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Titel: A Riemann Mapping Theorem for Bounded Symmetric Domains in Complex Banach Spaces....

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Jahr: 1983

PURL: https://resolver.sub.uni-goettingen.de/purl?266833020_0183|log39

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Math. Z. 183, 503–529 (1983)

Mathematische Zeitschrift
© Springer-Verlag 1983

A Riemann Mapping Theorem for Bounded Symmetric Domains in Complex Banach Spaces

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The Riemann mapping theorem we have in mind is the following result (compare 4.9): Every bounded symmetric domain D in a complex Banach space is biholomorphically equivalent to the open unit ball of a certain complex Banach space U uniquely determined by D up to a linear isometry.

Bounded symmetric domains in finite dimensions were introduced and completely classified by Cartan nearly 50 years ago. By the Harish Chandra embedding theorem every such domain can be realized as a bounded circled domain in $\mathbb{C}^n$ containing the origin. Moreover, the Hermann Convexity Theorem – proved much later – states that the Harish Chandra realization always is convex and hence is the open unit ball with respect to a certain norm on $\mathbb{C}^n$ (for details compare f.i. [20]).

In [8] we gave a complete classification of all bounded symmetric domains in complex Hilbert spaces – also every domain of this type admits a realization as open unit ball in a suitable complex Banach space, as follows by inspecting all occurring cases. In contrast to this the classification problem for bounded symmetric domains in arbitrary complex Banach spaces is far from being manageable. Nevertheless, Vigué showed that a Harish Chandra realization always exists, but the convexity could not be proved except for special cases.

By Harris the open unit ball of every C^*-algebra A is a symmetric domain and this remains true for every closed linear subspace $U \subset A$ invariant under the Jordan triple product

$$\text{(i)} \qquad \{x\, y^*\, z\} := \tfrac{1}{2}(x\, y^*\, z + z\, y^*\, x).$$

In [3] an extension to JB^*-algebras (=Jordan C^*-algebras) was given: The open unit balls of (unital) JB^*-algebras are precisely (i.e. up to biholomorphy) those bounded symmetric domains, which admit a realization as upper half plane. In this paper as a further generalization we introduce the concept of a JB^*-triple and show as our main result: *Every bounded symmetric domain in a complex Banach space is biholomorphically equivalent to the open unit ball of a JB^*-triple and in this way, the category of all bounded symmetric domains with base point is equivalent to the category of JB^*-triples.*

We define a JB^*-triple as a complex Banach space U together with a continuous, sesquilinear, operator valued product

$$U \times U \to \mathscr{L}(U),$$
$$(x, y) \mapsto x \square y^*$$

on U such that $\{x\, y^*\, z\} := x \square y^*(z)$ defines a Jordan triple system (over $\mathbb{R}$) and such that every $z \square z^*$ is a hermitian operator $\geqq 0$ on U satisfying the C^*-condition

$$\|z \square z^*\| = \|z\|^2.$$

The first to introduce Jordan methods into the theory of bounded symmetry domains in $\mathbb{C}^n$ was Koecher [10] – for a complete presentation of the finite dimensional case compare also [12]. The main tools in finite dimensions – the trace form $\operatorname{tr}(x \square y^*)$ and the Peirce decomposition do not apply in the general case. Nevertheless several ideas of [10–12] proved to be helpful also in our situation.

The paper is organized as follows: In § 1 we prove a Gelfand theorem for abelian J^*-triples. J^*-triples (= hermitian Jordan triple systems) characterize algebraically all symmetric complex Banach manifolds [7]. The importance of abelian J^*-triples stems from the fact that J^*-triples are abelian locally, i.e. every single element generates an abelian subtriple. In § 2 we determine explicitly all solutions of certain Riccati differential equations with constant coefficients on a J^*-triple. In § 3 we discuss the spectra of the operators $z \square z^*$ on a J^*-triple U – these spectra determine the geometric structure of the symmetric complex manifold associated with U. In § 4 we give an explicit realization of all simply connected symmetric complex Banach manifolds which belong to positive J^*-triples. Among these are all symmetric manifolds that are separable with respect to bounded holomorphic functions. In § 5 we discuss JB^*-triples and in § 6 we illustrate (without giving proofs) our constructions in terms of a simple example.

Notations. For complex Banach spaces U, V let $\mathscr{L}(U, V)$ be the Banach space of all bounded linear operators λ: $U \to V$. $U^t := \mathscr{L}(U, \mathbb{C})$ is the dual Banach space and $\mathscr{L}(U) := \mathscr{L}(U, U)$ is considered as a complex Banach algebra with maximal subgroup denoted by $\mathrm{GL}(U)$. $\mathscr{H}(U) \subset \mathscr{L}(U)$ is the $\mathbb{R}$-linear subspace of all hermitian operators on U, i.e. of all $\lambda \in \mathscr{L}(U)$ with $\exp(i t \lambda) \in \mathrm{GL}(U)$ isometry for all $t \in \mathbb{R}$. $\mathscr{J}(U) := \mathscr{H}(U) + i\mathscr{H}(U)$ is a closed complex Lie subalgebra of $\mathscr{L}(U)$ in the commutator product and admits a unique continuous involution $*$ with $\mathscr{H}(U) = \{\lambda \in \mathscr{J}(U)\colon \lambda^* = \lambda\}$ – compare [1]. $\sigma(\lambda) \subset \mathbb{C}$ is the spectrum of $\lambda \in \mathscr{L}(U)$ – in case the spectrum refers to a subalgebra $B \subset \mathscr{L}(U)$ we write $\sigma_B(\lambda)$. The inequalities $\lambda > r$, $\geqq r$, $\leqq r$ and $< r$ for $\lambda \in \mathscr{H}(U)$ and $r \in \mathbb{R}$ mean that the spectrum $\sigma(\lambda) \subset \mathbb{R}$ satisfies the corresponding inequality. $\mathscr{H}^+(U)$ denotes the convex cone of all $\lambda \geqq 0$ in $\mathscr{H}(U)$. The identity of $\mathrm{GL}(U)$, often simply denoted by 1, is an inner point of $\mathscr{H}^+(U)$.

Δ is the open unit disc in $\mathbb{C}$ and for every locally compact topological space S, $\mathscr{C}_0(S)$ is the complex Banach space of all continuous functions $S \to \mathbb{C}$ vanishing at infinity. For every topological group G the identity component is denoted by G^0.

§ 1. Commutative Gelfand Theory for J^*-Triples

Symmetric complex Banach manifolds can be algebraically characterized by hermitian Jordan triple systems (J^*-triples for short - compare [7]). By definition a J^*-triple is a complex Banach space U together with a continuous, sesquilinear, operator-valued product

$$U \times U \to \mathscr{L}(U)$$
$$(x, y) \mapsto x \square y^*$$

such that

(1.1) the triple product $\{x y^* z\} := x \square y^*(z)$ is symmetric in x, z,

(1.2) $\{z\{x a^* y\}^* w\} = \{\{a x^* z\} y^* w\} - \{a x^*\{z y^* w\}\} + \{z y^*\{a x^* w\}\}$,

(1.3) $z \square z^* \in \mathscr{H}(U)$

for all $a, x, y, z, w \in U$.

The J^*-triple U is called *positive*[1] (*negative* resp., *trivial* resp.) if $z \square z^* \geqq 0$ ($\leqq 0$ resp., $=0$ resp.) for all $z \in U$. By (1.2) the linear span of $U \square U^*$ is a complex Lie subalgebra of $\mathscr{J}(U)$ invariant under the involution $*$, and every λ in this span satisfies

$$\lambda\{x y^* z\} = \{(\lambda x) y^* z\} - \{x(\lambda^* y)^* z\} + \{x y^*(\lambda z)\}. \tag{1.4}$$

The J^*-triple U is called *abelian* if $U \square U^*$ is a commutative set of operators. In this case the closed linear span $B := B(U)$ of $U \square U^*$ in $\mathscr{L}(U)$ is a commutative complex Banach algebra and a $*$-algebra with respect to the involution of $\mathscr{J}(U)$ - compare [8 I, (1.17)]. Actually more is true

(1.5) **Proposition.** *For every abelian J^*-triple U the closed linear span $B(U)$ of $U \square U^*$ is a commutative C^*-algebra.*

Proof. Let A be the closed unital subalgebra of $\mathscr{L}(U)$ generated by B. Then $A \subset \mathscr{J}(U)$ is also $*$-invariant and a V-algebra, which implies the result by the Vidav-Palmer theorem [1, § 38].

As a C^*-algebra $B(U)$ can be identified with the space $\mathscr{C}_0(\Sigma)$, where $\Sigma = \Sigma(U)$ is the space of all non-trivial multiplicative linear forms on $B(U)$. In the following we study the analogue for J^*-triples: $\mathbb{C}$ is always considered as a J^*-triple with triple product defined by $\{x y^* z\} = x \bar{y} z$. If we define the *norm* $N = N(U)$ of the J^*-triple U by

$$N := \sup \|\{x y^* z\}\|,$$

where the sup is taken over all unit vectors $x, y, z \in U$, we get

(1.6) **Lemma.** *Every algebraic J^*-morphism λ: $U \to \mathbb{C}$ is continuous and satisfies*

$$\|\lambda\| \leqq \sqrt{N}.$$

---

[1] This definition differs slightly from the one given in [8 II]

Proof. Otherwise, there would be an $x \in U$ with $N\|x\|^2 < 1$ and $\lambda(x) = 1$. Then $y := \sum_{k=1}^{\infty} x_k$ for $x_1 := x$, $x_{k+1} := \{x\, x_k^*\, x\}$ exists in U and satisfies $x + \{x\, y^*\, x\} = y$, i.e. $1 + \overline{\lambda(y)} = \lambda(y)$, a contradiction.

For every J^*-triple U denote by $\Lambda = \Lambda(U)$ the set of all J^*-morphisms λ: $U \to \mathbb{C}$ with $\lambda \neq 0$. Considered as a subset of $U^t = \mathscr{L}(U, \mathbb{C})$ and endowed with the topology $\sigma(U^t, U)$ Λ is a locally compact space and a principal $\mathbb{T}$-bundle for the group

$$\mathbb{T} := \{t \in \mathbb{C}\colon\ |t| = 1\}.$$

The base $\Lambda/\mathbb{T}$ of this bundle can be identified with the space of all ideals $I \subset U$ such that U/I is J^*-isomorphic to $\mathbb{C}$.

$$A(U) := \mathscr{C}_0^{\mathbb{T}}(\Lambda) := \{f \in \mathscr{C}_0(\Lambda)\colon\ f(t\,\lambda) = t\, f(\lambda) \quad \text{for all } t \in \mathbb{T},\ \lambda \in \Lambda\}$$

is a subtriple of $\mathscr{C}_0(\Lambda)$ and $\hat{x}(\lambda) := \lambda(x)$ for $x \in U$, $\lambda \in \Lambda$ defines a continuous J^*-morphism

$$\hat{}\colon\ U \to A(U) \tag{1.7}$$

which we call the *Gelfand representation* of U. This representation depends functorially on U. Applying Stone-Weierstraß to the conjugation-invariant subalgebra of $\mathscr{C}_0(\Lambda)$ generated by $\hat{U}$ and using invariant integration over $\mathbb{T}$, the following is easily derived

(1.8) **Lemma.** *The Gelfand representation* (1.7) *has dense image.*

Note, that Λ may be empty – in this case we put $\mathscr{C}_0(\Lambda) = 0$.

Assume that U is abelian. Then $\Sigma = \Sigma(U)$ was defined before as the spectrum of $B = B(U)$ and for $\varepsilon = \pm$ we put

$$\Sigma^\varepsilon := \{t \in \Sigma\colon\ \varepsilon\, \tau(x \square x^*) \geqq 0 \text{ for all } x \in U\}.$$

(1.9) **Proposition.** *Σ is the disjoint union of Σ^+ and Σ^-, in particular Σ^+, Σ^- are open and closed in Σ. To every $\lambda \in \Lambda$ there exists a unique $\tau \in \Sigma^+$ with*

$$\tau(x \square y^*) = \lambda(x)\, \overline{\lambda(y)}$$

for every $x, y \in U$. Furthermore $\lambda \mapsto \tau$ induces a bijection $\Lambda/\mathbb{T} \approx \Sigma^+$.

Proof. Let $\lambda \in \Lambda$ be given and fix $\alpha \in U$ with $\lambda(\alpha) = 1$. $I := \lambda^{-1}(0)$ is an ideal in U and hence invariant under B. Therefore

$$J := \{b \in B\colon\ \lambda(b\,\alpha) = 0\}$$

is a maximal ideal of B with $\alpha \square \alpha^* \notin J$. For $\tau \in \Sigma$ defined by $\tau^{-1}(0) = J$ there is a constant $c \neq 0$ with

$$\tau(x \square y^*) = c\, \lambda(x)\, \overline{\lambda(y)}$$

for all $x, y \in U$. Evaluating $\tau(\alpha\square\alpha^*)^2 = \tau(\{\alpha\,\alpha^*\,\alpha\}\square\alpha^*)$ gives $c=1$ and $\tau \in \Sigma^+$. Now let $\tau \in \Sigma \setminus \Sigma^-$ be an arbitrary element. Then $\tau(\alpha\square\alpha^*)=1$ for some $\alpha \in U$ and

$$I := \{x \in U:\ \tau(x\square\alpha^*)=0\}$$

is an ideal of codimension 1 in U. Denote by λ: $U \to \mathbb{C}$ the unique linear form with $\lambda(I)=0$ and $\lambda(\alpha)=1$. Then $\alpha\square\alpha^*$ induces the identity on $U/I \approx \mathbb{C}$, i.e. $\lambda \in \Lambda$ and $\tau(x\square y^*) = \lambda(x)\,\overline{\lambda(y)}$ for all $x, y \in U$. Q.e.d.

(1.10) **Corollary.** *Suppose U is an abelian J^*-triple and $z \in U$. Then*

$$\hat{z}=0 \quad \textit{if and only if} \quad z\square z^* \leqq 0.$$

In particular, the kernel of the Gelfand representation is the maximal negative ideal in U.

(1.11) **Corollary.** *Suppose U is a positive abelian J^*-triple. Then the Gelfand representation $U \to A(U)$ is a surjective J^*-isomorphism if and only if there is a constant $c>0$ with*

(i) $$\|\{z\,z^*\,z\}\| \geqq c\,\|z\|^3$$

for all $z \in U$. ^ is an isometry if and only if

(ii) $$\|\{z\,z^*\,z\}\| = \|z\|^3$$

for all $z \in U$.

Proof. For $z \in U$ we have

$$\|z\square z^*\| \in \sigma(z\square z^*) = \sigma_B(z\square z^*),$$

and we may assume $z \neq 0$.

Therefore, if (i) is true, there are $\tau \in \Sigma^+$ and $\lambda \in \Lambda$ depending on z such that

$$|\hat{z}(\lambda)|^2 = |\lambda(z)|^2 = \tau(z\square z^*) = \|z\square z^*\| \geqq c\,\|z\|^2, \quad \text{i.e.}$$

(iii) $$\|\hat{z}\| \geqq \sqrt{c}\,\|z\| \quad \text{for all } z \in U.$$

This implies that ^ has closed image and hence is surjective by (1.8). Suppose (ii) is true but ^ is not an isometry. Then by (iii) there is a $z \in U$ with $\|z\| < \|\hat{z}\| = 1$. $z_1 := z$, $z_{k+1} := \{z_k\,z_k^*\,z_k\}$ defines a sequence in U with $\lim z_k = 0$, while $(\hat{z}_k)$ does not converge to $0 \in A(U)$, a contradiction. Q.e.d.

We give some applications: Suppose S, T are locally compact principal $\mathbb{T}$-bundles and φ: $U \to V$ is a J^*-morphism for $U := \mathscr{C}_0^{\mathbb{T}}(S)$, $V := \mathscr{C}_0^{\mathbb{T}}(T)$. Then there is a commutative diagram

$$\begin{tikzcd} U \arrow[r, "\varphi"] \arrow[d, "\wedge"'] & V \arrow[d, "\wedge"] \\ A(U) \arrow[r, "A(\varphi)"'] & A(V), \end{tikzcd}$$

where the vertical arrows represent J^*-isomorphisms by (1.11). Since $A(\varphi)$ is induced by the continuous map $\varphi^t\colon \Lambda(V)\to\Lambda(U)$, we get

(1.12) **Proposition.** *For every linear bijection $\varphi\colon \mathscr{C}_0^{\mathbb{T}}(S)\to\mathscr{C}_0^{\mathbb{T}}(T)$ with S, T locally compact principal $\mathbb{T}$-bundles the following are equivalent*

(i) *φ is a J^*-isomorphism*

(ii) *φ is an isometry*

(iii) *$\varphi(f)=f\circ\tau$ for some $\mathbb{T}$-equivariant homeomorphism $\tau\colon T\to S$.*

This generalizes the well-known fact, that for every locally compact space X the surjective isometries of $\mathscr{C}_0(X)$ are precisely the transformations $f\mapsto g\cdot(f\circ\tau)$ with $f\colon X\to X$ a homeomorphism and $g\colon X\to\mathbb{T}$ a continuous function – for $\mathscr{C}_0(X)\approx\mathscr{C}_0^{\mathbb{T}}(S)$ with $S:=X\times\mathbb{T}$ the trivial $\mathbb{T}$-bundle over X. Furthermore we get

(1.13) **Corollary.** *Let S be a locally compact principal $\mathbb{T}$-bundle. Then $\mathscr{C}_0^{\mathbb{T}}(S)$ is isometrically isomorphic to $\mathscr{C}_0(X)$ for some locally compact X if and only if S is a trivial $\mathbb{T}$-bundle.*

As an example the $(2n+1)$-sphere

$$S:=\{z\in\mathbb{C}^{n+1}\colon\ \|z\|_2=1\}$$

is a non-trivial principal $\mathbb{T}$-bundle over the complex projective space $\mathbb{P}_n(\mathbb{C})$.

For every J^*-triple U and every $\alpha\in U$ denote by U_α the smallest closed linear subspace containing α and invariant by $\alpha\square\alpha^*$. U_α is an abelian subtriple and is called the *subtriple generated by* α.

(1.14) **Lemma.** *Suppose the J^*-triple U is generated by a single element (i.e. $U=U_\alpha$ for some $\alpha\in U$). Then $A(U)$ is J^*-isomorphic to $\mathscr{C}_0(S)$ for some locally compact space S.*

Proof. Suppose U is generated by $\alpha\in U$. Then $A(U)=\mathscr{C}_0^{\mathbb{T}}(\Lambda)$ is generated by $\hat{\alpha}$, i.e. we may take $S:=\hat{\alpha}(\Lambda)\cap\mathbb{R}^+\subset\mathbb{R}$.

(1.15) **Corollary.** *Let U be the J^*-triple associated with a bounded symmetric domain D in a complex Banach space and $\alpha\in U$. Then U_α is J^*-isomorphic to $\mathscr{C}_0(S)$ for some locally compact space S.*

Proof. We may assume $U=U_\alpha$. By [7] (compare also § 4) $\{z\in U\colon z\square z^*<1\}$ may be considered as a subdomain of D. In particular, U is positive by (1.10) and we may assume that $D=\{z\in U\colon \|\hat{z}\|<1\}$. This implies that the Gelfand representation $U\to A(U)$ is a J^*-isomorphism and the result follows with (1.14).

Remark. (1.15) is already contained in [7, (4.8)]. The proof there uses Propositions 4.5 and 4.7, which in general are only true if the flat subsystem V is generated by a single element – compare also [8 II, (5.2)]. – A certain type of abelian J^*-triples is also considered in [23].

§ 2. Explicit Integration of Certain Polynomial Vector Fields

Let U be a J^*-triple and $\alpha \in U$. We want to find all solutions of the Riccati differential equation

$$\frac{dg_t}{dt} = f(g_t), \tag{2.1}$$

where the polynomial map $f\colon U \to U$ is given by

$$f(z) = \alpha - \{z\,\alpha^*\,z\}.$$

We study the problem in the more general setting of Jordan pairs [13]: Let U be a complex Banach space and let $\mathcal{Q}(U)$ be the complex Banach space of all continuous homogeneous quadratic maps $q\colon U \to U$, where the norm is given by $\|q\| = \sup_{\|z\| \leq 1} \|q(z)\|$. Every $q \in \mathcal{Q}(U)$ can be considered as a commutative complex algebra structure (non-associative in general) on U where the q-product of $z, w \in U$ is given by

$$\{z\,q\,w\} := \tfrac{1}{2}(q(z+w) - q(z) - q(w)). \tag{2.2}$$

For every $p, q \in \mathcal{Q}(U)$ and $a \in U$ we define $\{p\,a\,q\} \in \mathcal{Q}(U)$ by

$$\{z\{p\,a\,q\}\,w\} = \{\{a\,p\,z\}\,q\,w\} - \{a\,p\{z\,q\,w\}\} + \{z\,q\{a\,p\,w\}\} \tag{2.3}$$

for all $z, w \in U$ - compare (1.2).

We call a closed linear subspace $V \subset \mathcal{Q}(U)$ a *Jordan family* on U, if

$$\{v\,a\,w\} = \{w\,a\,v\} \in V \quad \text{for all } a \in U,\ v, w \in V. \tag{2.4}$$

Combining (2.3) with (2.4) implies

$$\{\{z\,v\,z\}\,w\,z\} = \{z\,v\{z\,w\,z\}\} \tag{2.5}$$

for all $z \in U$ and $v, w \in V$.

Jordan families can also be defined in terms of polynomial vector fields on U: Denote by $\mathfrak{P}$ the set of all polynomial vector fields on U, that is of all expressions $X = f(z)\frac{\partial}{\partial z}$ $\left(\text{or simply } f\frac{\partial}{dz}\right)$ where $f\colon U \to U$ is a continuous polynomial map. $\mathfrak{P}$ is a complex Lie algebra in the Lie product[2] defined by

$$\left[f(z)\frac{\partial}{\partial z}, g(z)\frac{\partial}{\partial z}\right] = (f'(z)\,(g(z)) - g'(z)\,(f(z))\frac{\partial}{\partial z},$$

where $f', g'\colon U \to \mathcal{L}(U)$ are the derivatives of f, g. With respect to the norm

$$\left\| f\frac{\partial}{\partial z} \right\| = \sup_{\|z\| \leq 1} \|f(z)\|$$

the Lie product is continuous on $\mathfrak{P}$.

[2] The definition differs from the usual convention by sign and is also used in [10–12]

$\mathfrak{P}$ admits a $\mathbb{Z}$-gradation

$$\mathfrak{P}=\bigoplus_{\nu\in\mathbb{Z}}\mathfrak{P}_\nu,$$

where $\mathfrak{P}_\nu$ is the Banach subspace of all homogeneous polynomial vector fields of degree $\nu+1$ and $\mathfrak{P}_\nu=0$ for $\nu<(-1)$. $\mathfrak{P}_\nu$ is also the ν-eigenspace of $\mathrm{ad}(-Z)$, where $Z:=z\dfrac{\partial}{\partial z}\in\mathfrak{P}_0$. By a simple computation one shows

(2.6) **Lemma.** *A closed linear subspace $V\subset\mathcal{Q}(U)$ is a Jordan family if and only if*

$$\mathfrak{l}:=\mathfrak{l}_{-1}\oplus\mathfrak{l}_0\oplus\mathfrak{l}_1$$

is a Lie subalgebra of $\mathfrak{P}$, where

$$\mathfrak{l}_{-1}:=U\frac{\partial}{\partial z}=\mathfrak{P}_{-1}$$

$$\mathfrak{l}_1:=V\frac{\partial}{\partial z}\subset\mathfrak{P}_1$$

and

$$\mathfrak{l}_0:=\{X\in\mathfrak{P}_0\colon\ [X,\mathfrak{l}_1]\subset\mathfrak{l}_1\}.$$

In particular for every Jordan family $V\subset\mathcal{Q}(U)$, (U,V) is a complex Jordan pair [13] with quadratic operator $Q\colon U\cup V\to\mathcal{L}(V,U)\cup\mathcal{L}(U,V)$ defined by

$$Q(z)\,v=\{z\,v\,z\}\quad\text{and}\quad Q(v)\,z=\{v\,z\,v\}\tag{2.7}$$

for all $z\in U$, $v\in V$. In particular, every member of a Jordan family is a Jordan algebra structure. On the other hand for every Jordan algebra structure $v\in\mathcal{Q}(U)$ the closure of all homotopes $\{v\,a\,v\}$, $a\in U$, in $\mathcal{Q}(U)$ is a Jordan family on U.

For every $a\in U$ and $v\in V$ we define operators $a\square v\in\mathcal{L}(U)$ and $v\square a\in\mathcal{L}(V)$ by

$$\begin{gathered}a\square v(z)=\{a\,v\,z\},\qquad v\square a(w)=\{v\,a\,w\}.\\ B(a,v):=1-2a\square v+Q(a)\,Q(v)\in\mathcal{L}(U)\end{gathered}\tag{2.8}$$

is called the *Bergman operator.*

Suppose $z\mapsto z^*$ is a conjugate linear mapping $*\colon U\to V$ satisfying (1.3) and

$$\{z\,a^*\,w\}^*=\{z^*\,a\,w^*\}\tag{2.9}$$

for all $a,z,w\in U$. Then U is a J^*-triple with respect to $x\square y^*$, $x,y\in U$. On the other hand, if U is a J^*-triple and if we define $a^*(z):=\{z\,a^*\,z\}$ for $a,z\in U$, the closure V of $\{a^*\colon a\in U\}$ in $\mathcal{Q}(U)$ is a Jordan family with (2.9). In this case V is called the *Jordan family associated with the J^*-triple U*. For every $a,b,z\in U$ then

$$Q(a)\,Q(b^*)\,z=\{a\{b\,z^*\,b\}^*\,a\}\in U$$

is clear from (2.9).

(2.10) *Example.* Let E, F be complex Banach spaces and $U=\mathscr{L}(E,F)$. Then $V=\mathscr{L}(F,E)$ can be considered as a closed subspace of $\mathscr{Q}(U)$ if we put

$$v(z):=zvz$$

for all $v\in V$, $z\in U$. V is a Jordan family. In case E, F are complex Hilbert spaces, the usual adjoint $*$: $U\to V$ makes U a J^*-triple. In general, V is not associated with a J^*-triple.

In the following let $V\subset\mathscr{Q}(U)$ be a Jordan family. Then

$$\Gamma:=\{g\in \mathrm{GL}(U)\colon\ gVg^{-1}=V\}$$

is a complex Banach Lie group [5, Theorem 1] which we call the *structure group of* V. $\mathfrak{l}=\mathfrak{l}_{-1}\oplus\mathfrak{l}_0\oplus\mathfrak{l}_1$ is a graded complex Banach Lie algebra (a binary Lie algebra in the sense of Koecher [10, 11]).

There exists in a natural way a complex manifold M containing U as a domain such that every $X\in\mathfrak{l}$ extends to a complete holomorphic vector field on M. The construction [7, § 3] is roughly as follows: ad: $\mathfrak{l}\to\mathrm{der}(\mathfrak{l})$ (=derivation algebra) is an isomorphism and hence there exists a commutative diagram

$$\begin{tikzcd} L \arrow[r, "\mathrm{Ad}"] & \mathrm{Aut}(\mathfrak{l}) \\ \mathfrak{l} \arrow[u, "\exp"] \arrow[r, "\mathrm{ad}"'] & \mathrm{der}(\mathfrak{l}), \arrow[u, "\exp"'] \end{tikzcd}$$

where L is a complex Banach Lie group with Lie algebra $\mathfrak{l}$ and Ad is a biholomorphic homomorphism. $H:=\{g\in L\colon \mathrm{Ad}(g)\mathfrak{h}=\mathfrak{h}\}$ is a complex subgroup with Lie algebra $\mathfrak{h}:=\mathfrak{l}_0\oplus\mathfrak{l}_1$. Then $M:=L/H$, and U is a domain in M if we identify $a\in U$ with $\exp\left(a\dfrac{\partial}{\partial z}\right)H\in M$. The group L acts holomorphically effectively and transitively on M by $g(hH):=(gh)H$ for all $g,h\in L$. Define closed complex subgroups $L_\nu\subset L$ by

$$L_\nu:=\exp(\mathfrak{l}_\nu)\qquad\text{for } \nu=\pm 1$$

and $L_0:=\{g\in L\colon \mathrm{Ad}(g)Z=Z\}$ for $\nu=0$. Then L_ν is commutative and exp: $\mathfrak{l}_\nu\to L_\nu$ is biholomorphic for $\nu=\pm 1$. Every $g\in L_0$ is linear on U and via $g\mapsto g|U$ the group L_0 can be identified with the structure group Γ.

Finding explicit expressions for $g\in L$ on U is the problem we started with: For every $X=f\dfrac{\partial}{\partial z}\in\mathfrak{l}$, $g_t:=\exp(tX)$ and $z\in U$ the function $g_t(z)$ solves (2.1) for $|t|$ small to the initial value $g_0(z)=z$. Using a method of Koecher [10, 11] we find the general solution of (2.1) for certain X: Fix $g\in L$ and put $W:=U\cap g^{-1}(U)$. For every $z\in W$ the derivative $g'(z)\in\mathrm{GL}(U)$ is well defined and every $X:=f(z)\dfrac{\partial}{\partial z}$ is related to $\mathrm{Ad}(g^{-1})(X)=:h(z)\dfrac{\partial}{\partial z}$ by

$$h(z)=g'(z)^{-1}f(g(z))\qquad\text{for } z\in W. \tag{2.11}$$

Then, if we define the polynomial maps of degree $\leqq 2$

(2.12) $$p\colon U \to U \quad \text{and} \quad q\colon U \to \mathscr{L}(U)$$

by

$$\mathrm{Ad}(g^{-1})\, a \frac{\partial}{\partial z} = (q(z)\, a) \frac{\partial}{\partial z} \quad \text{for all } a \in U$$

and

$$\mathrm{Ad}(g^{-1})\, z \frac{\partial}{\partial z} = p(z) \frac{\partial}{\partial z},$$

the Eq. (2.11) implies

$$g'(z)^{-1} = q(z) \quad \text{and} \quad g'(z)^{-1} g(z) = p(z)$$

for all $z \in W$, i.e.

(2.13) $$g(z) = q(z)^{-1} p(z) \quad \text{and} \quad g'(z) = q(z)^{-1}$$

for all $z \in W = U \cap g^{-1}(U)$. In this sense g is a quadratic fractional transformation.

(2.14) **Proposition.** *$W = U \cap g^{-1}(U) = \{z \in U\colon q(z) \in \mathrm{GL}(U)\}$. Furthermore, the derivative $g'(z) = q(z)^{-1}$ is in the structure group Γ for every $z \in W$.*

Proof. Suppose $a \in U$ satisfies $q(a) \in \mathrm{GL}(U)$. We have to show $g(a) \in U$ and $q(a) \in \Gamma$. We put $\theta := \mathrm{Ad}(g^{-1})$ and

$$t_c := \exp\left(c \frac{\partial}{\partial z}\right) \in L_{-1} \quad \text{and} \quad \tilde{t}_v := \exp\left(v \frac{\partial}{\partial z}\right) \in L_1$$

for every $c \in U$, $v \in V$. Replacing g by $g t_a^{-1}$ we may assume $a = 0$, i.e. $\lambda := q(0)^{-1} \in \mathrm{GL}(U)$ exists. Replacing again g by $t_b^{-1} g$ for $b := \lambda p(0)$ we may assume in addition that also $p(0) = 0$ is true, i.e. $\theta Z \in \mathfrak{h} = \mathfrak{l}_0 \oplus \mathfrak{l}_1$. But then θ applied to $\left[Z, a \frac{\partial}{\partial z}\right] = a \frac{\partial}{\partial z}$ gives

$$v \frac{\partial}{\partial z} := \theta Z - Z \in \mathfrak{l}_1.$$

From this we derive $\theta \mathfrak{l}_1 = \mathfrak{l}_1$ by applying θ to $[Y, Z] = Y$ for every $Y = f(z) \frac{\partial}{\partial z} \in \mathfrak{l}_1$. $\frac{1}{2}\left[c \frac{\partial}{\partial z}, \left[c \frac{\partial}{\partial z}, Y\right]\right] = f(c) \frac{\partial}{\partial z}$ implies furthermore that $\theta Y =: h(z) \frac{\partial}{\partial z}$ satisfies $\lambda h \lambda^{-1} = f$, i.e. $\lambda \in \Gamma \cong L_0$. Replacing finally g by $\tilde{t}_v^{-1} \lambda g$ we may also assume $q(0) = 1 \in \mathrm{GL}(U)$ and $\theta Z = Z$. But then $g = 1 \in L$ and in particular $g(0) = 0 \in U$. Q.e.d.

We consider two special examples. Let as above

(2.15) $$t_c := \exp\left(c \frac{\partial}{\partial z}\right) \in L_{-1}.$$

In this case $W=U$, $q\equiv 1\in \mathrm{GL}(U)$ and $p(z)=z+c$, i.e. t_c is the translation

$$t_c(z)=z+c, \qquad z\in U,$$

on U (a result, which clearly also follows at once by integrating (2.1)). Next consider

$$\tilde{t}_v := \exp\left(v\frac{\partial}{\partial z}\right)\in L_1. \tag{2.16}$$

In this case $q(z)=B(z,v)$ is the Bergman operator and $p(z)=z-Q(z)\,v$, i.e.

$$\tilde{t}_v(z)=B(z,v)^{-1}(z-Q(z)\,v) \tag{2.17}$$

for every $z\in U$ with $B(z,v)\in \mathrm{GL}(U)$, and the derivative there is given by

$$\tilde{t}_v'(z)=B(z,v)^{-1}. \tag{2.18}$$

In the sense of [13, p. 22] therefore the pair (z,v) is quasi-invertible if and only if $\tilde{t}_v(z)\in U$ and $\tilde{t}_v(z)$ is the quasi-inverse. In the special case $\|z\square v\|<1$, (2.17) can also be written as a linear fractional transformation

$$\tilde{t}_v(z)=(1-z\square v)^{-1}\,z \tag{2.19}$$

since $g_t(z):=(1-t\,z\square v)^{-1}\,t\,z$ solves

$$\frac{\partial}{\partial t}g_t(z)=\{g_t(z)\,v\,g_t(z)\}.$$

The proof of (2.14) includes in particular

(2.20) **Corollary.** *Every $g\in L$ with $g(0)\in U$ has a unique representation*

$$g=t_c\,\lambda\,\tilde{t}_v\in L_{-1}\,L_0\,L_1,$$

where $c=g(0)$, $\lambda=g'(0)$ and $v=g''(0)$ (in the sense that $v(z)=g''(0)\,(z,z)$). In particular $H=L_0\,L_1$.

Thus every $g\in L$ is uniquely determined by the three derivatives up to order 2 at a single point of M.

(2.21) **Proposition.** *Suppose $g\in\exp(\mathrm{I}_{-1}\oplus\mathrm{I}_1)$ satisfies $g=t_c\,\lambda\,\tilde{t}_v$ as in (2.20). Then*

$$\lambda^2=B(c,-v)\in \mathrm{GL}(U).$$

Proof. Put $h:=g^{-1}$ and $s:=\exp(i\pi Z)\in L_0$. Then $h=s\,g\,s$ and $s(z)=-z$ for $z\in U$, i.e. $h(0)=-c$ and $h'(0)=g'(0)=\lambda$. This implies by the chain rule

$$1=q'(h(0))\,h'(0)=\lambda\circ\tilde{t}_v'(-c)\circ\lambda\in \mathrm{GL}(U).$$

Then (2.18) together with $B(-c,v)=B(c,-v)$ gives the result.

c, v in (2.21) can be obtained by direct integration: Suppose $\alpha\in U$, $\beta\in V$, $X:=(\alpha+\beta)\dfrac{\partial}{\partial z}\in\mathrm{I}_{-1}\oplus\mathrm{I}_1$ are given anf $g:=\exp(X)$. Define the meromorphic

function $\mathfrak{t}$ on $\mathbb{C}$ by

$$s\,\mathfrak{t}(s^2)=\tan(s)$$

which we also denote by $\mathfrak{t}(s)=\dfrac{\tan\sqrt{s}}{\sqrt{s}}$. Then

$$\Omega:=\{s\in\mathbb{C}:\ s\in\mathbb{R}\Rightarrow s<\pi^2/4\} \tag{2.22}$$

is the maximal starlike domain in $\mathbb{C}$ on which $\mathfrak{t}$ is holomorphic. Now denote by A the smallest $\alpha\square\beta$-invariant closed linear subspace of U containing α and assume $\sigma(\alpha\square\beta)\subset\Omega$ (in (3.3) we will show that this follows already from the weaker condition $\sigma(\alpha\square\beta|A)\subset\Omega$). A is a commutative complex Banach algebra in the product β.

$$c_t:=t\,\mathfrak{t}(t^2\,\alpha\square\beta)\,\alpha\in A$$

is defined for $0\leqq t\leqq 1$ and solves the differential equation

$$\frac{\partial}{\partial t}c_t=\alpha+\beta(c_t)$$

to the unital value $c_0=0$, *i.e.*

$$c=\mathfrak{t}(\alpha\square\beta)\,\alpha\quad\text{in (2.21).}$$

It is clear that c depends holomorphically on the parameters α,β. v can be obtained in the dual way: Put $\tilde{\mathfrak{h}}:=\mathfrak{l}_{-1}\oplus\mathfrak{l}_0$ and $\tilde{H}:=\{g\in L:\ \mathrm{Ad}(g)\,\tilde{\mathfrak{h}}=\tilde{\mathfrak{h}}\}$. Then $\tilde{g}(\tilde{H}\,h):=\tilde{H}\,h\,g$ defines by $g\mapsto\tilde{g}$ a holomorphic antiaction of L on the complex manifold $\tilde{M}:=\tilde{H}\backslash L$. For every $w\in V$ we write $t_w\in\mathrm{Aut}(\tilde{M})$ instead of $\tilde{t}_w$. Identifying $w\in V$ and $t_w(\tilde{H})\in\tilde{M}$ we consider V as a domain in $\tilde{M}$. Then t_w is the translation $v\mapsto v+w$ on V. For $g=t_c\,\lambda\,\tilde{t}_v\in L$ as in (2.20) therefore $v=\tilde{g}(0)$ holds.

For every $Y\in\mathfrak{l}$ there exists a unique holomorphic vectorfield $\tilde{Y}$ on $\tilde{M}$ with

$$\tilde{g}=\exp(\tilde{Y})\quad\text{for } g=\exp(Y)\in L.$$

For every $\alpha\in U$, $\beta\in V$ we have

$$\widetilde{\{z\,\beta\,z\}\frac{\partial}{\partial z}}=\beta\frac{\partial}{\partial\tilde{z}}$$

$$\widetilde{\alpha\square\beta\frac{\partial}{\partial z}}=\beta\square\alpha\frac{\partial}{\partial\tilde{z}}$$

($\tilde{z}$ denotes the local coordinate on $V\subset\tilde{M}$).

Taking brackets we derive

$$\tilde{X}=(\beta+\{\tilde{z}\,\alpha\,\tilde{z}\})\frac{\partial}{\partial\tilde{z}}\quad\text{for } X=(\alpha+\{z\,\beta\,z\})\frac{\partial}{\partial z}.$$

Under the assumption $\sigma(\alpha\square\beta)\cup\sigma(\beta\square\alpha)\subset\Omega$ therefore $\tilde{g}_t(0)=t\mathfrak{t}(t^2\beta\square\alpha)\,\beta$ holds for $g_t:=\exp(tX)$ and $0\leqq t\leqq 1$. In particular

$$v=\mathfrak{t}(\beta\square\alpha)\,\beta$$

in (2.21) for $g=\exp X$. Summing up we have proved

(2.23) **Theorem.** *Let U be a complex Banach space, $V\subset\mathcal{Q}(U)$ a Jordan family and $\mathcal{D}:=\{(\alpha,\beta)\in U\oplus V\colon \sigma(\alpha\square\beta)\cup\sigma(\beta\square\alpha)\subset\Omega\}$ with Ω defined by (2.22). Then for every $(\alpha,\beta)\in\mathcal{D}$ the automorphism*

$$g:=\exp(\alpha+\beta(z))\frac{\partial}{\partial z}\in L$$

has a unique representation

$$g=t_c\,\lambda\tilde{t}_v,$$

where $c=\dfrac{\tan\sqrt{\alpha\square\beta}}{\sqrt{\alpha\square\beta}}\alpha\in U$, $v=\dfrac{\tan\sqrt{\beta\square\alpha}}{\sqrt{\beta\square\alpha}}\beta\in V$ and λ is uniquely determined by the following conditions:

(i) *λ is in the structure group $\Gamma\approx L_0$ and depends holomorphically on the parameter $(\alpha,\beta)\in\mathcal{D}$ and $\lambda=1\in\mathrm{GL}(U)$ for $\alpha=0$,*

(ii) *$\lambda^2=B(c,-v)$.*

Now suppose U is a J^*-triple and V is the corresponding Jordan family on U (=closure of U^* in $\mathcal{Q}(U)$). Then (2.9) implies

$$\sigma(\alpha^*\square\alpha)\subset\sigma(\alpha\square\alpha^*)\quad\text{for all } \alpha\in U. \tag{2.24}$$

With I, L, H, M as before we put $t^*_\alpha:=\tilde{t}_v$ for every $\alpha\in U$ and $v:=\alpha^*\in V$. As a corollary to (2.23) we get the solution to (2.1):

(2.25) **Theorem.** *Let U be a J^*-triple and*

$$\mathcal{A}=\{\alpha\in U\colon \sigma(\alpha\square\alpha^*)\subset(-\Omega)\}.$$

Then for every $\alpha\in\mathcal{A}$ the automorphism $g:=\exp(\alpha-\{z\alpha^*z\})\dfrac{\partial}{\partial z}\in L$ has a unique representation

$$g=t_cB_ct^*_{-c}$$

with

$$c=\frac{\tanh\sqrt{\alpha\square\alpha^*}}{\sqrt{\alpha\square\alpha^*}}\alpha=g(0)\in U$$

and $B_c=B(c,c^*)^{\frac{1}{2}}=g'(0)\in\Gamma\approx L_0$. $B(c,c^*)\in\mathrm{GL}(U)$ has positive real spectrum and the square root is understood in these sense of the holomorphic functional calculus on $\mathcal{L}(U)$.

Proof. We apply (2.23) for $\beta=-\alpha^*$ and obtain $g=t_c\lambda\tilde{t}_v$ with

$$-v=\mathfrak{t}(-\alpha^*\square\alpha)\,\alpha^*=(\mathfrak{t}(-\alpha\square\alpha^*)\,\alpha)^*=c^*.$$

Let U_α be the subtriple generated by α and $\theta:=\alpha\square\alpha^*|U_\alpha$. Then $U_c\subset U_\alpha$ and $c\square c^*|U_\alpha=(\tanh\sqrt{\theta})^2$. The complex function $s\mapsto(\tanh\sqrt{s})^2$ maps $-\Omega\cap\mathbb{R}$ onto $\{s\in\mathbb{R}: s<1\}$. By the spectral mapping theorem therefore $\sigma(c\square c^*|U_c)<1$ is true. In the next section (3.4) we show that then $B(c,c^*)$ has positive real spectrum. Now $\lambda=B(c,c^*)^{\frac{1}{2}}$ follows from (2.23).

Remark. The transformations g_c were used by Koecher [10, p. 115] in the case of bounded symmetric domains in $\mathbb{C}^n$. (2.25) in the finite dimensional case was already proved by Loos [12, (9.8)] using Peirce decompositions.

§ 3. Spectral Properties of J^*-Triples

The spectral properties of the operators $\alpha\square\alpha^*$ on a J^*-triple U determine the geometric structure of the symmetric complex Banach manifold associated with U. For every $\alpha\in U$ let U_α be the subtriple generated by α and put $S_\alpha:=\sigma(\alpha\square\alpha^*|U_\alpha)$. Then S_α is a compact subset of $\mathbb{R}$. In the following we want to estimate $\sigma(\alpha\square\alpha^*)$ in terms of S_α. We consider a well-known [12, p. 314].

(3.1) *Example.* Let U be a J^*-triple and suppose $\alpha\in U$ is a linear combination

$$\alpha=\sum_{i=1}^{n}\alpha_i e_i,\qquad \alpha_i\in\mathbb{C}$$

of an orthogonal set $\{e_1,\dots,e_n\}\subset U\setminus\{0\}$ of tripotents (i.e. $\{e_i e_i^* e_j\}=\delta_{ij}e_j$ for $1\leqq i,j\leqq n$). We put $\alpha_0:=0$ and assume for simplicity that $\alpha_i\neq\alpha_j$ for $0\leqq i<j\leqq n$. Then $U_\alpha=\sum\mathbb{C}e_i$ and $S_\alpha=\{\alpha_i\bar{\alpha}_i: 1\leqq i\leqq n\}$. A spectral decomposition of U with respect to $\alpha\square\alpha^*$ is given by the Peirce-decomposition

$$U=\bigoplus_{0\leqq i\leqq j\leqq n}U_{ij},$$

where $U_{ij}=\{z\in U: \{e_k e_k^* z\}=\frac{1}{2}(\delta_{ik}+\delta_{kj})\,z$ for $k=1,\dots,n\}$. The restriction of $\alpha\square\alpha^*$ to U_{ij} is just the multiplication by $\frac{1}{2}(\alpha_i\bar{\alpha}_i+\alpha_j\bar{\alpha}_j)$, i.e.

$$\sigma(\alpha\square\alpha^*)\subset\tfrac{1}{2}(S+S)$$

for $S:=S_\alpha\cup\{0\}$.

In general, a given J^*-triple does not contain any tripotent $e\neq 0$. Nevertheless we have

(3.2) **Proposition.** *For every J^*-triple U and every $\alpha\in U$ the spectrum of $\alpha\square\alpha^*$ satisfies*

$$\sigma(\alpha\square\alpha^*)\subset\tfrac{1}{2}(S+S),$$

where $S:=S_\alpha\cup\{0\}$.

The statement follows for $v=\alpha^*$ from the more general

(3.3) **Theorem.** *Let $V\subset\mathscr{Q}(U)$ be a Jordan family and $\alpha\in U$, $v\in V$. Denote by A the smallest closed $\alpha\square v$-invariant linear subspace of U which contains α. Then*

$$\sigma(\alpha\square v)\subset\tfrac{1}{2}(S+S)$$

for $S:=\sigma(\alpha\square v|A)\cup\{0\}$.

Proof. U is a complex Jordan algebra in the product $xy:=\{xvy\}$ and without loss of generality we may assume $\|xy\|\leqq\|x\|\,\|y\|$. Adjoining a unit e we obtain a unital complex Banach Jordan algebra $\hat{U}=U\oplus\mathbb{C}e$ which contains U as an ideal. For every $a\in\hat{U}$ denote by $L(a)\in\mathscr{L}(\hat{U})$ the left-multiplication by a and by

$$P(a):=2L(a)^2-L(a^2)\in\mathscr{L}(\hat{U})$$

the quadratic representation. $\hat{A}:=A\oplus\mathbb{C}e$ is a commutative, complex Banach algebra and $L(\hat{A})\subset\mathscr{L}(\hat{U})$ is a commutative set of operators. Denote by C a maximal commutative subalgebra of $\mathscr{L}(\hat{U})$ which contains $L(\hat{A})$. Then C is a unital complex Banach algebra with

$$\sigma(\alpha\square v)\subset\sigma(L(\alpha))=\sigma_C(L(\alpha)).$$

Now fix an element $z\in\sigma(\alpha\square v)$. Using an idea of [15, Theorem 1.2] we show $2z\in(S+S)$: There exists a multiplicative linear functional $\lambda\neq 0$ on C with $\lambda(L(\alpha))=z$. Determine $z_1,z_2\in\mathbb{C}$ such that

$$2z=(z_1+z_2)\quad\text{and}\quad\lambda(P(\alpha))=z_1z_2.$$

Then we obtain

$$\lambda(P(a_j))=\lambda(P(\alpha)-2z_jL(\alpha)+z_j^2)=0$$

for $a_j:=\alpha-z_je\in\hat{A}$ and $j=1,2$. Consequently $P(a_j)$ is not invertible in C and by the maximality of C also not in $\mathscr{L}(\hat{U})$. Therefore a_j is not invertible in $\hat{A}$ since otherwise $P(a_j^{-1})=P(a_j)^{-1}$ would be true. Therefore

$$z_j\in\sigma_{\hat{A}}(\alpha)=\sigma_A(\alpha)\cup\{0\}=S$$

for $j=1,2$. Q.e.d.

Recall from § 2 the definition of the operators $Q(\alpha)$, $Q(v)$ and the Bergman operator $B(\alpha,v)$ for every $\alpha\in U$, $v\in V$.

(3.4) **Corollary.** *With the notation of* (3.3) *we have*

(i) $\sigma(Q(\alpha)\,Q(v))\subset SS$ *and*

(ii) $\sigma(B(\alpha,v))\subset(1-S)\,(1-S)$ *for all* $\alpha\in U$, $v\in V$.

Proof. The proof of (3.3) implies

$$\sigma(P(\alpha))\subset\{z=z_1z_2\colon z_j\in S\}=SS$$

and

$$\sigma(1-2L(\alpha)+P(\alpha))\subset\{1-(z_1+z_2)+z_1z_2\}=(1-S)(1-S).$$

$Q(\alpha)\,Q(v)=P(\alpha)|\,U$ implies the result.

(3.5) **Corollary.** *For every J^*-triple U, every closed subtriple W and every $\alpha\in W$ the operators $\alpha\square\alpha^*\in\mathscr{L}(U)$ and $\alpha\square\alpha^*|\,W\in\mathscr{L}(W)$ have the same norm.*

Proof. By (3.2)

$$\|(\alpha\square\alpha^*|\,W)\|=\sup\{|s|:\,s\in S_\alpha\}$$

since $\alpha\square\alpha^*|\,W\in\mathscr{L}(W)$ is hermitian.

(3.6) **Lemma.** *A J^*-triple U is positive if and only if $W\square W^*=0$ for every negative subtriple W of U.*

Proof. Suppose U is positive and $W\subset U$ is a negative subtriple. W is also positive, i.e. $W\square W^*=0$ by (3.2). Now suppose that U is not positive. Then there exists by (3.2) an $\alpha\in U$ such that also U_α is not positive - hence we may assume that U is abelian and $\Sigma^-\neq\varnothing$ in the notation of § 1. Therefore there exists a function $f\in B(U)\approx\mathscr{C}_0(\Sigma)$ with support in Σ^- and $f\neq 0$. $\beta:=f(\alpha)$ then satisfies $\beta\square\beta^*\leqq 0$ and $W:=U_\beta$ is a negative subtriple with $W\square W^*\neq 0$. Q.e.d.

For the rest of this section we assume that the J^*-triple U is *positive* and define the function $\|\ \|_\infty$ on U by

$$\|\alpha\|_\infty^2:=\|\alpha\square\alpha^*\|$$

for all $\alpha\in U$. By (3.2) then $\|\alpha\|_\infty^2=\sup S_\alpha$ is true, i.e. $\|\ \|_\infty$ does not depend on the norm of U - only on the topology of U.

(3.7) **Lemma.** *$\|\ \|_\infty$ is a continuous semi-norm on U.*

Proof. It is enough to show that $\|\alpha\|_\infty\leqq 1$ and $\|\beta\|_\infty\leqq 1$ imply $\|\alpha+\beta\|_\infty\leqq 2$. But this is obvious from the identity

$$4-(\alpha+\beta)\square(\alpha+\beta)^*=2(1-\alpha\square\alpha^*)+2(1-\beta\square\beta^*)+(\alpha-\beta)\square(\alpha-\beta)^*\in\mathscr{H}^+(U).$$

(3.8) **Proposition.** *$r(\alpha\square\beta^*)\leqq\|\alpha\|_\infty\,\|\beta\|_\infty$ for every $\alpha,\beta\in U$, where r is the spectral radius.*

Proof. Suppose $\|a\|_\infty=\|\beta\|_\infty=1$ and denote by $V(\alpha\square\beta^*)\subset\mathbb{C}$ the numerical range of $\alpha\square\beta^*$, compare [1]. Then

$$\alpha\square\beta^*=\sum_{\varepsilon^4=1}\varepsilon\left(\frac{\alpha+\varepsilon\beta}{2}\right)\square\left(\frac{\alpha+\varepsilon\beta}{2}\right)^*$$

and (3.2) imply

$$V(\alpha\square\beta^*)\subset Q:=\{x+iy\in\mathbb{C}:\,|x|\leqq 1,\,|y|\leqq 1\}.$$

Replacing $\alpha\in U$ by $\lambda\alpha\in U$ for $\lambda\in\mathbb{C}$ with $|\lambda|=1$ we get

(3.9) $V(\alpha\square\beta^*)\subset\bigcap_{|\lambda|=1}\lambda Q=\{z\in\mathbb{C}:\,|z|\leqq 1\}$ and the result follows. Q.e.d.

For every $\alpha, \beta \in U$ the operator $\alpha \square \beta^* + \beta \square \alpha^* \in \mathscr{L}(U)$ is hermitian and (3.9) implies $\|(\alpha \square \beta^* + \beta \square \alpha^*)\| \leqq 2\|\alpha\|_\infty \|\beta\|_\infty$. Replacing α by $\lambda\alpha$ for $|\lambda|=1$ and taking convex combinations we derive

$$\|(\alpha \square \beta^* + \lambda \beta \square \alpha^*\| \leqq 2\|\alpha\|_\infty \|\beta\|_\infty \tag{3.10}$$

for every $\lambda \in \mathbb{C}$ with $|\lambda| \leqq 1$ and in particular

$$\|\alpha \square \beta^*\| \leqq 2\|\alpha\|_\infty \|\beta\|_\infty \quad \text{for every } \alpha, \beta \in U.$$

(3.11) *Problem.* Is $\|\alpha \square \beta^*\| \leqq \|\alpha\|_\infty \|\beta\|_\infty$ or equivalently $\|\{\alpha\beta\gamma\}\| \leqq \|\alpha\|_\infty \|\beta\|_\infty \|\gamma\|$ true for every $\alpha, \beta, \gamma \in U$?

(3.9) and (3.10) are strong evidence that (3.11) has a positive answer. In case that $\alpha \square \beta^*$ is hermitian (3.11) is also true as a consequence of (3.7).

§ 4. Positive J^*-Triples and Symmetric Domains

Let U be a fixed J^*-triple in the following. $V \subset \mathscr{Q}(U)$ denotes the corresponding Jordan family and $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$, $L, H, M = L/H$ are defined as in § 2. Consider the real Banach Lie algebra

$$\mathfrak{g} := \mathfrak{k} \oplus \mathfrak{p}$$

where

$$\mathfrak{p} := \left\{ (\alpha - \alpha^*) \frac{\partial}{\partial z} : \alpha \in U \right\} \subset \mathfrak{l}_{-1} \oplus \mathfrak{l}_1$$

and

$$\mathfrak{k} := \left\{ X \in i\mathscr{H}(U) \frac{\partial}{\partial z} : [X, \mathfrak{p}] \subset \mathfrak{p} \right\} \subset \mathfrak{l}_0.$$

Let $G \subset L$ be the subgroup generated by $\exp(\mathfrak{g})$. Then G is a connected real Banach Lie group and the canonical injection $G \hookrightarrow L$ is an analytic immersion. Put

$$P := \exp(\mathfrak{p}) \quad \text{and} \quad K := G \cap H. \tag{4.1}$$

From the implicit function theorem it follows that $0 \in U \subset M$ is an inner point of $P(0)$, i.e. the orbit

$$N := G(0) \subset M$$

is a domain in M. Using $iz \dfrac{\partial}{\partial z} \in \mathfrak{k}$ it follows $K \subset L_0 \cong \Gamma$.

$$D := \{a \in U : a \square a^* < 1\}$$

is a balanced domain in U with $\sigma(B(c, c^*))$ positive real for every $c \in D$. In particular $B_c := B(c, c^*)^{\frac{1}{2}} \in \Gamma \subset \mathrm{GL}(U)$ exists for every $c \in D$ and

$$g_c := t_c B_c t^*_{-c} \in P \subset G \tag{4.2}$$

satisfies $g_c(0) = c$, $g_c'(0) = B_c$, $g_{-c} = g_c^{-1}$.

(4.3) **Proposition.** (i) *For every* $c \in N = G(0)$ *there exists a unique involutory* $s_c \in G$, *such that* c *is an isolated fixed point of* s_c.

(ii) $c \mapsto s_c$ *induces an analytic mapping* $N \to G$.

(iii) D *is a connected component of* $U \cap N$.

Proof. (i) It is enough to consider the case $c=0$. Here the statement follows from $K \subset \mathrm{GL}(U)$.

(ii) $s_c = g_c s_0 g_{-c} = g_c^2 s_0$ for $c \in D$.

(iii) D is connected and contained in $U \cap N$. Suppose, there exists a boundary point $a \in \partial D$ with $a \in N$. Then there exists a connected neighborhood $A \subset U \cap N$ of a with

$$s_c(d) \in U \qquad \text{for all } c, d \in A.$$

Let $W := U_a$ be the subtriple generated by a. Then

$$s_c(d) = (1-(2d-c)\square c^*)^{-1}(2c-(1+c\square c^*)\,d)$$

and hence

$$(1-(2d-c)\square c^*)^2 \,|\, W = -s_c'(d)^{-1} \circ (1-c\square c^*)^2 \,|\, W \tag{4.4}$$

for all $c, d \in W$ near a. By analytic continuation (4.4) remains true for all $c, d \in A \cap W$. Choose $c, d \in W \cap A$ with $c \in D$ and

$$(2d-c)\square c^* = a \square a^*.$$

Then (4.4) implies $(1-a\square a^*)|W \in \mathrm{GL}(W)$, i.e. $a\square a^* | W < 1$ and hence by (3.2) also $a \square a^* < 1$, a contradiction. Q.e.d.

Let TN be the tangent bundle of the complex manifold N. We identify the tangent space at $0 \in N$ in a natural way with the complex Banach space U. Then the norm $\| \ \|$ of U can be extended to a G-invariant norm v: $TN \to \mathbb{R}$ (making N to a symmetric complex Banach manifold in the sense of [7; 8]) if and only if $K \subset \mathrm{GL}(U)$ consists of isometries of the complex Banach space U. From the definition of $\mathfrak{g}$ it follows, that the connected component K^0 of K leaves the norm of U invariant, i.e. G/K^0 is a covering of N and a symmetric complex Banach manifold. In particular, the universal covering $\tilde{N}$ of N is the simply connected symmetric complex Banach manifold associated with the J^*-triple U as considered in [7]. Since D is simply-connected, we may consider D as a domain in $\tilde{N}$, the norm v on the tangent bundle $TD \cong D \times U$ is given by

$$v(c, z) = \|B_c^{-1} z\| \tag{4.5}$$

as a consequence of (4.2).

Now suppose the J^*-triple U is positive and the seminorm $\| \ \|_\infty$ on U is defined as in § 3. Then we can also write

$$D = \{a \in U : \|a\|_\infty < 1\}.$$

In particular, D is balanced and convex.

(4.6) **Proposition.** *Suppose U is positive. Then D endowed with the norm* (4.5) *is the simply connected symmetric complex Banach manifold associated with the J^*-triple U. Every $g \in G$ is an isometric automorphism of D. Furthermore $s_a \neq s_c$ for all $a \neq c$ in D, and $G = KP = PK$, where $K := \{g \in G: g(0) = 0\} = \{g \in G: gs_0 = s_0 g\}$ and $P := \exp(\mathfrak{p}) = \{g^2: g \in G$ and $gs_0 = s_0 g^{-1}\}$. For every $c \in D$, $g_c = t_c B_c t^*_{-c}$ is the unique element of P mapping 0 to c.*

Proof. For every $c, z \in D$ we have $\sigma(z \square c^*) \subset \Delta$ by (3.8) and hence $\sigma(B(z, -c^*)) \subset (1-\Delta)(1-\Delta) \subset \mathbb{C}^*$ by (3.4). Therefore $t^*_{-c}(D) \subset U$ and also $g_c(D) \subset U$ for all $c \in D$. In particular, $g_c(D)$ is a connected subset of $U \cap N$ with $c \in D \cap g_c(D)$, i.e. $g_c(D) \subset D$ by (4.3) and hence $G(D) = D$. The mapping $U \to D$ given by $\alpha \mapsto \exp\left((\alpha - \alpha^*)\frac{\partial}{\partial z}\right)0$ is a real-analytic diffeomorphism with inverse $c \mapsto \frac{\tanh^{-1}\sqrt{c \square c^*}}{\sqrt{c \square c^*}}c$. Therefore $s_0 = s_c$ only for $c = 0$ and consequently $P(0) = D$, i.e. $G = PK = KP$. Now suppose $g \in G$ satisfies $gs_0 = s_0 g^{-1}$. Then for $c := g(0)$ we have $g_{-c}g \in K$ and therefore $g_{-c}g = s_0 g_{-c} g s_0 = g_c g^{-1}$, i.e. $g^2 = g_c^2 \in P$. Q.e.d.

In the following let M be an arbitrary symmetric complex Banach manifold as defined in [7; 8]. Then in particular M is a complex manifold with a fixed norm $v = v_M$ on the tangent bundle TM. Denote by c_M: $TM \to \mathbb{R}$ the *Carathéodory semi-norm* on M defined by

$$c_M := \sup \omega \circ Tf,$$

where ω: $T\Delta \to \mathbb{R}$ is the usual invariant Poincaré metric on the unit disc $\Delta \subset \mathbb{C}$ and the sup is taken over all holomorphic mappings f: $M \to \Delta$. c_M is lower semi-continuous and invariant under all biholomorphic automorphisms of M.

Now fix a point $a \in M$ (called *base point*). Then the tangent space $U := T_a$ at a is a complex Banach space, a J^*-triple in a natural way (compare [7; 8]) and we have

(4.7) **Proposition.** *For every symmetric complex Banach manifold M with base point a and corresponding J^*-triple U the following conditions are equivalent*

(i) *The bounded holomorphic functions separate the points of M.*

(ii) *c_M is a norm on every tangent space T_x, $x \in M$.*

(iii) *U is positive and $\| \ \|_\infty$ is a norm on U.*

In either case the restriction of c_M to $U = T_a$ coincides with $\| \ \|_\infty$ and there exists an isometric biholomorphism of M onto the balanced, convex domain

$$D := \{z \in U: \|z\|_\infty < 1\}$$

with norm given by (4.5).

Proof. Suppose (iii) is not true. Then

$$\tilde{D} := \{z \in U: z \square z^* < 1\}$$

can be considered as a domain in the universal covering $\tilde{M}$ of M. There exists a negative subtriple $W \subset U$ with $W \neq 0$ - compare (3.6). $W \subset \tilde{D}$ and Liouville then imply that also (i) and (ii) cannot be true. Now suppose that (iii) is true. Then (4.6) applies and in particular D can be considered as the universal covering of M. By [8, II, (1.9)] the corresponding covering map $D \to M$ is a biholomorphic isometry. The theorem of Hahn-Banach implies that the restriction of c_D to the tangent space U at the origin coincides with $\| \ \|_\infty$, i.e. (ii) is true since D is homogeneous. Again by Hahn-Banach the points of D can be separated by continuous linear functions on U that are bounded on D. Q.e.d.

(4.8) **Proposition.** *With the notations of* (4.7) *also the following conditions are equivalent*

(i) *M is biholomorphically equivalent to a bounded domain.*

(ii) *U is positive and $\| \ \|_\infty$ is an equivalent norm on U. In either case U is also a J^*-triple with respect to the norm $\| \ \|_\infty$.*

Proof. For every bounded domain $B \subset U$ the restriction of c_B to every tangent space $T_b \approx U$, $b \in B$, is an equivalent norm. With this the statement follows immediately from (4.7). As a corollary we get

(4.9) **Theorem.** *For every bounded symmetric domain D in a complex Banach space and every $a \in D$ there exists a complex Banach space U and a biholomorphic mapping φ from D onto the open unit ball of U with $\varphi(a) = 0$. U and φ are uniquely determined up to a linear isometry.*

(4.10) **Corollary.** *Let U be a complex Banach space and $D \subset U$ a homogeneous bounded, circular* (i.e. *$\lambda D = D$ for $\lambda \in \mathbb{C}$ with $|\lambda| = 1$*) *domain with $0 \in D$. Then D is convex.*

Proof. D is symmetric. Therefore there is a complex Banach space with open unit ball B and a biholomorphic mapping $\varphi: D \to B$ with $\varphi(0) = 0$. Then φ is linear by Cartan's uniqueness theorem.

§ 5. JB^*-Triples

By the Gelfand-Neumark theorem C^*-algebras can be characterized abstractly as B^*-algebras, that is by complex Banach star algebras satisfying

$$\|zz^*\| = \|z\|^2.$$

As an analogue of this C^*-condition let us consider the condition

$$\|z \square z^*\| = \|z\|^2 \tag{5.1}$$

for J^*-triples. It is obvious that for every J^*-triple U satisfying (5.1) for every $z \in U$ the same is true for the dual triple ${}^-U$ (obtained from the J^*-triple U by multiplying the triple product with the factor -1 - compare [8]). On the other hand every C^*-algebra A becomes a positive J^*-triple satisfying (5.1)

if the triple product is defined by

$$\{\alpha\beta^*\gamma\} = 1/2(\alpha\beta^*\gamma + \gamma\beta^*\alpha). \tag{5.2}$$

(5.3) **Proposition.** *For every J^*-triple U the following conditions are equivalent*

(i) *U is positive and $\|\alpha\square\alpha^*\| = \|\alpha\|^2$ for all $\alpha\in U$.*

(ii) *U is positive and $\|\{\alpha\alpha^*\alpha\}\| = \|\alpha\|^3$ for all $\alpha\in U$.*

(iii) *For every $\alpha\in U$ there exists an isometric J^*-isomorphism from U_α onto a commutative C^*-algebra.*

Proof. (i)$\Rightarrow$(iii). Suppose (i) is true. Then $\|\ \| = \|\ \|_\infty$ and by §4 U is the J^*-triple of a bounded symmetric domain. By (1.15) there exists a locally compact space S and a J^*-isomorphism $\varphi\colon U_\alpha \to \mathscr{C}_0(S)$ for given $\alpha\in U$. For every $\beta\in U_\alpha$ and $b := \varphi(\beta)$ the hermitian operators $\beta\square\beta^*|U_\alpha$ and $b\square b^*$ have the same spectrum and hence the same norm. Using (3.5) we derive $\|\beta\|^2 = \|\beta\square\beta^*\| = \|b\square b^*\| = \|b\|^2$, i.e. φ is an isometry. (iii)$\Rightarrow$(ii) is trivial, (ii)$\Rightarrow$(iii) follows from (1.11), and (iii)$\Rightarrow$(i) is a consequence of (3.2) and (3.5). Q.e.d.

We call a J^*-triple U a *JB^*-triple*[3] if one of the equivalent conditions in (5.3) is satisfied. It is clear that every closed subtriple V of a JB^*-triple U again is a JB^*-triple and also U/V, if V is an ideal in addition. For every family $(U_i)_{i\in I}$ of JB^*-triples also the ∞-sum (compare [8, p. 464])

$$U := \bigoplus_{i\in I}^{\infty}$$

is a JB^*-triple. On a JB^*-triple U the norm coincides with $\|\ \|_\infty$. By §4 therefore the open unit ball $D = \{z\in U\colon \|z\| < 1\}$ is a bounded symmetric domain and has U as J^*-triple. On the other hand if M is a bounded symmetric domain with base point a then the tangent space $U = T_a$ is a JB^*-triple in the Carathéodory norm, i.e.

(5.4) **Theorem.** *The category of all bounded symmetric domains with base point is equivalent to the category of JB^*-triples.*

Suppose U, $\tilde{U}$ are JB^*-triples with open unit balls D, $\tilde{D}$ and $g\in\mathscr{L}(U,\tilde{U})$ is a J^*-morphism. Then $g(D)\subset\tilde{D}$ and $\|g\|\leqq 1$. In particular every J^*-isomorphism $U\to\tilde{U}$ is an isometry. Suppose on the other hand, that $g\colon U\to\tilde{U}$ is a linear, surjective isometry. Then $g\colon D\to\tilde{D}$ is biholomorphic and by Cartan's uniqueness theorem $s_{gc} = g s_c g^{-1}$ holds for all $c\in D$, i.e. g is a J^*-isomorphism. Specializing $\tilde{U} = U$ we get - compare also [7, (5.4) and (5.6)]:

(5.5) **Proposition.** *Let U be a JB^*-triple. Then $g\in\mathrm{GL}(U)$ is an isometry of U if and only if g is a J^*-isomorphism of U,* i.e. *if*

(i) *$g\{xy^*z\} = \{(gx)(gy)^*(gz)\}$ for all $x, y, z\in U$. $\lambda\in\mathscr{L}(U)$ is a hermitian operator on U if and only if $i\lambda$ is a J^*-derivation of U,* i.e. *if*

(ii) *$\lambda\{xy^*z\} = \{(\lambda x)y^*z\} - \{x(\lambda y)^*z\} + \{xy^*(\lambda z)\}$ for all $x, y, z\in U$.*

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[3] Called C^*-triple in [7]

By (5.5.i) the group $\mathrm{GL}(D)$ of all linear surjective isometries of U is a real Banach Lie group in the induced topology from $\mathrm{GL}(U)$ [5] and $i\mathscr{H}(U)$ is the corresponding Lie algebra. $\mathrm{GL}(D)$ is also contained in the structure group

$$\Gamma=\{g\in\mathrm{GL}(U)\colon g\circ U^*\circ g^{-1}=U^*\},$$

which is a complex Banach Lie group in the topology induced from $\mathrm{GL}(U)$ [5]. For every $g\in\Gamma$ there exists a unique $g^*\in\Gamma$ with

$$(g^*a)^*=g^{-1}\circ a^*\circ g \quad \text{for all } a\in U.$$

For $h:=(g^{-1})^*=(g^*)^{-1}$ and all $a, x, y, w, z\in U$ therefore

$$g\{xa^*y\}=\{(gx)(ha)^*(gx)\}$$

holds, and twice application of (1.2) gives

$$\begin{aligned}\{(gz)(h\{xa^*y\})^*(gw)\}&=g\{z\{xa^*y\}^*w\}\\&=\{(gz\{(hx)(ga)^*(hy)\}^*(gw)\},\end{aligned}$$

i.e.
$$\{(hx)((h^{-1})^*a)^*(hy)\}=\{(hx)(ga)^*(hy)\}$$

i.e.
$$(h^{-1})^*=g \quad \text{and hence } g^{**}=g.$$

Summing up we get

(5.6) **Proposition.** *Let U be a JB^*-triple. Then the structure group Γ of U is a complex Banach Lie group in the induced topology from* $\mathrm{GL}(U)$ *with Lie algebra $\mathscr{J}(U)\cdot g\mapsto g^*$ induces an antiholomorphic involution of Γ with*

$$\mathrm{GL}(D)=\{g\in\Gamma\colon g^*g=1\}.$$

Now let for $V=U^*\subset\mathscr{Q}(U)$ the Lie algebras $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}\subset\mathfrak{l}=\mathfrak{l}_{-1}\oplus\mathfrak{l}_0\oplus\mathfrak{l}_1$ and the Lie groups $K\subset H$, $G\subset L\approx\mathrm{Aut}(\mathfrak{l})$ be as in § 2. Then by (5.5) and (5.6)

$$\mathfrak{k}=i\mathscr{H}(U)\frac{\partial}{\partial z},\qquad \mathfrak{l}_0=\mathscr{J}(U)\frac{\partial}{\partial z}$$

and therefore $\mathfrak{l}=\mathfrak{g}\oplus i\mathfrak{g}$ is true. In particular there is a unique continuous and antilinear involution $X\mapsto X^*$ of $\mathfrak{l}$ with

$$\mathfrak{g}=\{X\in\mathfrak{l}\colon X^*=-X\}.$$

Obviously $\mathfrak{l}_\nu^*=\mathfrak{l}_{-\nu}$ for all ν, i.e. $\mathfrak{l}$ is a symmetric Lie algebra in the sense of [10]. The corresponding antiholomorphic involution on L is given by

$$\mathrm{Ad}(g^*)X=(\mathrm{ad}(g^{-1})X^*)^*$$

for all $X\in\mathfrak{l}$, $g\in L$. Then

(5.7) $$\hat{G}:=\{g\in L\colon g^*g=1\}=\{g\in L\colon \mathrm{Ad}(g)\mathfrak{g}=\mathfrak{g}\}$$

is an algebraic real subgroup of $L\approx\mathrm{Aut}(\mathfrak{l})\subset\mathrm{GL}(\mathfrak{l})$ with identity component G (= the subgroup generated by $\exp(\mathfrak{g})$).

Denote by $\mathrm{Aut}(D)$ the group of all biholomorphic automorphisms of D. As a consequence of Cartan's uniqueness theorem $\mathrm{GL}(D)$ is the isotropy subgroup at the origin $0 \in D$. On the other hand we know that $P = \exp(\mathfrak{p}) = \{g_c : c \in D\}$ is transitive on D, i.e.

$$\begin{aligned} \mathrm{Aut}(D) &= P \cdot \mathrm{GL}(D) \\ &= \{g \in \hat{G} : g(0) \in D\} \end{aligned} \tag{5.8}$$

is an open subgroup of the algebraic real group $\hat{G}$ and has G as identity component and $\mathfrak{g}$ as Lie algebra.

JB^*-triples generalize JB^*-algebras (=Jordan C^*-algebras, compare [21; 3; 22]). By definition a JB^*-algebra is a complex Jordan algebra A with unit element e, conjugate linear involution $*$ and complete norm such that

(i) $\|e\| = 1$, (ii) $\|zw\| \leqq \|z\| \cdot \|w\|$ and

(iii) $\|\{zz^*z\}\| = \|z\|^3$ holds for every $z, w \in A$, where

$$\{xy^*z\} := (xy^*)z + (zy^*)x - (zx)y^*$$

for every $x, y, z \in A$. With this as triple product, every JB^*-algebra is a JB^*-triple. On the other hand for every JB^*-triple U and every unitary element $e \in U$ (i.e. $e \square e^* = 1$) a JB^*-algebra with unit e is obtained if product and involution are defined by

$$zw := \{ze^*w\} \quad \text{and} \quad z^* := \{ez^*e\}.$$

In this sense JB^*-algebras are precisely the JB^*-triples with distinguished unitary element.

JB^*-triples are just the positive J^*-triples U with $\| \ \| = \| \ \|_\infty$. Therefore (3.11) reads here

(5.9) *Problem.* Is $\|x \square y^*\| \leqq \|x\| \cdot \|y\|$ or equivalently $\|\{xy^*z\}\| \leqq \|x\| \cdot \|y\| \cdot \|z\|$ true in every JB^*-triple U?

From a result of Wright [21] it follows, that the inequalities in (5.9) hold in case U is J^*-isomorphic to a subtriple of a JB^*-algebra.

(5.10) *Problem.* Is every JB^*-triple J^*-isomorphic to a subtriple of a JB^*-algebra?

§ 6. An Illustrating Example

The following example for a JB^*-triple is easily accessable for two reasons: (i) The Jordan triple product is obtained from an associative product, and (ii) the elements are operators on Hilbert spaces:

Let $p, q \geqq 1$ be arbitrary cardinal numbers and E, F complex Hilbert spaces of dimensions p, q. Then $U := \mathscr{L}(E, F)$ is a JB^*-triple in the triple product

$$\{xy^*z\} = \tfrac{1}{2}(xy^*z + zy^*x).$$

Contrary to the abstract case, for every $x, y \in U$ the operators $y^* x \in \mathscr{L}(E)$ and $x y^* \in \mathscr{L}(F)$ have a meaning by themselves and

$$\sigma(x \square y^*) \subset \tfrac{1}{2}(\sigma(x y^*) + \sigma(y^* x)),$$

which is essentially (3.3), since for the smallest $x \square y^*$-invariant, closed linear subspace X of U containing x

$$\sigma(x \square y^* \mid X) \cup \{0\} = \sigma(x y^*) \cup \{0\} = \sigma(y^* x) \cup \{0\}$$

is true. The Bergman operator is given by

$$B(x, y)\, z = (1 - x y^*)\, z (1 - y^* z). \tag{6.1}$$

The open unit ball D of U is homogeneous. For every $c \in D$ the automorphism $g_c = t_c B_c t^*_{-c} \in \mathrm{Aut}(D)$ has the explicit form

$$\begin{aligned} g_c(z) &= (1 - c c^*)^{1/2} (1 + z c^*)^{-1}\, z (1 - c^* c)^{1/2} + c \\ &= (1 - c c^*)^{-1/2} (z + c)\, (1 + c^* z)^{-1} (1 - c^* c)^{1/2}. \end{aligned} \tag{6.2}$$

This transformation plays an important role in the work of Harris [4]. In the case of matrix domains in finite dimension g_c is already contained in [16].

The group $\mathrm{GL}(D)$ of isometries of $U = \mathscr{L}(E, F)$ has at most two connected components. The identity component is given by [9, Satz 4]:

$$\mathrm{GL}(D)^0 = \{[z \mapsto a z d] \in \mathrm{GL}(U) \colon a \in \mathscr{L}(F),\ d \in \mathscr{L}(E) \text{ unitary}\}. \tag{6.3}$$

Consequently

$$\mathscr{J}(U) = \{[z \mapsto a z + z d] \in \mathscr{L}(U) \colon a \in \mathscr{L}(F),\ d \in \mathscr{L}(E)\} \tag{6.4}$$

with involution given by

$$[z \mapsto a z + z d]^* = [z \mapsto a^* z + z d^*].$$

The complex Lie algebra $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$ therefore can be realized in the following way: Put $B = F \oplus E$ as orthogonal sum and realize every $\lambda \in \mathscr{L}(B)$ in the obvious way as an operator matrix

$$\lambda = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with $a \in \mathscr{L}(F)$, $b \in \mathscr{L}(E, F)$, $c \in \mathscr{L}(F, E)$, $d \in \mathscr{L}(E)$. Then $\lambda \mapsto (b + (a z - z d) - z c z) \dfrac{\partial}{\partial z}$ induces an isomorphism of complex Banach Lie algebras

$$\mathscr{L}(B)/\text{center} \approx \mathfrak{l}.$$

The projective linear group

$$\mathrm{PGL}(B) := \mathrm{GL}(B)/\text{center}$$

acts on $\mathscr{L}(B)$ by inner automorphisms and induces a biholomorphic isomorphism of complex Lie groups

$$\mathrm{PGL}(B) \approx \mathrm{Aut}(\mathrm{I})^0 \approx L^0.$$

In this sense we can represent every $g \in L^0$ by a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(B)$ and the action of g on M is given by the *Möbius transformation*

$$g(z) = (az+b)(cz+d)^{-1}$$

for all $z \in U \cap g^{-1}(U) \subset M$. The full group L can be described as follows: Fix a conjugation j on B, i.e. a conjugate-linear involutary isometry of B. Then $[\lambda \mapsto j\lambda^* j] \in \mathrm{GL}(\mathscr{L}(B))$ induces a *transposition* $\tau = \tau_j \in L \approx \mathrm{Aut}(\mathrm{I})$ and

$$L = L^0 \cup \tau L^0.$$

τ is not contained in L^0 (except for the case $p=q=1$, where L is connected). The complex manifold M and the holomorphic action of L on M can be interpreted geometrically as follows: Denote by $\mathbb{G} = \mathbb{G}(B)$ the Graßmann manifold of all closed linear subspaces of B. Then, for instance,

$$\mathbb{G}_{p,q} := \{A \in \mathbb{G} \colon \dim A = p,\ \mathrm{codim}\, A = q\}$$

is the connected component of $\mathbb{G}$ containing the element $E \in \mathbb{G}$. The natural action of $L^0 \approx \mathrm{PGL}(B)$ on $\mathbb{G}$ extends to a holomorphic action of L on $\mathbb{G}$ if we put

$$\tau_j(A) := j(A^\perp) = (jA)^\perp$$

for every transposition $\tau_j \in L$ and $A \in \mathbb{G}$. If we identify $g(0) \in M$ with $g(E) \in \mathbb{G}$ for every $g \in L$ we may write

$$M = \mathbb{G}_{p,q} \cup \mathbb{G}_{q,p}.$$

Every $z \in U = \mathscr{L}(E,F) \subset M$ in this sense is identified with the corresponding graph

$$\{(x+zx) \colon x \in E\} \in \mathbb{G}_{p,q} \subset M.$$

The group $\mathrm{Aut}(D)$ has always index 2 in the algebraic group $\hat{G}$:

$$\hat{G} = \mathrm{Aut}(D) \cup \tau\, \mathrm{Aut}(D)$$

for every transposition $\tau = \tau_j$ with $j(E) = E$, $j(F) = F$, and every such τ maps D onto the domain

$$D^\perp := \{z^\perp \colon z \in D\} \subset \mathbb{G}_{q,p} \subset M.$$

We distinguish three cases:

$p=q=1$. $L, M, \mathrm{Aut}(D)$ and $\mathrm{GL}(D)$ are connected. $M = \mathbb{G}_{1,1}$ is the Riemann sphere, D is the lower and $D^\perp$ the upper hemisphere in M. $\hat{G} \cong \mathrm{PGL}(2,\mathbb{R})$ and $\mathrm{Aut}(D) \cong \mathrm{PSL}(2,\mathbb{R})$.

$p=q>1$. $M=\mathbb{G}_{p,p}$ is connected. GL(D), Aut(D) and the structure group $\Gamma\cong L_0$ have precisely two connected components. Each of these groups is generated by its identity component and a transposition τ_j with $j(E)=F$.

$p\neq q$. M has two connected components. The groups GL(D), Aut(D) and $\Gamma\cong L_0$ are connected.

Now consider the dual triple ${}^-U$, i.e. U endowed with the triple product changed by sign

$$\{xy^*z\}=-\tfrac{1}{2}(xy^*z+zy^*x).$$

Then (5.1) remains true for all z but ${}^-U$ is negative. The corresponding Jordan family $V\approx\mathscr{L}(F,E)$, the Lie algebras $\mathfrak{k},\mathfrak{l}$, the groups H,L and the complex manifold $M=\mathbb{G}_{p,q}\cup\mathbb{G}_{q,p}$ remain unchanged. Changed are the Lie algebra $\mathfrak{g}$ and the Lie groups G, $\hat{G}$. The involution of $\mathfrak{l}=\mathscr{L}(B)$/center is now induced by the usual adjoint of $\mathscr{L}(B)$. In particular, the algebraic real group $\hat{G}$ acts transitively on M. Consider the connected complex mainifold $N:=\mathbb{G}_{p,q}$. Then by [9]

$$\mathrm{Aut}(N)=\{g\in L\colon g(N)=N\}\supset L^0$$

is the group of all biholomorphic automorphisms of N. The group G has index 2 in $\hat{G}$ (except in the case $p=q=1$) and N is a symmetric complex Banach manifold with J^*-triple ${}^-U$ such that $\hat{G}\cap\mathrm{Aut}(N)$ is the group of all biholomorphic isometries of N.

Suppose $r:=\min(p,q)$ is not finite. Then there is a closed linear subspace $A\subsetneqq E$ with $A\in N$. It is easily verified, that such an A is not contained in $L_1(U)\subset N$, i.e. N cannot be obtained in the form $U\oplus V/\sim$ as in finite dimensions - compare [14].

References

1. Bonsall, F.F., Duncan, J.: Numerical ranges of operators on normed spaces and of elements of normed algebras. Cambridge: Cambridge University Press 1971
2. Braun, H., Koecher, M.: Jordan-Algebren. Berlin-Heidelberg-New York: Springer 1966
3. Braun, R., Kaup, W., Upmeier, H.: A holomorphic characterization of Jordan C^*-algebras. Math. Z. **161**, 277–290 (1978)
4. Harris, L.A.: Bounded symmetric homogeneous domains in infinite dimensional spaces. In: Proceedings on infinite dimensional Holomorphy (Kentucky 1973); pp. 13–40. Lecture Notes in Math. **364**. Berlin-Heidelberg-New York: Springer 1974
5. Harris, L.A., Kaup, W.: Linear algebraic groups in infinite dimensions. Illinois J. Math. **21**, 666–674 (1977)
6. Jacobson, N.: Structure and representations of Jordan algebras. Amer. Math. Soc. Colloq. Publ. **39**. Providence: Amer. Math. Soc. 1968
7. Kaup, W.: Algebraic characterization of symmetric complex manifolds. Math. Ann. **228**, 39–64 (1977)
8. Kaup, W.: Über die Klassifikation der symmetrischen hermiteschen Mannigfaltigkeiten unendlicher Dimension I, Math. Ann. **257**, 463–483 (1981); II, Math. Ann. **262**, 57–75 (1983)
9. Kaup, W.: Über die Automorphismen Graßmannscher Mannigfaltigkeiten unendlicher Dimension. Math. Z. **144**, 75–96 (1975)
10. Koecher, M.: An elementary approach to bounded symmetric domains. In: Proceedings of a Conference on Complex Analysis (Houston 1969). Houston: Rice University 1969

11. Koecher, M.: Gruppen und Lie-Algebren von rationalen Funktionen. Math. Z. **109**, 349–392 (1969)
12. Loos, O.: Bounded symmetric domains and Jordan pairs. Mathematical Lectures. Irvine: University of California at Irvine 1977
13. Loos, O.: Jordan pairs. Lecture notes in Math. **460**. Berlin-Heidelberg-New York: Springer 1975
14. Loos, O.: Homogeneous algebraic varieties defined by Jordan pairs. Monatsh. Math. **86**, 107–129 (1978)
15. Moreno, J.M.: JV-algebras. Math. Proc. Cambridge Philos. Soc. **87**, 47–50 (1980)
16. Potapov, V.P.: The multiplicative structure of J-contractive matrix-functions. Amer. Math. Soc. Transl. **15**, 131–243 (1960)
17. Upmeier, H.: Über die Automorphismengruppen von Banachmannigfaltigkeiten mit invarianter Metrik. Math. Ann. **223**, 279–288 (1976)
18. Vigué, J.P.: Le groupe des automorphismes analytiques d'un domaine borné d'un espace de Banach complexe. Ann. Sci. École Norm. Sup. (4) **9**, 203–282 (1976)
19. Vigué, J.P.: Sur la convexité des domaines bornés cerclés homogènes. Seminaire Lelong-Skoda. Lecture Notes in Math. **822**, pp. 317–331. Berlin-Heidelberg-New York: Springer 1980
20. Wolf, J.: Fine structure of hermitian symmetric spaces. Symmetric spaces, pp. 271–357. New York: Decker 1972
21. Wright, J.D.M.: Jordan C^*-algebras. Michigan Math. J. **24**, 291–302 (1977)
22. Youngson, M.A.: Non-unital Banach Jordan algebras and C^*-triple systems. Proc. Edinburgh Math. Soc. **24**, 19–29 (1979)

Added in proof:

23. Friedman, Y., Russo, B.: Function Representation of Commutative Operator Triple Systems. Preprint

Received September 10, 1982; received in final form March 4, 1983