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The Unstable Decomposition of $\Omega^2\Sigma^2 X$ and Its Applications

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§1. Introduction

Suppose that a space Y stably splits into a bouquet as $A \vee B$. One might ask whether this splitting is realized after some finite number of suspensions and if so, what is the minimum number of suspensions which realizes this decomposition? This question is of particular interest for $\Omega^n\Sigma^n X$. In this paper we obtain information and resulting applications in case $n=2$. At the request of the editor we include some of our additional motivation for this question. For convenience we restrict attention to spaces having the homotopy type of a CW complex.

A theorem of V. Snaith gives that $\Omega^n\Sigma^n X$ stably splits into a certain bouquet $\bigvee_{k\geq 0} D_k(R^n, X)$ if X is path-connected [26, 8]. We proved in [8] that if $n<\infty$ and k is fixed, there are choices of maps $\Sigma^L\Omega^n\Sigma^n X \to \Sigma^L D_k(R^n, X)$ for $L<\infty$ which induce Snaith's stable decomposition. A short elementary proof is given in an appendix here. Specializing the question in the first paragraph to $\Omega^n\Sigma^n X$, we would like to find minimal values for L.

We conjecture that L is the embedding dimension of a certain manifold $B(R^n, k)$ defined below if X runs over all spheres. In case $k=2$, $B(R^n, 2)$ is homeomorphic to $\mathbb{R}P^{n-1}\times\mathbb{R}^{n+1}$ and this conjecture has been verified in [6] using the methods of Sect. 6 here provided $n=2,4,8$, or $1+2^a$. In general, $B(R^n, k)$ is the space of unordered k-tuples of distinct points in $\mathbb{R}^n$ [14]. The point of all this is that desuspension questions for $\Omega^n\Sigma^n X$ are intimately related to natural embedding questions.

In case all spaces are localized at an odd prime p, then the minimum values for L seem different. In particular, $D_p(R^2, S^{2n-1})$ splits off $\Omega^2 S^{2n+1}$ in three (possibly two) suspensions. An immediate corollary is P. Selick's theorem that p annihilates the p-primary component of $\pi_* S^3$ [25]. We consider the following more general statement: There is a natural map $\tilde{\theta}_n$: $B(R^n, P)\to\Omega^n S^n$ which

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extends to a map $\bar{\theta}_n$: $\Omega^n\Sigma^n B(R^n, P)\to\Omega^n S^n$ and which, if S^n is an H-space, admits a further extension θ_n: $\Omega^{n+1}\Sigma^{n+1}B(R^n, P)\to\Omega^n S^n$. We conjecture that $\bar{\theta}_n$ (and θ_n) induce epimorphisms on the p-primary component of homotopy groups in case n is odd. Observe that this conjecture is correct for θ_1 and θ_∞. Furthermore, this conjecture would provide an analogue of the Kahn-Priddy theorem for each odd dimensional sphere. If θ_n induces an epimorphism on the p-primary components, then the known result that p^n annihilates the p-primary component of $\pi_* S^{2n+1}$, $p>2$, follows directly. All of these follow from appropriate desuspensions. Rather than elaborate on these points here, we consider desuspension questions which have been useful for computations.

Let W_n denote the homotopy theoretic fibre of the double suspension E^2: $S^{2n-1}\to\Omega^2 S^{2n+1}$. In M. Mahowald's papers [19–21] relations amongst the W_n are studied. Odd primary analogues are given in work of J. Harper and H. Miller [15]. In particular, a secondary analogue of the double suspension is studied.

In this paper we give a geometric construction of the secondary suspension (localized at a prime p)

$$\sigma_n\colon W_n\to\Omega^{2p}W_{n+1}$$

which is degree one on the bottom cell. We show that this map may be obtained by destabilizing V. Snaith's stable decomposition for $\Omega^n\Sigma^n X$ [26]; modifications for the case $p=2$ are required. The destabilization results obtained here are best possible (when localized at 2). An immediate corollary is a mild improvement of James' result on exponents of spheres at 2 [29]. Finally, some useful observations concerning homological properties of maps f: $\Omega^2 S^{2n+1}\to Z$ where Z is any "reasonable" space are included.

Theorem 1.1 (Gauss). *If X is path-connected, there exist maps*

$$\tilde{h}_k\colon \Sigma^{2k}\Omega^2\Sigma^2 X\to\Sigma^{2k}D_k(R^2, X)$$

which induce Snaith's stable decomposition of $\Omega^2\Sigma^2 X$.

In a sense, the maps $\tilde{h}_k$ are best possible. The following proposition will be proven in [6].

Proposition 1.2. *If X is S^{2n+1}, k is a power of 2, and f: $\Sigma^L\Omega^2\Sigma^2 X\to\Sigma^L D_k(R^2, X)$ gives an epimorphism in homology, then $L\geqq 2k$. Hence the maps $\tilde{h}_k$ of 1.1 do not desuspend whenever k is a power of 2.*

Localize all spaces at a fixed prime p. The maps of 1.1 (with a modification in case $p=2$) give a secondary analogue of the double suspension.

Proposition 1.3. *There exists a map*

$$\sigma_n\colon W_n\to\Omega^{2p}W_{n+1}$$

which is degree one on the bottom cell. If $n\geqq p$, then σ_n is an equivalence through dimension $2np^2-4$. If $p>2$, then σ_n may be chosen to be an H-map.

Let $P^j(r)$ be $S^{j-1}\cup_r e^j$. Consider the mapping telescope X_n given by

$$W_n\xrightarrow{\sigma_n}\Omega^{2p}W_{n+1}\xrightarrow{\Omega^{2p}\sigma_{n+1}}\Omega^{4p}W_{n+2}\to\dots\,.$$

Proposition 1.4. *X_n is homotopy equivalent to $\Omega^\infty\Sigma^\infty P^{2np-2}(p)$.*

We remark that the odd primary analogues of 1.3 and 1.4 can also be given using the methods of [9, 10, 25].

The maps $\tilde{h}_k$ have additional external structure; the following proposition will be proven in [4].

Proposition 1.5. *$\prod_{k=0}^{\infty}\Omega^{2k}\Sigma^{2k}D_k(R^2,X)$ may be given the structure of a two-fold loop space such that*

$$H\colon \Omega^2\Sigma^2 X\to\prod_{k=0}^{\infty}\Omega^{2k}\Sigma^{2k}D_k(R^2,X)$$

is a two-fold loop map where H is given by the diagonal composed with the product of the adjoints of the $\tilde{h}_k$.

We include a remark here. It would be quite interesting to iterate the procedure in 1.3. In particular, denote the homotopy theoretic fibre of σ_n by Y_n. Is there a map $Y_n\to\Omega^{2p^2}Y_{n+2}$ which is degree one on the bottom cell? Iterations of these sorts of maps would be of computational use.

§ 2. Statements 1.1, 1.3, and 1.4

By the result of [8] or by the analogues in the appendix here, there exist maps

$$\Sigma^L\Omega^n\Sigma^n X\to\Sigma^L D_k(R^n,X)$$

inducing the stable decomposition of $\Omega^n\Sigma^n X$ where L is the embedding dimension of $B(R^n,k)$. In case $n=2$, L has been computed by Gauss: if $\mathbb{C}$ denotes the complex numbers, then the k-fold symmetric product $\mathbb{C}^k/\Sigma_k$ is homeomorphic to $\mathbb{C}^k$. (An explicit proof may be found in the appendix to [13].) Hence $B(\mathbb{R}^2,k)$ is homeomorphic to an open submanifold of $\mathbb{C}^k$ and 1.1 follows.

To prove 1.3, observe that 1.1 provides a map

$$\tilde{h}_p\colon \Sigma^{2p}\Omega^2\Sigma^2 X\to\Sigma^{2p}D_p(R^2,X)$$

which gives an epimorphism in homology. Specialize to $X=S^{2n-1}$ and recall the calculations of [5, p. 225] to see that $\Sigma^{2p}D_p(R^2,S^{2n-1})$ localized at p is $P^{2np+2p-1}(p)$. Hence the adjoint of $\tilde{h}_p$ gives a map

$$h_p\colon \Omega^2 S^{2n+1}\to\Omega^{2p}\Sigma^{2p}P^{2np-1}(p)$$

such that h_p induces an isomorphism on $H_{2np-2}(\ ;Z/pZ)$. We use h_p to give a map

$$g_n\colon \Omega^2 S^{2n+1}\to\Omega^{2p-1}W_{n+1}.$$

To construct g_n, localize all spaces at a fixed odd prime p and postpone the 2-primary analogues to Sect. 3. Adams [1] has shown that S^{2n-1} may be given the structure of an H-space such that $E^2\colon S^{2n-1}\to\Omega^2 S^{2n+1}$ is an H-map. Hence W_{n+1} is a p-local H-space. Notice that W_{n+1} is $(2np+2p-4)$-connected and $\pi_{2np+2p-3} W_{n+1} \cong Z/pZ$. Let $\iota: P^{2np+2p-2}(p) \to W_{n+1}$ be the generator of $\pi_{2np+2p-3} W_{n+1}$ extended over the Moore space. Since W_{n+1} is an H-space, ι may be extended over $\Omega \Sigma P^{2np+2p-2}(p)$ to give

$$j: \Omega P^{2np+2p-1}(p) \to W_{n+1}.$$

Set $g_n = \Omega^{2p-1} j \circ h_p$.

Since $g_n \circ E^2$ is null-homotopic, there is an induced morphism of fibration sequences

$$\begin{array}{ccc} W_n & \xrightarrow{\sigma_n} & \Omega^{2p} W_{n+1} \\ \downarrow & & \downarrow \\ S^{2n-1} & \longrightarrow & * \\ \downarrow & & \downarrow \\ \Omega^2 S^{2n+1} & \xrightarrow{g_n} & \Omega^{2p-1} W_{n+1}. \end{array}$$

Next, consider the induced map of fibration sequences

$$\begin{array}{ccc} \Omega S^{2n-1} & \longrightarrow & * \\ \downarrow & & \downarrow \\ \Omega^3 S^{2n+1} & \xrightarrow{\Omega g_n} & \Omega^{2p} W_{n+1} \\ \downarrow & & \downarrow \\ W_n & \xrightarrow{\sigma_n} & \Omega^{2p} W_{n+1}. \end{array}$$

Since Ωg_n induces a mod-p homology epimorphism through dimension $2np^2 - 4$ if $n \geqq p$, and both $H_*(W_n; Z/p)$ and $H_*(\Omega^{2p} W_{n+1}; Z/p)$ are of the same dimension for $* \leqq 2np^2 - 4$, it follows that σ_n induces a mod-p homology isomorphism through dimension $2np^2 - 4$. σ_n is the map of Proposition 1.3. That $h_p: \Omega^2 S^{2n+1} \to \Omega^{2p} P^{2np+2p-1}(p)$ may be chosen to be an H-map if $p > 2$ is checked in [4].

We remark that if W_{n+1} were known to be an H-space localized at 2, then the above procedure would give the 2-primary analogue of σ_n; an alternative construction of σ_n is given in Sect. 3 here. M. Mahowald conjectures that W_{n+1} is a loop space [12, pp. 5–6]. Larry Taylor points out that the above procedure gives a map $W_n \to \Omega^8 W_{n+2}$ which is degree one on the bottom cell.

To prove 1.4, consider the map

$$\phi_n: W_n \to \Omega^{2p+1} P^{2np+2p-1}(p)$$

obtained from the following homotopy commutative diagram

$$\begin{array}{ccc} W_n & \xrightarrow{\phi_n} & \Omega^{2p+1} P^{2np+2p-1}(p) \\ \downarrow & & \downarrow \\ S^{2n-1} & \longrightarrow & * \\ \downarrow & & \downarrow \\ \Omega^2 S^{2n+1} & \xrightarrow{h_p} & \Omega^{2p} P^{2np+2p-1}(p), \end{array}$$

and observe that the composite

$$W_n \xrightarrow{\phi_n} \Omega^{2p+1} P^{2np+2p-1}(p) \xrightarrow{\text{stabilize}} \Omega^\infty \Sigma^\infty P^{2np-2}(p)$$

induces a mod-p homology isomorphism through dimension $2np^2+2p-5$. Let E^{2p}: $P^{2np-2}(p) \to \Omega^{2p}\Sigma^{2p}P^{2np-2}(p)$ be the $(2p)$-fold suspension and observe that the following diagram homotopy commutes

$$\begin{array}{ccc} W_n & \xrightarrow{\sigma_n} & \Omega^{2p} W_{n+1} \\ \downarrow & & \downarrow \\ \Omega^{2p+1}\Sigma^{2p+1}P^{2np-2}(p) & \xrightarrow{\Omega^{2p+1}E^{4p+1}} & \Omega^{4p+1}\Sigma^{4p+1}P^{2np-2}(p). \end{array}$$

Hence, there is a map

$$f\colon X_n \to \Omega^\infty \Sigma^\infty P^{2np-2}(p)$$

which by the above remarks is a homotopy equivalence.

An alternative construction for a map ϕ_n satisfying the above homological requirements is obtained from the composite

$$\Omega^2 S^{2n+1} \xrightarrow{h'} \Omega^2 S^{2np+1}\{p\} \xrightarrow{\Omega^2 h_m} \Omega^3 P^{2np+2}\{p\} \xrightarrow{\Omega^3\Sigma^{2p-3}} \Omega^{2p} P^{2np+2p-1}(p)$$

where h' is a lift of Ωh_p with h_p given by the p^{th} James-Hopf invariant [25], h_m is the map in [10, Lemma 11.4] and Σ^{2p-3} is the $(2p-3)$-fold suspension.

§ 3. The Map $\sigma_n: W_n \to \Omega^4 W_{n+1}$

We give the 2-primary modifications required in our construction of the map σ_n: $W_n \to \Omega^4 W_{n+1}$ of Proposition 1.3. Lemma 3.1 is a slight modification of results due to I.M. James [29] and J.C. Moore [23].

Let h_2: $\Omega S^{n+1} \to \Omega S^{2n+1}$ be the second James-Hopf invariant [8]; h_2 is given explicitly in [29]. Let 2: $\Omega S^{n+1} \to \Omega S^{n+1}$ be the H-space squaring map (2 is the composite $\mu \circ \Delta$ where Δ is the diagonal map and μ is the multiplication on ΩS^{n+1}). Let $(\Omega S^{n+1})\{2\}$ denote the homotopy theoretic fibre of 2. In the next three sections assume that all spaces are localized at 2.

Lemma 3.1. *If n is even $\Omega(2 \circ h_2)$ is null-homotopic. Hence, there is a lift h of Ωh_2 to $(\Omega^2 S^{n+1})\{2\}$ which gives an isomorphism on $H_{4n-2}(\;;Z/2Z)$.*

Let $S^n\{q\}$ denote the homotopy theoretic fibre of the degree q map on S^n. A check of the Hilton-Milnor theorem [27] gives

Lemma 3.2. *There is a map*

$$i_n\colon (\Omega S^{n+1})\{2\} \to \Omega^2(S^{n+2}\{2\})$$

which is degree one on the bottom cell.

We remark that $(\Omega S^{n+1})\{2\}$ and $\Omega(S^{n+1}\{2\})$ are frequently not homotopy equivalent.

Proposition 3.3. *Any map* $f\colon (\Omega^2 S^{4n+1})\{2\} \to \Omega^2(S^{4n+1}\{2\})$ *is zero on* $H_{4n-2}(\ ; Z/2Z)$. *Hence, there does not exist a homotopy equivalence* $g\colon (\Omega S^{4n+1})\{2\} \to \Omega(S^{4n+1}\{2\})$. *Furthermore* $\Omega^2 S^3\langle 3\rangle$ *is not a retract of* $\Omega^2(S^5\{2\})$.

It seems reasonable that $(\Omega S^{2n+1})\{2\}$ is homotopy equivalent to $\Omega(S^{n+1}\{2\})$ precisely when $n=0, 1$, or 3, but we have not checked this.

Splicing EHP sequences twice gives the following lemma due to M. Mahowald [19].

Proposition 3.4. *There exists a map*

$$\gamma_n\colon S^{4n-2}\{2\} \to W_n$$

which is degree one on the bottom cell.

Using the above, we define $\sigma_n\colon W_n \to \Omega^4 W_{n+1}$. First let $g_n\colon \Omega^2 S^{2n+1} \to \Omega^3 W_{n+1}$ be the composite $(\Omega^3 \gamma_{n+1}) \circ (\Omega i_{4n}) \circ h$. Clearly $g_n \circ E^2$ is null-homotopic. Hence there is an induced morphism of fibration sequences

$$\begin{array}{ccc} W_n & \longrightarrow & \Omega^4 W_{n+1} \\ \downarrow & & \downarrow \\ S^{2n-1} & \longrightarrow & * \\ \downarrow{\scriptstyle E^2} & & \downarrow \\ \Omega^2 S^{2n+1} & \longrightarrow & \Omega^3 W_{n+1}. \end{array}$$

Fix a choice of map $\sigma_n\colon W_n \to \Omega^4 W_{n+1}$ induced on fibres; this is the σ_n advertised in Proposition 1.3. The proof of 1.3 is analogous to that given in Sect. 2 for odd primes.

Similarly, the proof of 1.4 is analogous to that given in Sect. 2 for odd primes.

Since the composite

$$\Omega^2 S^{2n+1} \xrightarrow{2} \Omega^2 S^{2n+1} \xrightarrow{\Omega h_2} \Omega^2 S^{4n+1}$$

is null, there is a lift of $\Omega^2 S^{2n+1}$ to the fibre of Ωh_2 which after localizing at 2 is ΩS^{2n}. Using the unit tangent bundle of S^{2n} we obtain a map $\Omega S^{2n} \to S^{2n-1}$ which is degree two on the bottom cell. If S^{2n-1} is an H-space, there is a degree one map $\Omega S^{2n} \to S^{2n-1}$. Taking composites we obtain a map $\pi\colon \Omega^2 S^{2n+1} \to S^{2n-1}_{(2)}$ which is degree four on the bottom cell and degree two if $n=2$ or 4. Using the methods of [10, Cor. 13.2] we have the following where ${}_2\pi_* S^{2n+1}$ denotes the 2-primary component of $\pi_* S^{2n+1}$.

Corollary 3.5. $2 \cdot 4^{n-1}\, {}_2\pi_* S^{2n+1} = 0$ *if* $n \geqq 2$ *and* $4^{n-1}\, {}_2\pi_* S^{2n+1} = 0$ *if* $n \geqq 4$.

This corollary is a slight improvement of James' work [29].

Mahowald conjectures that there ought to be a map $\pi\colon \Omega^2 S^{4n+1} \to S^{4n-1}_{(2)}$ which is degree 2 on the bottom cell for all n. Furthermore he conjectures that the fibre of π ought to be a delooping of the fibre of the double suspension $E^2\colon S^{2n-1} \to \Omega^2 S^{2n+1}$ [12].

Lemma 3.1 is true in slightly more generality. In particular, if $g\colon \Omega S^{2n+1}\to\Omega S^{4n+1}$ is any map which induces an isomorphism on $H_{4n}(\ ;Z)$ and fits into two homotopy commutative diagrams

$$\begin{array}{ccc} \Omega S^{2k+1} & \xrightarrow{g} & \Omega S^{4k+1} \\ \big\downarrow{\scriptstyle \Omega\Sigma f} & & \big\downarrow{\scriptstyle \Omega\Sigma(f\wedge f)} \\ \Omega S^{2n+1} & \xrightarrow{g} & \Omega S^{4n+1}, \end{array} \quad \text{and} \quad \begin{array}{ccc} \Omega S^{2n+1} & \searrow^{g} & \\ \big\downarrow{\scriptstyle \Omega(-1)} & & \Omega S^{4n+1} \\ \Omega S^{2n+1} & \nearrow_{g} & \end{array}$$

for all n, then Ωg has order 2 in $[\Omega^2S^{2n+1},\Omega^2S^{4n+1}]$. Indeed, h_2 may be replaced by g in 3.1.

§ 4. Proofs of Statements 3.2–3.4

Statement 3.4 is proven in [19, Lemma 4.8]; a proof follows. Consider the homotopy commutative diagram

$$\begin{array}{ccc} \Omega S^{2n} & \xrightarrow{\Omega E} & \Omega^2 S^{2n+1} \\ \big\downarrow & & \big\downarrow \\ \Omega S^{4n-1} & \longrightarrow & * \end{array}$$

where h_2 is the second James-Hopf invariant. As in [9] extend this homotopy commutative square to a morphism of fibration sequences

$$\begin{array}{ccccc} W_n & \longrightarrow & S^{2n-1} & \xrightarrow{E^2} & \Omega^2 S^{2n+1} \\ \big\downarrow & & \big\downarrow{\scriptstyle \Sigma} & & \big\downarrow \\ \Omega^3 S^{4n+1} & \longrightarrow & \Omega S^{2n} & \xrightarrow{\Omega E} & \Omega^2 S^{2n+1} \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle h_2} & & \big\downarrow \\ \Omega S^{4n-1} & \longrightarrow & \Omega S^{4n-1} & \longrightarrow & * \end{array}$$

since the fibre of the suspension $E\colon S^{2n}\to\Omega S^{2n+1}$ is Ω^2S^{4n+1}. A check of the boundary homomorphism in the long exact sequence in homotopy for the fibration π gives that π is degree 2 on the bottom cell. Hence there is a morphism of fibrations

$$\begin{array}{ccc} S^{4n-2}\{2\} & \longrightarrow & W_n \\ \big\downarrow & & \big\downarrow \\ S^{4n-2} & \xrightarrow{E^3} & \Omega^3 S^{4n+1} \\ \big\downarrow & & \big\downarrow \\ S^{4n-2} & \xrightarrow{E} & \Omega S^{4n-1} \end{array}$$

and a lift $\gamma_n\colon S^{4n-2}\{2\}\to W_n$ is the map of 3.4.

To prove 3.3, observe that there is a map

$$\iota_n : S^{n+1}\{2\} \to \Omega P^{n+2}(2)$$

which is degree one on the bottom cell [10]. Such maps can be obtained from the map of fibration sequences given by

$$\begin{array}{ccc} S^{n+1}\{2\} & \longrightarrow & \Omega P^{n+2}(2) \\ \downarrow & & \downarrow \\ S^{n+1} & \longrightarrow & * \\ \downarrow{\scriptstyle 2} & & \downarrow \\ S^{n+1} & \longrightarrow & P^{n+2}(2). \end{array}$$

Let $f\colon (\Omega^2 S^{4n+1})\{2\} \to \Omega^2(S^{4n+1}\{2\})$ be any map which is degree one on the bottom cell. Then the composite $v = \Omega^2(\iota_{4n}) \circ f \circ h$ is a map $\Omega^2 S^{2n+1} \to \Omega^3 P^{4n+2}(2)$ which induces an isomorphism on $H_{4n-2}(\ ; Z/2Z)$. This contradicts Proposition 1.2 because $D_2(R^2, S^{2n-1})$ has the homotopy type of $P^{4n-1}(2)$ [5, p. 225]. Hence f must be degree zero on the bottom cell. To finish 3.3, observe that $(\Omega^2 S^{4n+1})\{2\}$ is homotopy equivalent to $\Omega[(\Omega S^{4n+1})\{2\}]$. Details are similar for $\Omega^2 S^3\langle 3\rangle$.

To prove 3.2 first recall the Hilton-Milnor theorem specialized to spheres [27, p. 264]. Let $H_2 : \Omega S^{2n+1} \to \Omega S^{4n+1}$ be the second Hilton-Hopf invariant and let $w\colon S^{4n+1} \to S^{2n+1}$ denote the Whitehead product $[\iota, \iota]$ where ι is the fundamental class of S^{2n+1}. Since Whitehead products of length at least 3 are zero in $\pi_* S^{2n+1}$, there is a homotopy commutative diagram

$$\begin{array}{ccccc} \Omega S^{2n+1} & & \xrightarrow{\quad\Omega(2)\quad} & & \Omega S^{2n+1} \\ \downarrow{\scriptstyle \Delta^3} & & & & \uparrow{\scriptstyle m} \\ (\Omega S^{2n+1})^3 & \xrightarrow[1\times H_2]{} & (\Omega S^{2n+1})^2 \times \Omega S^{4n+1} & \xrightarrow[1\times \Omega w]{} & (\Omega S^{2n+1})^3 \end{array}$$

where $m(x, y, z) = (x\,y)\,z$.

A similar but slightly more complicated diagram exists when $2n$ is replaced by $2n-1$.

Next, notice that the square

$$\begin{array}{ccc} S^{n+1} & \xrightarrow{E} & \Omega S^{n+2} \\ \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle \Omega(2)} \\ S^{n+1} & \xrightarrow{E} & \Omega S^{n+2} \end{array}$$

homotopy commutes. Since Whitehead products in $\pi_* S^{n+1}$ are in the kernel of the suspension, we obtain a homotopy commutative square

$$\begin{array}{ccc} \Omega S^{n+1} & \xrightarrow{\Omega E} & \Omega^2 S^{n+2} \\ \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle \Omega^2(2)} \\ \Omega S^{n+1} & \longrightarrow & \Omega^2 S^{n+2} \end{array}$$

and an induced map of fibration sequences

$$\begin{array}{ccc} (\Omega S^{n+1})\{2\} & \xrightarrow{i_n} & \Omega^2(S^{n+2}\{2\}) \\ \downarrow & & \downarrow \\ \Omega S^{n+1} & \xrightarrow{\Omega E} & \Omega^2 S^{n+2} \\ \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle \Omega^2(2)} \\ \Omega S^{n+1} & \xrightarrow{\Omega E} & \Omega^2 S^{n+2}. \end{array}$$

The map i_n induced on fibres is the map of 3.2. A check of the long exact sequence in homotopy gives that i_n is degree one on the bottom cell.

§5. Proof of 3.1

The proof given here is an adaptation of a calculation due to I.M. James [29] and J.C. Moore [23].

Let $f: X \to Y$ be a map. By naturality of the James-Hopf invariants [29], we have a homotopy commutative diagram

$$\begin{array}{ccc} \Omega\Sigma X & \xrightarrow{\Omega\Sigma f} & \Omega\Sigma Y \\ \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h_2} \\ \Omega\Sigma X^{[2]} & \xrightarrow[\Omega\Sigma f^{[2]}]{} & \Omega\Sigma Y^{[2]} \end{array}$$

where h_2 is the second James-Hopf invariant and $X^{[2]}$ denotes the two-fold smash product $X \wedge X$. Let $\theta: Z \to \Omega\Sigma X$ be any map. Then in the group $[Z, \Omega\Sigma Y^{[2]}]$ of pointed homotopy classes of maps we have the equation

$$h_2 \circ \Omega\Sigma f \circ \theta = \Omega\Sigma f^{[2]} \circ h_2 \circ \theta.$$

If $\Sigma f^{[2]}$ is null-homotopic, then

$$h_2 \circ \Omega\Sigma f \circ \theta = 0$$

in $[Z, \Omega\Sigma Y^{[2]}]$.

Notice that the Whitehead product $[\iota, \iota] = w$ in $\Pi_{4n+1} S^{2n+1}$ is a suspension; in particular, the suspension $E_*: \Pi_{4n} S^{2n} \to \Pi_{4n+1} S^{2n+1}$ is an epimorphism [28]. Set $f = \Sigma_*^{-1} w$, $X = S^{4n}$ and $Y = S^{2n}$. Furthermore, notice that $\Sigma f \wedge f$ is null-homotopic. Hence, by the above remarks $h_2 \circ \Omega\Sigma f \circ \theta = 0$ in $[Z, \Omega S^{4n+1}]$.

Next, consider a map $-1: S^{2n+1} \to S^{2n+1}$ of degree -1 and consider the composite Φ given by

$$S^{2n+1} \xrightarrow{\text{pinch}} S^{2n+1} \vee S^{2n+1} \xrightarrow{-1 \vee 1} S^{2n+1} \vee S^{2n+1} \xrightarrow{\text{fold}} S^{2n+1}.$$

Clearly Φ is null-homotopic.

As in Sect. 4 we obtain a homotopy commutativity diagram by looping Φ and expanding $\Omega S^{2n+1} \vee S^{2n+1}$ according to the Hilton-Milnor theorem:

$$\begin{array}{ccccc} \Omega S^{2n+1} & & \xrightarrow{\Omega\Phi} & & \Omega S^{2n+1} \\ \downarrow{\scriptstyle \Delta^3} & & & & \uparrow{\scriptstyle m} \\ (\Omega S^{2n+1})^3 & \xrightarrow[\Omega(-1)\times\Omega(1)\times H_2]{} & (\Omega S^{2n+1})^2 \times \Omega S^{4n+1} & \xrightarrow[1^2\times\Omega w']{} & (\Omega S^{2n+1})^3 \end{array}$$

Here H_2, w, and m are as given in Sect. 4 with $w' = w \circ (-1)$.

By the above paragraph $\Omega(-1) + \Omega(1) + \Omega w' \circ H_2 = 0$ in $[\Omega S^{2n+1}, \Omega S^{2n+1}]$. Hence

$$h_2 \circ (\Omega(-1) + \Omega(1) + \Omega w' \circ H_2) = 0 \quad \text{in} \quad [\Omega S^{2n+1}, \Omega S^{4n+1}].$$

Since $h_2 \circ \Omega w' \circ H_2 = 0$ by previous remarks, it follows that

$$(\Omega h_2) \circ (\Omega^2(-1) + \Omega^2(1)) = 0 \tag{1}$$

in $[\Omega^2 S^{2n+1}, \Omega^2 S^{4n+1}]$. But by naturality of h_2, we have the equation

$$h_2 \circ \Omega(-1) = h_2 \tag{2}$$

in $[\Omega S^{2n+1}, \Omega S^{4n+1}]$. Combining (1), (2), and the fact that $2 \cdot (\Omega h_2) = (\Omega h_2) \cdot 2$ gives that $2 \cdot \Omega h_2 \cdot \Omega^2(1) = 0$ in $[\Omega^2 S^{2n+1}, \Omega^2 S^{4n+1}]$ and 3.1 follows. Notice that our calculations were done in $[\Omega^2 S^{2n+1}, \Omega^2 S^{4n+1}]$ rather than in $[\Omega S^{2n+1}, \Omega S^{4n+1}]$ in order to have the equation $2 \cdot (\Omega h_2) = (\Omega h_2) \cdot 2$.

§ 6. Calculations of $h_{p^k *}$

In this section we give some homological information related to the first sections. In particular, if p is a prime and $\tilde{h}$: $\Sigma^L \Omega^2 \Sigma^2 X \to \Sigma^L D_{p^k}(R^2, X)$ induces a mod-p homology epimorphism, then information about the homological behavior of the adjoint of $\tilde{h}, h$: $\Omega^2 \Sigma^2 X \to \Omega^L \Sigma^L D_{p^k}(R^2, X)$, is studied. We give such calculations in case p is an odd prime; 2-primary calculations with more definitive results are given in [6]. A remark is appropriate here. Specialize to $X = S^{2n-1}$. By Theorem 1.1 here there is a map $\tilde{h}_2$: $\Sigma^4 \Omega^2 S^{2n+1} \to \Sigma^4 D_2(R^2, S^{2n-1})$ which gives an epimorphism on mod-2 homology. By the calculations in [6] $\tilde{h}_2$ does not desuspend. Using the techniques of [25], there is a p-local map $\tilde{h}_p$: $\Sigma^3 \Omega^2 S^{2n+1} \to \Sigma^3 D_p(R^2, S^{2n-1})$ with $p > 2$ giving an epimorphism in homology. Odd primary calculations here give that $\tilde{h}_p$ does not desuspend twice. Hence, we know the smallest integer L such that there is a p-local map $\Sigma^L \Omega^2 S^{2n+1} \to \Sigma^L D_p(R^2, S^{2n-1})$ giving an epimorphism in mod-p homology precisely when $p = 2$; the case $p > 2$ is open.

Related 2-primary calculations are given in the theses of P. Kirley (Northwestern, 1976) and N. Kuhn (University of Chicago, 1980); these calculations apply to certain specific combinatorial maps given in [2, 8].

To begin recall the following theorem where all homology groups are taken with Z/pZ-coefficients for p an odd prime.

Theorem 6.1 [24, 5].

$$H_*\Omega^2 S^{2n+1} \cong \bigotimes_{k\geq 0} \Lambda[Y_{2np^k-1}] \bigotimes_{k\geq 1} P[X_{2np^k-2}]$$

as a Hopf algebra. The Steenrod operations (in homology) are specified by

1. $\beta y_{2np^k-1} = x_{2np^k-2}$, $k \geq 1$,
2. $P_*^r y_{2np^k-1} = 0$,
3. $P_*^1 x_{2np^k-2} = -(x_{2np^{k-1}-2})^p$, $k>1$, *and*
4. $P_*^{p^r} x_{2np^k-2} = 0$, $r \geq 1$.

Next, observe that if Z is any space, $H_* Z$ is a differential graded coalgebra with differential given by the homology Bockstein. Let B_n denote the differential graded Hopf algebra $\Lambda[y_{2n+1}] \otimes P[x_{2n}]$ where y_{2n+1} and x_{2n} are of degree $2n+1$ and $2n$ respectively with differential specified by $d y_{2n+1} = x_{2n}$.

Lemma 6.2. *If f: $\Omega^2 S^{2n+1} \to Z$ is any map and π: $H_* Z \to B_{np^k-1}$ is any map of differential graded coalgebras such that $\pi \cdot f_*(x_{2np^k-2}) \neq 0$, then $f_*(x^{p^s}_{2np^j-2}) \neq 0$ for all $j \geq k$ and all $s \geq 0$.*

Corollary 6.3. *If Z is any H-space such that $H_* Z \cong B_{np^k-1} \otimes C$ as a differential graded Hopf algebra, π: $H_* Z \to B_{np^k-1}$ is the natural projection, and f: $\Omega^2 S^{2n+1} \to Z$ is any map with $\pi \cdot f_*(x_{2np^k-2}) \neq 0$, then $f_*(x^{p^s}_{2np^k-2})$ has $(f_* x_{2np^k-2})^{p^s}$ as a non-trivial summand.*

The point of 6.3 is that, homologically, maps of $\Omega^2 S^{2n+1}$ to certain H-spaces behave much like H-maps. Somewhat more is true. If h: $\Omega^2 S^{2n+1} \to \Omega^\infty \Sigma^\infty D_{p^k}(R^2, S^{2n-1})$, then h_* behaves much like a 2-fold loop map.

Recall from [5, p. 225] that there is a map

$$q\colon D_{p^k}(R^2, S^{2n-1}) \to P^{2np^k-1}(p)$$

which induces an epimorphism in homology. If the composite $\Omega^\infty \Sigma^\infty(q) \circ h = g$ gives an epimorphism in homology in degree $2np^k-2$, then g behaves very much like a 2-fold loop map in homology. To describe this property let Q^{I_r} denote the Dyer-Lashof operation $Q^{np^{k+r}} Q^{np^{k+r-1}} \dots Q^{np^{k+1}} Q^{np^k}$. By the calculations in [5, p. 225] $y_{2np^{k+r}-1} = Q^{I_r} y_{2np^k-1}$.

Corollary 6.4. *If $g_* y_{2np^k-1} \neq 0$, then $g_*(Q^{I_r} y_{2np^k-1})$ has $Q^{I_r} g_*(y_{2np^k-1})$ as a non-trivial summand.*

We remark that the calculations of 6.4 are consistent with the existence of a p-local map $\Sigma^2 \Omega^2 S^{2n+1} \to \Sigma^2 D_{p^k}(R^2, S^{2n-1})$ which induces an epimorphism in mod-p homology.

Corollary 6.5. *If $\tilde{f}$: $\Sigma^{2p^k} \Omega^2 \Sigma^2 X \to \Sigma^{2p^k} D_{p^k}(R^2, X)$ gives a* mod-p *homology epimorphism for every X, then $\tilde{f}$ does not desuspend $(2p^k-1)$ times.*

We do not know whether $\tilde{f}$ can desuspend $(2p^k-2)$ times.

§ 7. Proof of Statements 6.2–6.5

The following observation proven in [7] is useful.

Lemma 7.1. *Let ϕ: $B_n \to B_n$ be any morphism of differential graded coalgebras with $\phi(x_{2n}) \neq 0$. Then ϕ is an isomorphism of differential graded coalgebras.*

Proof of 6.2. Let i: $B_{np^k-1} \to H_* \Omega^2 S^{2n+1}$ be the morphism of differential graded Hopf algebras specified by $i(y_{2np^k-1}) = y_{2np^k-1}$. If f: $\Omega^2 S^{2n+1} \to Z$ and π: $H_* Z \to B_{np^k-1}$ are as given in the hypotheses of 6.2, then the composite $g = \pi \cdot f_* \cdot i$ satisfies $g(x_{2np^k-2}) \neq 0$. Since f_* is induced by a map of spaces, g is a morphism of differential graded coalgebras and 7.1 applies to give that g is an isomorphism. Hence $g(x^r_{2np^k-2}) \neq 0$ for all r.

To finish the proof of 6.2, notice that by 6.1 we have the formula

$$P^I_*(x^{p^s}_{2np^{k+r}-2}) = (-1)^{rs} x^{p^{r+s}}_{2np^k-2}$$

where $P^I_* = P^{p^{rs}}_* \cdot P^{p^{s(r-1)}}_* \cdot P^{p^{s(r-2)}}_* \dots P^{p^s}_*$. By naturality of the Steenrod operations, $f_*(x^{p^s}_{2np^{k+r}-2}) \neq 0$ since $f_*(x^{p^{r+s}}_{2np^k-2}) \neq 0$.

Proof of 6.3. The hypotheses of 6.3 allow the application of 6.2 and the result follows.

Proof of 6.4. Let Z be any connected space. The calculations in [5, p. 128] give that $H_* \Omega^\infty \Sigma^\infty Z$ is a free commutative algebra with generators $Q^I x$ where x runs over a basis for $\bar{H}_* Z$ and I runs over certain admissible sequences. If each element of $\bar{H}_* Z$ is primitive, then each element $Q^I x$ is primitive and so $H_* \Omega^\infty \Sigma^\infty Z$ is a primitively generated Hopf algebra.

Write $g = \Omega^\infty \Sigma^\infty(q) \circ h$ where q and h are as given in 6.4. Let u and v denote $g_*(x_{2np^k-2})$ and $g_*(y_{2np^k-1})$ respectively. By hypothesis and the above paragraph, we have that $H_* \Omega^\infty \Sigma^\infty P^{2np^k-1}(p)$ is a primitively generated Hopf algebra with algebra generators given by $Q^I u$ and $Q^J v$.

Let $Q^{I_r} = Q^{np^{k+r}} \dots Q^{np^{k+1}} Q^{np^k}$. Since $Q^{I_r} y_{2np^k-1} = y_{2np^{k+r}-1}$, $g_* Q^{I_r} y_{2np^k-1}$ is non-zero by Lemma 6.3. An application of 6.3 together with naturality of the Bockstein gives that $g_*(\beta Q^{I_r} y_{2np^k-1})^p$ has $(g_* \beta Q^{I_r} y_{2np^k-1})^p$ as a non-trivial summand. Further, notice that $g_* Q^{I_r} y_{2np^k-1}$ is a non-trivial odd degree primitive and so we have the formula

$$g_*(Q^{I_r} y_{2np^k-1}) = \sum_I Q^I u + \sum_J Q^J v$$

(up to non-zero coefficients) for some choice of I and J.

Applying naturality, the Nishida relation $P^1_* \beta Q^s = (s-1)(\beta Q^{s-1} + Q^{s-1} \beta)$ [5, p. 181] and the above remarks, we see that $(P^1_* \beta)(g_* Q^{I_r} y_{2np^k-1})$ must have a non-trivial p-th power summand. A check of definitions gives that $(P^1_* \beta) Q^I u$ does not have $(\beta Q^{I_{r-1}} v)^p$ as a summand. If $P^1_* \beta Q^J v$ has $(\beta Q^{I_{r-1}} v)^p$ as a non-trivial summand then a similar check of definitions gives that $Q^J v = Q^{I_r} v$ up to non-zero coefficients.

Proof of 6.5. Let $\bar{f}$: $\Sigma^{2p^k} \Omega^2 \Sigma^2 X \to \Sigma^{2p^k} D_{p^k}(R^2, X)$ be any map which induces a mod-p homology epimorphism. If $\bar{f} = \Sigma^{2p^k-1} \hat{f}$, then passage to adjoints gives a homotopy commutative diagram

$$\begin{array}{ccc} & \xrightarrow{\operatorname{adj}(\hat{f})} & \Omega \Sigma D_{p^k}(R^2, X) \\ \Omega^2 \Sigma^2 X & & \downarrow \Omega \Sigma^{2p^k - 1} \\ & \xrightarrow{\operatorname{adj}(\tilde{f})} & \Omega^{2p^k} \Sigma^{2p^k} D_{p^k}(R^2, X). \end{array}$$

Specialize to $X = S^{2n-1}$ and observe by 6.4 that $(\operatorname{adj} \tilde{f})_*(y_{2np^{k+1}-1})$ has $Q^{np^k}(\operatorname{adj} \tilde{f}_*)(y_{2np^k-1})$ as a summand. But $Q^{np^k}(\operatorname{adj} \tilde{f}_*)(y_{2np^k-1})$ is not in the image of $(\Omega \Sigma^{2p^k-1})_*$.

Appendix

We describe the combinatorial maps giving the desuspension results of Theorem 1.1. An outline of a proof of Snaith's theorem [26] is included.

Assume that X is a non-degenerately based, path-connected, compactly generated Hausdorff space with basepoint $*$; assume that Y is compactly generated and Hausdorff. Let $C(Y, X)$ be the equivalence classes of pairs (S, f) where S is a finite subset of $Y, f\colon S \to X$, and with the equivalence relation generated by setting (S, f) equivalent to $(S - \{q\}, f|s - \{q\})$ if $f(q) = *$. The space $C(Y, X)$ was introduced in [11] to study the cohomology of function spaces and Gelfand-Fuks cohomology. In case $Y = R^n$, $C(R^n, X)$ has the weak homotopy type of $\Omega^n \Sigma^n X$ [22]. The functional notation was given in [17].

Let $F(Y, k)$ denote the configuration space $\{(y_1, \ldots, y_k) \in Y^k | y_i \neq y_j \text{ if } i \neq j\}$ [14]. The symmetric group Σ_k acts freely on $F(Y, k)$ by permuting coordinates (if $F(Y, k) \neq \phi$). Σ_k acts on the k-fold smash, $X^{[k]}$, by permuting coordinates. Let $D_k(Y, X)$ denote the quotient $F(Y, k) \times_{\Sigma_k} X^{[k]} / F(Y, k) \times_{\Sigma_k} *$ where Σ_k acts diagonally on $F(Y, k) \times X^{[k]}$. Let $B(Y, k) = F(Y, k)/\Sigma_k$. Observe that $B(Y, k)$ is the space of subsets of Y having cardinality k and that $D_k(Y, X)$ is the equivalence classes of pairs (S, f) where S is a subset of Y having cardinality $k, f\colon S \to X$, and (S, f) is equivalent to the basepoint if $f(q) = *$ for some q in S.

$C(Y, X)$ is filtered by setting $F_j C(Y, X)$ equal to the subspace of points represented by (S, f) where the cardinality of S is at most j. For simplicity, assume that $C(Y, S^0) = \amalg_{j \geqq 0} B(Y, j)$ is a subspace of $\mathbb{R}^\infty$. (This assumption is not really necessary, but makes the exposition cleaner.) Set $C = C(Y, X)$, $D = \bigvee_{k \geqq 0} D_k(Y, X)$, $2^C = C(R^\infty, D)$, $F_j D = \bigvee_{k=0}^{j} D_k(Y, X)$, and $D_k = D_k(Y, X)$. Snaith's theorem [26] states that C and D are stably equivalent.

Define maps

$$h_k\colon C \to C(B(Y, k), D_k(Y, X)) \quad \text{and}$$
$$H\colon C \to 2^C$$

by the formulas a) $h_k(S, f) = (\{T\}, g)$ where T runs over the subset of S having cardinality k and $g\{T\} = (T, f|_T)$, and b) $H(S, f) = (\{T\}, g)$ where T runs over all subsets of and $g\{T\} = (T, f|_T)$. h_k and H are clearly continuous. Notice that $\prod_{k=0}^{\infty} C(B(Y, k), D_k(Y, X))$ is naturally a subspace of 2^C and H is the composite

$$C \xrightarrow{\text{diagonal}} \prod_0^\infty C \xrightarrow{\pi h_k} \prod_0^\infty C(B(Y, k), D_k(Y, X)) \xrightarrow{\text{inclusion}} 2^C.$$

Furthermore, notice that H restricted to the filtrations of C gives a commutative diagram

$$\begin{array}{ccc} F_{j-1}C & \xrightarrow{H|F_{j-1}} & C(R^\infty, F_{j-1}D) \\ \downarrow i & & \downarrow \iota \\ F_jC & \xrightarrow[H|F_j]{} & C(R^\infty, F_jD) \end{array}$$

where the map ι is induced by the standard inclusion of $F_{j-1}D$ in F_jD. Let p: $C(R^\infty, F_jD) \to C(R^\infty, D_j)$ be the map induced by collapsing F_{j-1} in F_jD to a point. By definition $(p \circ \iota \circ H|_{F_{j-1}})(F_{j-1}C) = *$. Hence, we obtain a strictly commutative diagram

$$\begin{array}{ccc} F_{j-1}C & \xrightarrow{H|F_{j-1}} & C(R^\infty, F_{j-1}D) \\ \downarrow & & \downarrow \iota \\ F_jC & \xrightarrow{H|F_j} & C(R^\infty, F_jD) \\ \downarrow & & \downarrow p \\ D_j & \xrightarrow{\phi} & C(R^\infty, D_j) \end{array}$$

where ϕ is homotopic to the standard inclusion. We replace all spaces $C(R^\infty, Z)$ by $\Omega^\infty\Sigma^\infty(Z)$ up to homotopy type to obtain a homotopy commutative diagram where the vertical right-hand maps are infinite loop maps:

$$\begin{array}{ccc} F_{j-1}C & \longrightarrow & \Omega^\infty\Sigma^\infty F_{j-1}D \\ \downarrow & & \downarrow \\ F_jC & \longrightarrow & \Omega^\infty\Sigma^\infty F_jD \\ \downarrow & & \downarrow \\ D_j & \longrightarrow & \Omega^\infty\Sigma^\infty D_j \end{array}$$

Applying $\Omega^\infty\Sigma^\infty$ to the left-hand side, we obtain another homotopy commutative diagram

$$\begin{array}{ccc} \Omega^\infty\Sigma^\infty F_{j-1}C & \longrightarrow & \Omega^\infty\Sigma^\infty F_{j-1}D \\ \downarrow & & \downarrow \\ \Omega^\infty\Sigma^\infty F_jC & \longrightarrow & \Omega^\infty\Sigma^\infty F_jD \\ \downarrow & & \downarrow \\ \Omega^\infty\Sigma^\infty D_j & \longrightarrow & \Omega^\infty\Sigma^\infty D_j \end{array}$$

where the bottom arrow is a homotopy equivalence. The vertical sides are quasifibrations (because the inclusion of $F_{j-1}C$ in F_jC is a cofibration with cofibre D_j [8]). That $\Omega^\infty\Sigma^\infty F_jC \to \Omega^\infty\Sigma^\infty F_jD$ induces an isomorphism on homotopy groups follows inductively because the bottom arrow is a homotopy equivalence and $F_0 C = F_0 D = *$. Hence the induced map $\Omega^\infty\Sigma^\infty C \to \Omega^\infty\Sigma^\infty D$ gives an isomorphism on homotopy groups.

We remark that the above gives that the adjoint of the power set map $H: C \to 2^C$ is the stable homotopy equivalence of Snaith. The unstable maps promised in Theorem 1.1 are obtained from the maps $h_k: C \to C(B(Y,k), D_k(Y,k))$: if $B(Y,k)$ embeds in R^L, then there is an induced map $C \to \Omega^L\Sigma^L D_k(Y,X)$. The proof given here is that given in [8].

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