retract of B, then A is normal-convex in B.

1.3.3. If $A \subset B \subset C$, and A is normal-convex in C, then A is normal-convex in B.

Another easy fact is:

1.3.4. If $A \subset B \subset C$ and A is normal-convex in B and B is normal-convex in C, then A is normal-convex in C.

An example of a normal-convex subgroup of countable rank in a free group of rank 2 is given in [2].

1.4. If $\alpha, \beta_1, \ldots, \beta_n \in G$, we say α is *dependent* on $(\beta_1, \ldots, \beta_n)$ *in* G when

$$\alpha \in [\beta_1]^G \cdot [\beta_2]^G \cdot \ldots \cdot [\beta_n]^G.$$

That is to say, α is a product of n terms, the i^{th} one being of the form $g_i \beta_i g_i^{-1}$ for some $g_i \in G$.

1.5. If G is a group, a set of *equations over* G in $\{x_1, \ldots, x_n\}$ is a set

$$\{w_1 = 1, \ldots, w_k = 1\}$$

where $w_1, \ldots, w_k$ are elements of the free product $G * X$, where X has free basis $\{x_1, \ldots, x_n\}$.

We say the set of equations has a *solution* if there is a group $H \supset G$ and elements $\xi_1, \ldots, \xi_n \in H$, such that, under the *evaluation map*

$$G * X \to H$$

taking $G \to H$ by the inclusion and $x_i \to \xi_i$, the images of $w_1, \ldots, w_k$ are all $= 1$. This is the same as saying that the composition

$$G \to G * X \to G * X / \langle w_1, \ldots, w_k \rangle_{G*X}$$

is injective.

If, as above, $w \in G * X$, then $e_i(w)$ denotes the *exponent sum* of x_i in w. That is, retract $G * X$ onto the cyclic group generated by x_i, by sending G and all x_j for $j \neq i$ to 1; the image of w is a power of x_i, the exponent being $e_i(w)$.

A set of equations over G, $\{w_1 = 1, \ldots, w_k = 1\}$, is said to be *non-singular* when the integer matrix $[e_i(w_j)]$ has rank $k =$ the number of equations. In this case, the number k of equations is equal to or less than the number n of unknowns. If $k = n$, we have a *square* set of equations, whose *determinant* $\det[e_i(w_j)]$ is a non-zero integer if the set of equations is non-singular.

2. *The p-adic Topology*

2.1. Let p be a prime number. For any group G there is a central series which descends fastest subject to the condition that the successive quotients are Z_p-

vector-spaces. This is the *p-central series* $\{{}_pG_n\}$

$$ {}_pG_1 = G $$
$$ {}_pG_{n+1} = ({}_pG_n)^p[G, {}_pG_n] $$

The *p-derived series* $\{{}_pG_{[n]}\}$ is defined as the fastest descending normal series whose successive quotients are Z_p-vector-spaces:

$$ {}_pG_{[1]} = G $$
$$ {}_pG_{[n+1]} = ({}_pG_{[n]})^p[{}_pG_{[n]}, {}_pG_{[n]}]. $$

We also denote the ω^{th} term of the p-central series thus:

$$ {}_pG_\omega = \bigcap_{n=1}^{\infty} {}_pG_n. $$

A fact easily proved is this:

$$ {}_p({}_pG_{[k]})_{[l]} = {}_pG_{[k+l-1]}. $$

If, as will be common, a prime number is fixed (we shall not be concerned with interactions of these ideas for different primes, in general), we omit the p from the notation and say, for example:

$$ G_n \supset G_{[n]} $$
$$ (G_{[k]})_{[l]} = G_{[k+l-1]} $$

These series consist of functorial, and, in particular, fully invariant subgroups.

2.2. If G is finitely generated, it is easy to prove, inductively, that both G_n and $G_{[n]}$ are of finite index in G, both indices being powers of p. It is well known that if H is a finite p-group, there exists n such that $H_n = \{1\}$. Therefore, when G is finitely generated, for any integer k, there is n such that $(G/G_{[k]})_n = \{1\}$. This is the same as $G_n \subset G_{[k]}$.

2.3. The *p-adic topology* on a group G is the topology where a basis of neighborhoods of 1 is the p-central series $\{G_n\}$. By (2.2), if G is finitely generated, the same topology is defined by the p-derived series.

The *completion* of G in the p-adic topology is denoted ${}_p\hat{G}$ or simply $\hat{G}$. The natural map $G \to \hat{G}$ has kernel G_ω. If G is finitely generated, then $\hat{G}$ is compact, totally disconnected, metric.

Two classical theorems are:

2.4. (Magnus-Iwasawa [4]). If F is a free group, then F_ω is trivial.

2.5. (Stallings [7]). If $\phi\colon E \to F$ is a homomorphism, E and F free groups, and $\phi_*\colon H_1(E; Z_p) \to H_1(F; Z_p)$ is an isomorphism, then for all finite n, ϕ includes an isomorphism

$$ E/E_n \approx F/F_n, $$

ϕ is injective, and the induced map on p-adic completions is an isomorphism $\hat{E} \approx \hat{F}$.

2.6 **Lemma.** *Let G be finitely generated, $N \lhd G$, G/N a finite p-group. Then for all n there exists k such that $G_k \subset N_n$.*

Proof. There exists l such that

$$(G/N)_l = \{1\}, \quad \text{or} \quad G_l \subset N.$$

By (2.2), there exists k such that

$$G_k \subset G_{[l+n-1]}.$$

Then:

$$G_k \subset G_{[l+n-1]} = (G_{[l]})_{[n]}$$
$$\subset (G_l)_{[n]} \subset N_{[n]} \subset N_n.$$

2.7 **Theorem.** *Let G be finitely generated, N a subnormal subgroup of index a power of p. Then:*

N is an open subgroup of G in the p-adic topology.

$\hat{N}$ is an open subgroup of $\hat{G}$.

The natural map of right cosets $G/N \to \hat{G}/\hat{N}$ is bijective.

Proof. This all follows from (2.6) by induction on the length of the subnormal series connecting N to G.

3. *Conjugacy in a Free Group*

In this section we prove Proposition I.4.8 in Lyndon-Schupp [5], which is attributed to Baumslag, Taylor, and Higman. We follow Lyndon and Schupp's proof, filling in the gaps; Lyndon-Schupp, in particular use the word "conjugate" loosely (instead of "conjugate in S"), and we think the confusion needs cleaning up.

3.1. Let $N \lhd G$, $\alpha, \beta \in G$. We say that α and β are *conjugate* $\operatorname{mod} N$ *in* G, if the images of α and β are comjugate in G/N. That is, there exist $u \in G$, $v \in N$ such that

$$\alpha = u \beta u^{-1} v.$$

Recall that a prime p is fixed, $\{G_n\}$ is the p-central series of G, and $\hat{G}$ is the p-adic completion of G.

3.2 **Lemma.** *Suppose that $\alpha, \beta \in G$, a finitely generated group. Then: α and β are conjugate* $\operatorname{mod} G_n$ *in G for all n, if and only if the images of α and β are conjugate in $\hat{G}$.*

Proof. That conjugacy in $\hat{G}$ implies conjugacy $\operatorname{mod} G_n$ for all n is clear. Conversely, if α and β are conjugate $\operatorname{mod} G_n$ in G for all n, there exist $u_n \in G$, $v_n \in G_n$ such that $\alpha = u_n \beta u_n^{-1} v_n$. Since $\hat{G}$ is a compact metric space the image of $\{u_n\}$ in $\hat{G}$ contains a convergence subsequence, converging to $\gamma \in \hat{G}$; then $\alpha = \gamma \beta \gamma^{-1}$.

3.3 **Lemma.** *If N is a subnormal subgroup of a finitely generated group G, of index a power of p, and $\alpha \in N$, and $\{x_1, \ldots, x_k\}$ is a set of representatives of the* *right cosets Ng, and $\beta \in G$, and if α is conjugate to β in $\hat{G}$, then there is a representative x_i such that $x_i \beta x_i^{-1} \in N$ and α is conjugate to $x_i \beta x_i^{-1}$ in $\hat{N}$.*

Proof. There is $\gamma \in \hat{G}$ such that $\alpha = \gamma \beta \gamma^{-1}$. By (2.7) we can identify G/N with $\hat{G}/\hat{N}$, and so $\gamma = v x_i$ for $v \in \hat{N}$ and some x_i. Then

$$\alpha = v(x_i \beta x_i^{-1}) v^{-1}$$

and $x_i \beta x_i^{-1} \in \hat{N} \cap G = N$.

3.4. We give some definitions concerning the well known matter of subgroups of a free group ([5], page 103).

Let F be a free group with basis X, and $S \subset F$ a subgroup.

A *Schreier transversal* T of S in F relative to X is a set of representatives of the right cosets Sg, satisfying the *tree condition* that whenever $t \in T$, then every left segment of t (written as an X-word) belongs to T.

The corresponding *Schreier basis* Y of S consists of those elements of the form

$$t_1 x t_2^{-1}, \qquad \text{for } x \in X, \ t_1, t_2 \in T$$

which belong to S and are $\neq 1$ in F.

Given $s \in S$,

$$s = \prod_1^n x_i^{\varepsilon_i}, \qquad x_i \in X, \ \varepsilon_i = \pm 1,$$

there is a unique *rewriting* of s into a word in Y:

$$s = \prod_1^n t_i x_i^{\varepsilon_i} t_{i+1}^{-1}$$

with $t_1 = t_{n+1} = 1$, the t_i chosen to make each term $t_i x_i^{\varepsilon_i} t_{i+1}^{-1} = y_i^{\varepsilon_i}$ belong to S, hence $y_i \in Y$ or $y_i = 1$.

The length of $w \in F$ in terms of the basis X is denoted by $|w|_X$.

3.5 **Lemma.** *Let F be free with basis X, S a subgroup of F, T a Schreier transversal of S, Y the corresponding Schreier basis of S. Then:*

(i) *If $\beta \in F$, $t_1, t_2 \in T$, $t_1 \beta t_2^{-1} \in S$, then $|t_1 \beta t_2^{-1}|_Y \leqq |\beta|_X$.*

(ii) *If $\alpha \in S$, $\alpha = t w$, $t \in T$, $|w|_X < |\alpha|_X$, then $|\alpha|_Y < |\alpha|_X$.*

Proof. In rewriting $t_1 \beta t_2^{-1}$ in terms of Y, all terms involving X-letters of t_1 and t_2^{-1} yield trivial terms 1 because of the tree condition on T. Each letter of β gives one term which may or may not be 1. This proves (i).

For (ii), taking $t_1 = t$, $t_2 = 1$, $\beta = w$, part (i) says

$$|\alpha|_Y \leqq |w|_X < |\alpha|_X.$$

3.6 **Theorem** ([5]). *Let F be a finitely generated free group, $\alpha, \beta \in F$. Suppose that α and β are conjugate and mod F_n in F for all n, where $\{F_n\}$ is the p-central series of F. Then α and β are conjugate in F.*

Note. In the non-finitely generated case, the hypothesis that α and β are conjugate in F mod every normal subgroup of finite index *a* power of p yields the same conclusion. This is easily seen to be equivalent to this theorem.

Proof. The proof is a sort of "tower" argument.

A *situation* $\sum=(S, X, \alpha_1, \beta_1, u, v)$ consists of

S, a subnormal subgroup of F of index a power of p.
X, a free basis of S.
$\alpha_1, \beta_1 \in S$, such that α_1 and β_1 are conjugate in $\hat{S}$.
$u, v \in F$, such that $\alpha_1 = u\alpha u^{-1}$, $\beta_1 = v\beta v^{-1}$.

We define the complexity of Σ:

$$c(\Sigma) = |\alpha_1|_X + |\beta_1|_X.$$

There exist situations. For example, if X is a basis of F, then $\Sigma=(F, X, \alpha, \beta, 1, 1)$ is a situation, because, by (3.2), α and β are conjugate in $\hat{F}$. Therefore there exists a situation Σ of minimal complexity, which will be chosen for the rest of the argument.

First, α_1 and β_1 are cyclically reduced in terms of the basis X. Otherwise, complexity could be decreased by conjugating them within S.

Second, α_1 must be a power of a single element of X. Otherwise, if both x and y occur in α_1, we can arrange a homomorphism ϕ: $S \to Z_p$ with $\phi(\alpha_1)$ trivial and ϕ non-trivial on x or y or both. Suppose $\phi(x)$ non-trivial; rotate α_1, changing u, so that x or x^{-1} is the first letter of α_1; if x^{-1}, change X to X^{-1}; this does not change $c(\Sigma)$. Thus, $\alpha_1 = xw$ and $|w|_X < |\alpha_1|_X$.

Let $S' = \ker\phi$. Take $\{1, x, \ldots, x^{p-1}\}$ as a Schreier transversal of S' in S. Let X' be the corresponding Schreier basis of S'. By construction, $\alpha_1 \in S'$. By (3.3), there is k, $0 \leqq k < p$, such that α_1 and $x^k\beta_1 x^{-k} \in S'$ are conjugate in $\hat{S}'$. Then

$$\Sigma' = (S', X', \alpha_1, x^k\beta_1 x^{-k}, u, x^k v)$$

would be a situation, which, (by 3.5), would have less complexity than that of Σ.

Similarly, β_1 is a power of a single element of X.

Now, $\alpha_1 = x^k$, $\beta_1 = y^l$, with $x, y \in X$. We have α_1 conjugate to β_1 in $\hat{S}$, and so by (3.2), α_1 and β_1 are conjugate in S mod any normal subgroup of index a power of p; and so, α_1 and β_1 are equal in the abelian groups

$$S^{\mathrm{ab}} \otimes Z_{p^r} \quad \text{for all } r.$$

This implies $x=y$, $k=l$, $\alpha_1=\beta_1$. We then have

$$\alpha = (u^{-1}v)\beta(u^{-1}v)^{-1}, \quad u, v \in F \quad \text{QED.}$$

3.7. The Magnus-Iwasawa theorem (2.4), in the form, "$\alpha=1$ in $\hat{F}$ implies $\alpha=1$ in F", follows from (3.6) by taking $\beta=1$.

An interesting corollary of the proof of (3.6) is this:

3.7.1. If F is free, $\alpha \in F$, $\alpha \neq 1$, p a prime number, then there is a subnormal subgroup $S \subset F$, of index a power of p, and a basis X of S, with $\alpha \in S$ and $x \in X$, such that $\alpha = x^k$ and $k \not\equiv 0 \bmod p$.

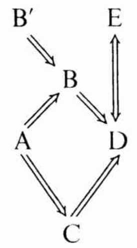
3.7.2. And so, if α is not a proper power in a free group F, then for every prime p, there is a subnormal subgroup $S \subset F$ of index a power of p, such that α is a primitive element (i.e., belongs to a basis) of S.

3.8 *Restatment of* (3.6). If F is a free group of finite rank, $\alpha \in F$, p a prime number, then the conjugacy class $[\alpha]^F$ is a closed set in the p-adic topology on F.

Proof. If $\beta \notin [\alpha]^F$, then (3.6) says there exists n such that $\beta F_n \cap [\alpha]^F = \varnothing$, so that β has a neighborhood βF_n disjoint from $[\alpha]^F$.

4. *Group-Theoretic Conjectures*

We state some group-theoretic conjectures, called A, B, B′, C, D, E. The diagram of implications is roughly



The conjecture (E) is the famous one on non-singular sets of equations in a group. A special case of (A) has just been proved, as (3.8); hence a special case of (B) is a *fact*, Theorem (5.3) below. The conjecture (B′) has an analogue in the theory of 3-manifolds, which is proved later, as Theorem (9.1). Some of these conjectures, e.g. (C), sound rather unlikely, but no counterexample is known. The implications described in the above diagram will be proved in the next section. Some of the conjectures have a parameter p, a prime number; and some have a parameter n, a positive integer.

4.1 **Conjecture $\mathbf{A}_n(p)$** [Closure problem]. *In a finitely generated free group F, the product of n conjugacy classes is a closed set in the p-adic topology.*

4.2 **Conjecture $\mathbf{B}_n(p)$** [Dependency problem]. *If S is a subgroup of a finitely generated free group F and $H_1(S; Z_p) \to H_1(F; Z_p)$ is an isomorphism, then for all $\alpha, \beta_1, \ldots, \beta_n \in S$, if α is dependent on $(\beta_1, \ldots, \beta_n)$ in F, then α is dependent on $(\beta_1, \ldots, \beta_n)$ in S.*

4.3. To state conjecture B′, we need a definition. Given $\beta_0, \beta_1, \ldots, \beta_n \in G$, we say that

$$\operatorname{genus}_G(\beta_0, \ldots, \beta_n) \leqq k$$

if there exist $g_i \in G$, $i = 0, \ldots, n$, and $x_j, y_j \in G$, $j = 1, \ldots, k$, such that

$$\prod_{i=0}^{n} g_i \beta_i g_i^{-1} = \prod_{j=1}^{k} [x_j, y_j].$$

4.4 **Conjecture $B'_n(p)$** [Genus problem]. *If S is a subgroup of a finitely generated free group F and $H_1(S;Z_p)\to H_1(F;Z_p)$ is an isomorphism, then, for all $\beta_0,\dots,\beta_n\in S$,*

$$\text{if} \quad \text{genus}_F(\beta_0,\dots,\beta_n)\leqq k$$
$$\text{then} \quad \text{genus}_S(\beta_0,\dots,\beta_n)\leqq k.$$

4.5 **Conjecture C(p)** [Normal-convexity in p-adic completion]. *If F is a finitely generated free group, then F is normal-convex in its p-adic completion $\hat{F}$.*

4.6 **Conjecture D(p)** [Normal-convexity of p-isomorphisms]. *If S is a subgroup of a finitely generated free group F and $H_1(S;Z_p)\to H_1(F;Z_p)$ is an isomorphism, then S is normal-convex in F.*

4.7 **Conjecture E(p)** [Solvability of non-singular sets of equations]. *If G is any group, and $\{w_1=1,\dots,w_k=1\}$ is a square set of equations (i.e., $w_i\in G*X$, X free on $\{x_1,\dots,x_k\}$) whose determinant, $\det[e_i(w_j)]$, is not divisible by the prime p, then the set of equations has a solution (i.e., $G\to G*X/\langle w_1,\dots,w_k\rangle_{G*X}$ is injective).*

The history of this problem began with the theorem of Gerstenhaber-Rothaus [1] who proved conjecture E (for all primes p) in case G is locally residually finite. Howie [3] proved it for the case that G is locally indicible. A simple consequence of E (for any one prime p) is the conjecture that a group of the form $G*Z$ cannot be killed by adding a single relation unless G is trivial; this conjecture has been credited to various people; the oldest reference I can find, [6, p. 403], credits it to Kervaire.

4.8 Let us describe conjectures $B_n(0)$, $B'_n(0)$, $D_n(0)$, as those obtained from $B_n(p)$, etc., by replacing the assumption

$$\text{“}H_1(S;Z_p)\to H_1(F;Z_p) \quad \text{is an isomorphism”}$$

by

$$\text{“}H_1(S;Q) \to H_1(F;Q) \quad \text{is an isomorphism”}$$

where Q is the additive group of rational numbers.

It is easy to see, since F is assumed to be finitely generated, that we have these implications:

(*) If $B_n(p)$ is true for infinitely many primes p,
then $B_n(0)$ is true.

And similarly for B'_n, D_n. This happens because $H_1(S;Q)\to H_1(F;Q)$ is an isomorphism if and only if $H_1(S;Z_p)\to H_1(F;Z_p)$ is an isomorphism for all but a finite number of primes p.

There is also a conjecture E(0), in which the condition that the determinant is not divisible by p is replaced by the condition that the determinant is non-zero. An implication similar to (*) holds relating $E(p)$ and E(0).

4.9 We can apparently strengthen conjectures $B_n(p)$, $B'_n(p)$, and $D_n(p)$ by relaxing the homology isomorphism condition to the condition

$$\text{“}H_1(S;Z_p)\to H_1(F;Z_p) \quad \text{is injective”}.$$

However, from this condition we can find a free group T and a map $T \to F$ such that $S * T \to F$ induces homology isomorphisms mod p. The original conjectures now assert something about $S * T$ in F; (by (2.5) $S * T$ is a subgroup of F); the corresponding assertion about S in F will follow by retracting $S * T$ onto S.

In the case of conjecture E the similar phenomenon is this: Replace the assumption that we have a square set of equations with determinant non-zero mod p, by hypothesizing a set of equations, such that the rank of the matrix $[e_i(w_j)]$ reduced mod p is equal to the number of equations. This apparently stronger conjecture (i.e., less likely to be true) follows from the original one, by adding extra equations to fill out a square set. To solve the larger set of equations yields, *a fortiori*, a solution of the smaller set.

4.10. In other words, at the expense of making the comments (4.8) and (4.9), we can dispense with making the more grandiose-sounding versions of the conjectures B, B', D, and E, involving characteristic zero and/or injectivity instead of isomorphism. The grandiose versions are related to the conjectures as stated by easy arguments.

4.11. Here are some specific questions. Let F have basis $\{x, y\}$, and S be the subgroup generated by $\{x, yxyx^{-1}y^{-1}\}$.

Question 1. Suppose $\alpha, \beta \in F$ such that $[\alpha, \beta] \in S$. Are there $a, b \in S$ such that $[a, b] = [\alpha, \beta]$? (This would follow from B'_0.)

Question 2. Suppose $a, b \in S$, $\gamma, \delta \in F$, and $\gamma a \delta b \delta^{-1} \gamma^{-1} \in S$. Are there $c, d \in S$ such that $cadbd^{-1}c^{-1} = \gamma a \delta b \delta^{-1} \gamma^{-1}$? (This would follow from B_2.)

A positive answer to either question might shed some light on the situation.

4.12. I feel that a proof of conjecture B might involve a construction, perhaps a tower such as used in the proof of (3.6), which would leap out of the genus zero situation, and therefore conjecture B′ might be the one that should be attempted.

5. *Implications*

5.1. $B'_n(p)$ implies $B_n(p)$.

Proof. Given $S \subset F$ satisfying the mod p homology isomorphism and $\alpha, \beta_1, \ldots, \beta_n \in S$, with α dependent on $(\beta_1, \ldots, \beta_n)$ in F, define $\beta_0 = \alpha^{-1}$. Then $\text{genus}_F(\beta_0, \ldots, \beta_n) \leqq 0$. By $B'_n(p)$ we then get $\text{genus}_S(\beta_0, \ldots, \beta_n) \leqq 0$, which translates to say that α is dependent on $(\beta_1, \ldots, \beta_n)$ in S.

5.2. $A_n(p)$ implies $B_n(p)$.

Proof. Suppose $S \subset F$ satisfies the condition that $H_1(S; Z_p) \to H_1(F; Z_p)$ is an isomorphism. By (2.5), for each finite k,

$$S/S_k \approx F/F_k.$$

Now, if $\alpha, \beta_1, \ldots, \beta_n \in S$ and α is dependent on $(\beta_1, \ldots, \beta_n)$ in F, then the image of α is dependent on the images of $(\beta_1, \ldots, \beta_n)$ in $F/F_k \approx S/S_k$. That is, for all k we can say

$$\alpha S_k \cap ([\beta_1]^S \cdot [\beta_2]^S \ldots [\beta_n]^S) \neq \varnothing.$$

That is, α is in the p-adic closure of

$$[\beta_1]^S \cdot \ldots \cdot [\beta_n]^S$$

and this is a closed subset of S if $A_n(p)$ is true. Thus α would be dependent on $(\beta_1, \ldots, \beta_n)$ in S.

5.3 **Theorem.** *If $S \subset F$, where F is a finitely generated free group, and $S^{ab} \to F^{ab}$ is injective, then the induced map from conjugacy classes in S to conjugacy classes in F is injective.*

Proof. There exists a prime p such that $H_1(S; Z_p) \to H_1(F; Z_p)$ is injective. We can find a free group T and a map $T \to F$, such that

$$H_1(S * T; Z_p) \to H_1(F; Z_p) \quad \text{is an isomorphism.}$$

By (2.5), we can think of $S * T$ as a subgroup of F. By (3.8) $A_1(p)$ is true, and so by (5.2), $B_1(p)$ is true. In the case at hand, $B_1(p)$ says that two elements α, β of $S \subset S * T$, which are conjugate in F, are conjugate in $S * T$. On retracting $S * T$ to S, we see that α and β are conjugate in S. QED.

5.4. If, for all n $A_n(p)$ is true, then $C(p)$ is true.

Proof. Let F be a finitely generated free group and $\hat{F}$ its p-adic completion. Let $R \lhd F$ and $\alpha \in \langle R \rangle_{\hat{F}} \cap F$. Then there are $\beta_1, \ldots, \beta_n \in R$ and $\gamma_1, \ldots, \gamma_n \in \hat{F}$ such that

$$\alpha = \prod_1^n \gamma_i \beta_i \gamma_i^{-1}.$$

For any k, there are $\gamma_{i,k} \in F$ such that $\gamma_{i,k} \equiv \gamma_i \bmod \hat{F}_k$. (Cf. (2.7).)

Then

$$\alpha_k = \sum_1^n \gamma_{i,k} \beta_i \gamma_{i,k}^{-1}$$

has the property:

$$\alpha_k \equiv \alpha \bmod F_k.$$

Hence α_k is a sequence of elements of $[\beta_1]^F \cdot \ldots \cdot [\beta_n]^F$ converging to α. If $A_n(p)$ is true, this product of conjugacy classes is closed, and so

$$\alpha \in [\beta_1]^F \cdot \ldots \cdot [\beta_n]^F \subset R \qquad \text{QED.}$$

5.5. If, for all n $B_n(p)$ is true, then $D(p)$ is true.

Proof. Suppose $S \subset F$ induces an isomorphism on homology mod p. Let $R \lhd S$ and $\alpha \in \langle R \rangle_F \cap S$. Then there are $\beta_1, \ldots, \beta_n \in R$ such that α is dependent on $(\beta_1, \ldots, \beta_n)$ in F. If $B_n(p)$ is true, then α is dependent on $(\beta_1, \ldots, \beta_n)$ in S and hence $\alpha \in R$.

5.6. $C(p)$ implies $D(p)$.

Proof. Suppose $S \subset F$ induces an isomorphism on homology mod p. By (2.5) the inclusion induces an isomorphism on p-adic completions

$$\hat{S} \approx \hat{F}.$$

If C(p) is true, then S is normal-convex in $\hat{S}$ and hence, by the above isomorphism, S is normal-convex in $\hat{F}$. Now, if $R \lhd S$, then

$$\langle R \rangle_F \cap S \subset \langle R \rangle_{\hat{F}} \cap S = R,$$

the equality by normal-convexity in $\hat{F}$. Therefore S is normal-convex in F.

5.7. D(p) implies E(p).

Proof. If there were a counterexample to E(p), we could find a finitely generated counterexample. Thus, we can suppose $G = \Phi/R$, where $R \lhd \Phi$ and Φ is a finitely generated free group. Let X be free on $\{x_1, \ldots, x_k\}$ and let $w_1, \ldots, w_k \in G * X$ such that $\det[e_i(w_j)] \not\equiv 0 \bmod p$.

Let $\tilde{w}_1, \ldots, \tilde{w}_k \in \Phi * X$ be such that their images under $\Phi \to G$ are $w_1, \ldots, w_k$ and such that $e_i(\tilde{w}_j) = e_i(w_j)$. Let W be the free group with basis $\{\hat{w}_1, \ldots, \hat{w}_k\}$, and map W to $\Phi * X$ via $\hat{w}_j \to \tilde{w}_j$. Then the evaluation map

$$\Phi * W \to \Phi * X$$

yields an isomorphism on homology mod p. By (2.5) this map is injective, so that we can suppose $\Phi * W$ is a subgroup of $\Phi * X$, with $\hat{w}_j = \tilde{w}_j$.

Now, if D(p) is true, then $\Phi * W$ is normal-convex in $\Phi * X$. Therefore, on factoring out by the normal closures of $R \cup W$, we get an injective homomorphism, which is exactly

$$G \to G * X / \langle w_1, \ldots, w_k \rangle.$$

5.8. E(p) implies D(p).

Proof. Let $S \subset F$ induce a homology isomorphism mod p, where F is free of rank k. It follows that S has rank k. Suppose $\{s_1, \ldots, s_k\}$ is a basis of S and $\{x_1, \ldots, x_k\}$ a basis of F. If $e_i(w)$ is the exponent-sum of x_i in w, then $\det[e_i(s_j)] \not\equiv 0 \bmod p$.

Let $R \lhd S$ and let $G = S/R$. Let σ_j be the image in G of s_j. To compute $F/\langle R \rangle_F$ we can take the free product of G and F and identify $\sigma_j \in G$ with $s_j \in F$. That is, let $w_j = \sigma_j^{-1} s_j$ in $G * F$; then

$$G * F / \langle w_1, \ldots, w_k \rangle \approx F / \langle R \rangle_F.$$

If E(p) is true, then, since $e_i(w_j) = e_i(s_j)$, E(p) applies to this case, and

$$G \to G * F / \langle w_1, \ldots, w_k \rangle$$

is injective. This is just

$$S/R \to F/\langle R \rangle_F$$

which is therefore injective.

Since R was an arbitrary normal subgroup of S, this shows S is normal-convex in F.

5.9. Note that (5.8) and (5.7) imply that the non-singular equation conjecture is equivalent to the conjecture that a particularly simple kind of equation-set can be solved. That is, all we have to do is to solve the equations of this form: $\{s_j = \sigma_j\}$ for $s_j \in X$ and $\sigma_j \in G$, where $\det[e_i[s_j]] \neq 0 \bmod p$.

II. Three-Manifolds

6. Definitions

6.1. We investigate 3-manifolds and surfaces in the polyhedral situation. Everything is implicitly supposed to be tame. "Manifold" means manifold with possibility empty boundary. Compactness and connectedness are assumed only if this is made explicit.

The boundary of a manifold M is denoted BdM. For M to be oriented means that each component of M is oriented; if M is oriented, then BdM is oriented.

If $A \subset B$ is a pair of 3-manifolds, we say that A is a *proper submanifold* if A is a closed subset of B and the frontier of A in B is bicollared and equal to the boundary of A. The closure of X in some understood ambient space is denoted $\bar{X}$.

6.2. Select a prime number p.

For $p \neq 2$, *p-oriented* means oriented, while *2-oriented* means nothing. For M to be a p-oriented n-manifold means, then, that the sheaf of local homology groups $H_n(M, M-\{x\}; Z_p)$ over the interior of M has a nowhere zero specified continuous cross-section.

Abbreviate the notation by writing $H_n(X)$ for $H_n(X; Z_p)$, and similarly for cohomology.

$\beta_n(X)$ is the rank of $H_n(X)$ over Z_p. $\chi(X) = \sum (-1)^n \beta_n(X)$. These are the Betti numbers and Euler characteristic. By the universal coefficient theorem, $\beta_n(X)$ is the rank of $H^n(X)$. Lefschetz duality says that if M is a compact p-oriented n-manifold, then $\beta_k(M) = \beta_{n-k}(M, \mathrm{Bd}\, M)$. If M is a compact, p-oriented n-manifold, whose boundary is partitioned into disjoint compact parts A and B, another version of Lefschetz duality says $\beta_k(M, A) = \beta_{n-k}(M, B)$.

6.3. If T is a compact 2-manifold, we define genus T by the formula:

$$\chi(T) + \beta_0(\mathrm{Bd}\, T) = 2(\beta_0(T) - \text{genus}\ T).$$

If T is connected and oriented, genus T is the usual thing, the maximum number of pairwise disjoint simple closed curves in the interior of T whose complement is connected. If T is connected, non-orientable, then genus T is one-half the usual definition. In the general case, genus T is the sum of the genera of the components of T.

A compact surface of genus 0, for instance, is a finite union of components which may be spheres, disks, and disks with various numbers of holes.

7. Elementary Facts About 3-Manifolds

A prime number p remains selected.

7.1. If A is a compact p-orientable 3-manifold, then $\chi(\mathrm{Bd}\, A) = 2\chi(A)$.

Proof. Lefschetz duality implies that $\chi(A, \mathrm{Bd}\,A) = -\chi(A)$. The exact homology sequence of $(A, \mathrm{Bd}\,A)$ implies that $\chi(\mathrm{Bd}\,A) = \chi(A) - \chi(A, \mathrm{Bd}\,A)$.

7.2. If A is a compact connected p-orientable 3-manifold, then $\beta_0(\mathrm{Bd}\,A) \leqq \beta_2(A) + 1$.

Proof. The exact cohomology sequence

$$H^0(A) \to H^0(\mathrm{Bd}\,A) \to H^1(A, \mathrm{Bd}\,A)$$

implies $\beta_0(\mathrm{Bd}\,A) \leqq \beta_0(A) + \beta_1(A, \mathrm{Bd}\,A)$ and Lefschetz duality says $\beta_1(A, \mathrm{Bd}\,A) = \beta_2(A)$. Finally, $\beta_0(\mathrm{A}) = 1$ since A is connected.

7.3. If A is a compact, p-orientable 3-manifold, then genus $\mathrm{Bd}\,A \leqq \beta_1(A)$.

Proof. The general case follows, as a sum of inequalities, from the case that A is connected. It is clear when $\mathrm{Bd}\,A = \varnothing$. When A is connected and $\mathrm{Bd}\,A \neq \varnothing$, then $\beta_3(A) = 0$, and (7.1) says

$$2(\beta_0(\mathrm{Bd}\,A) - \text{genus}\,\mathrm{Bd}\,A) = 2(1 - \beta_1(A) + \beta_2(A))$$

and the inequality (7.2) finishes the proof.

7.4 **Lemma.** *Let A be a compact connected p-orientable 3-manifold. Let $E \subset \mathrm{Bd}\,A$ be a compact surface, and define $D = \overline{\mathrm{Bd}\,A - E}$. If $H_2(A, E) = 0$, then* genus $E \leqq$ genus D.

Proof. Let A' be A with 2-handles attached along $\mathrm{Bd}\,E = \mathrm{Bd}\,D$. Let E' and D' be E and D with 2-cells attached along each boundary component. Then genus D = genus D', genus E = genus E', and $\mathrm{Bd}\,A$ is the disjoint union of D' and E'.

By excision, $H_2(A, E) \approx H_2(A', E \cup 2\text{-handles})$, and by the homotopy axiom, $H_2(A', E') \approx H_2(A', E \cup 2\text{-handles})$. By Lefschetz duality, $\beta_1(A', D') = \beta_2(A', E')$ which was just shown to be 0. The exact homology sequence of (A', D') then implies

$$\beta_1(A') \leqq \beta_1(D').$$

By (7.3),

$$\text{genus}\,E' + \text{genus}\,D' \leqq \beta_1(A').$$

These two inequalities, and the fact that

$$\beta_1(D') = 2\,\text{genus}\,D'$$

imply that

$$\text{genus}\,E' \leqq \text{genus}\,D'.$$

Note. The genus of the boundary of A' is an integer, and so genus D − genus E is an integer. This comment may be useful, but only for $p = 2$.

7.5 **Major Lemma.** *Let A be a p-orientable, connected (not necessarily compact) 3-manifold, such that $H_2(A, \mathrm{Bd}\,A; Z_p) = 0$. Let D be a compact, connected, p-orientable surface in A such that $\mathrm{Bd}\,D = D \cap \mathrm{Bd}\,A$.*

Then D separates A into two components whose closures A_1 and A_2 are 3-manifolds. At least one of these, say A_1, is compact. Let $E = \overline{\mathrm{Bd}\,A_1 - D}$; then, as *p-oriented 1-manifolds,* $\mathrm{Bd}\,E=\mathrm{Bd}\,D$;; *and* E *is a compact surface contained in* $\mathrm{Bd}\,A$; *and* genus $E \leqq$ genus D.

Proof. The number of components of $A-D$ is at most two since A and D are connected and D separates its tubular neighborhood in A into at most two components.

The orientation class δ of D represents a 2-cycle in $(A, \mathrm{Bd}\,A)$, and so, since $H_2(A, \mathrm{Bd}\,A)=0$, is the boundary of a 3-chain α of $(A, \mathrm{Bd}\,A)$. The restriction of α to the orientation sheaf of A changes by one on going through D and is constant on each component of $A-D$; hence $A-D$ is not connected and so has two components.

Now α is a finite chain; so either A is compact, or α is zero on one component of $A-D$ and non-zero on the other component, whose closure A_1 is compact. If A_1 is the compact closure of a component of $A-D$, clearly A_1 is a compact 3-manifold, and $A_1 \cap \mathrm{Bd}\,A = \overline{\mathrm{Bd}\,A_1 - D} = E$ is a compact surface, and $D \subset \mathrm{Bd}\,A_1$.

The exact homology sequence of the triple $(A,\ A_1 \cup \mathrm{Bd}\,A,\ \mathrm{Bd}\,A)$ contains

$$H_3(A, \mathrm{Bd}\,A) \to H_3(A, A_1 \cup \mathrm{Bd}\,A) \to H_2(A_1 \cup \mathrm{Bd}\,A, \mathrm{Bd}\,A) \to H_2(A, \mathrm{Bd}\,A).$$

The right-most group, $H_2(A, \mathrm{Bd}\,A)=0$ by hypothesis.

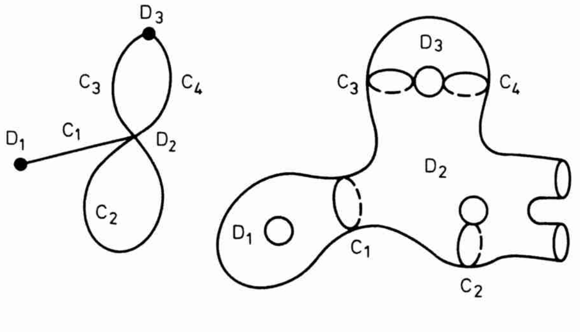
The left-most map, $H_3(A, \mathrm{Bd}\,A) \to H_3(A, A_1 \cup \mathrm{Bd}\,A)$, is an isomorphism. Either both are zero when A is non-compact, or both are Z_p since A and A_2 $=$the closure of the other component of $A-D$, are connected; in the latter, compact case, we check that it is an isomorphism by restricting to a local homology group at a point in the interior of A_2.

We conclude that $H_2(A_1 \cup \mathrm{Bd}\,A, \mathrm{Bd}\,A)=0$. Hence, by excision, $H_2(A_1, E)=0$. Now apply (7.4) to A_1 to conclude that genus $E \leqq$ genus D.

8. *Graphs of Surfaces*

Let T be a compact p-oriented surface. Let $\mathfrak{C}=(C_1, \ldots, C_n)$ be a finite collection of pairwise-disjoint, two-sided simple closed curves in the interior of T. Let $N(\mathfrak{C})$ be a tubular neighborhood of $\cup\mathfrak{C}$ in T. Let $\mathfrak{P}=(D_1, \ldots, D_k)$ be the collection of components of $T-N(\mathfrak{C})$.

Define a graph (a finite 1-complex) $\Gamma(T, \mathfrak{C})$. The vertices are the elements of $\mathfrak{P}$; the edges are the elements of $\mathfrak{C}$. An edge C_i connects the vertices which have the two curves parallel to C_i on $\overline{\mathrm{Bd}\,N(\mathfrak{C})}$ in their boundaries. This may be explained by a picture:



8.1. Genus $T=\sum_{i=1}^{k}\text{genus}\,D_i+\beta_1(\Gamma(T,\mathfrak{C}))$.

Proof. By adding 2-cells to $\text{Bd}\,T$ we do not change the graph, genus T, the genera of the D_i. So we can suppose that $\text{Bd}\,T=\phi$.

Now, $\beta_0(T)=\beta_0(\Gamma(T,\mathfrak{C}))$, and $\chi(T)=\sum_1^k \chi(D_i)$, since T is obtained by gluing the components D_i along curves C_j having $\chi(C_j)=0$.

If k is the number of vertices, n the number of edges, then

$$\chi(\Gamma(T,\mathfrak{C}))=\beta_0(\Gamma(T,\mathfrak{C}))-\beta_1(\Gamma(T,\mathfrak{C}))=k-n.$$

Furthermore, each curve C_j is counted exactly twice in $\sum_1^k \beta_0(\text{Bd}\,D_i)$, so that

$$2n=\sum_1^k \beta_0(\text{Bd}\,D_i).$$

Then compute

$$\begin{aligned}\text{genus}\,T&=\beta_0(T)-\tfrac{1}{2}\chi(T)\\&=\beta_0(T)-\tfrac{1}{2}\sum_1^k\chi(D_i)\\&=\beta_0(T)-\tfrac{1}{2}\sum_1^k[2-2\,\text{genus}\,D_i-\beta_0(\text{Bd}\,D_i)]\\&=\beta_0(T)-\tfrac{1}{2}\left[2k-2\sum_1^k\text{genus}\,D_i-2n\right]\\&=\beta_0(\Gamma(T,\mathfrak{C}))-k+n+\sum_1^k\text{genus}\,D_i\\&=\beta_1(\Gamma(T,\mathfrak{C}))+\sum_1^k\text{genus}\,D_i.\end{aligned}$$

8.2. Let $\phi\colon \Gamma'\to\Gamma$ be a map of finite graphs which is bijective on edges. Then $\beta_1(\Gamma')\leqq\beta_1(\Gamma)$.

Proof. A *forest* is a graph containing no simple circuit. In Γ, let F be a maximal forest. It is easy to see that $\phi^{-1}(F)$ is a forest, which can be enlarged to a maximal forest F' in Γ'. Now, $\beta_1(\Gamma)$ is the number of edges of $\Gamma-F$, and $\beta_1(\Gamma')$ is the number of edges $\Gamma'-F'$. So $\beta_1(\Gamma')\leqq\beta_1(\Gamma)$.

8.3. An *elementary simplification* of $(T,\mathfrak{C})$ means this: Select a vertex D_i; replace it abstractly by a not necessarily connected, compact, p-oriented surface E having $\text{Bd}\,E=\text{Bd}\,D_i$, and genus $E\leqq$ genus D_i.

This changes T to $T'=(T-D_i)\cup E$, on which exactly the same system of curves $\mathfrak{C}$ exists. The single vertex D_i of the graph has been replaced by possibly several vertices, the components of E. In this situation there is a map of graphs

$$\Gamma(T',\mathfrak{C})\to\Gamma(T,\mathfrak{C})$$

which is bijective on edges.

8.4 **Lemma.** *If* $(T, \mathfrak{C})$ *is changed by an elementary simplification (replacing* D_i *by* E *with* genus $E \leqq$ genus D_i*) to get* $(T', \mathfrak{C})$*, then*

$$\text{genus } T' \leqq \text{genus } T.$$

Proof. This follows from a computation using (8.1) and (8.2).

9. The Main Theorem

9.1 **Theorem.** *Let* p *be a prime number. Suppose* $B \subset A$ *is a pair of* p*-oriented* 3*-manifolds with* B *a proper submanifold of* A*. Suppose*

$$H_2(A, B; Z_p) = 0.$$

Let T *be a compact* p*-oriented surface contained in* A*, with* Bd $T \subset$ *the interior of* B*.*

Then there is a compact p*-oriented surface* $S \subset B$*, with* Bd $S =$ Bd T *as a* p*-oriented* 1*-manifold, and*

$$\text{genus } S \leqq \text{genus } T.$$

Proof. We can assume that T meets Bd B transversely. Thus $T \cap \text{Bd}\, B$ is a finite collection $\mathfrak{C} = (C_1, \ldots, C_n)$ of two-sided, pairwise-disjoint simple closed curves. Let $\{D_1, \ldots, D_l\}$ be the closures of the components of $T - B$; this is a subset of the set of vertices of the graph $\Gamma(T, \mathfrak{C})$. The proof is by induction on l; for $l = 0$ we take $S = T$.

We can remove any of the boundary of A not in B after first pushing T away from it. Let A' be the closure of a component of $A - B$; then $\text{Bd}\, A' = A' \cap B = A' \cap \text{Bd}\, B$. Using excision, we conclude from $H_2(A, B) = 0$ that $H_2(A', \text{Bd}\, A') = 0$. If $D_i \subset A'$, we can apply (7.5) to conclude that there is a compact 3-manifold $A_i \subset A'$, whose boundary is $D_i \cup E_i$, with $E_i = A_i \cap \text{Bd}\, A'$. If there is a D_j that intersects A_i, then $D_j \subset A_i$ and we can arrange for $A_j \subset A_i$. So, there is an innermost D_i, say D_1. Then there is a compact, connected 3-manifold A_1 that is contained in $\overline{A - B}$; $\text{Bd}\, A_1 = D_1 \cup E_1$, $E_1 = A_1 \cap B$, and $A_1 \cap T = D_1$. Furthermore, (7.5) says that

$$\text{genus } E_1 \leqq \text{genus } D_1.$$

Now, do an elementary simplification on $(T, \mathfrak{C})$ by replacing D_1 by E_1, getting $(T', \mathfrak{C})$. We have, by using the innermost argument, arranged it so that T' is in fact embedded in A. By (8.4), genus $T' \leqq$ genus T. By a slight push in the neighborhood of E_1 in the direction of the interior of B, we make T' transverse to Bd B. The components of $T' - B$ will consist of $\{D_2, \ldots, D_l\}$. This finishes the inductive step.

9.2. If A is compact and orientable, we have implications

$$H_2(A, B; Z_p) = 0 \Rightarrow H_2(A, B; Z) = 0$$
$$\Rightarrow \text{there exists } p \text{ such that } H_2(A, B; Z_p) = 0.$$

Hence, the "p" in the theorem is a bit of a red herring, but it proves useful to carry it along consistently in the proof.

The "proper" hypothesis is essential. The proof makes an essential use of Bd B. One cannot merely assume B is an open subset of A; two counterexamples come to mind in $A = \mathbb{R}^3$; B could be Whitehead's contractible manifold, or the non-simply-connected component of the complement of Alexander's horned sphere; in both cases, there are curves in B that bound disks in A but not in B.

The conditions that T and S are embedded, not just mapped into A, are essential in the proof, but I do not know if a theorem about, for example, immersed surfaces is true or false. If A and B were handlebodies, thus having free fundamental group, then (9.1) in that case, for immersed surfaces, would say exactly that conjecture $B'_n(p)$, the genus conjecture, is true for all n.

I do not know yet of a good use for (9.1) other than to suggest that the group-theoretic conjectures I propose *might* be true.

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Added in Proof

The original assertion of Theorem 3.6 is a remark attributed to G. Higman on page 278 in G. Baumslag, Residual nilpotence and relations in free groups, J. Algebra **2**, 271–282 (1965).