is injective. The assertion now follows by a simple diagram chase, taking into account that δ is surjective.

A *weak Kähler metric* for f (or for X relative S) is by definition a weak Kähler metric for $X \times_S S_{\mathrm{real}}$ relative S_{real}. If such a metric exists, f is called *weakly Kähler*. Obviously any Kähler map is weakly Kähler. (5.8) implies a converse for some cases:

(5.9) **Corollary.** *Let the assumptions and notations be as in* (5.8). *If there exists a real analytic weak Kähler metric for f which is horizontal, i.e. is in* $\Gamma(X, \mathscr{H}_{X/S})^{\triangledown}$, *then f is Kähler.*

§ 6. *Deformations of Kähler Spaces*

In this section we first give a criterion for a local deformation of a compact Kähler space to be Kähler.

(6.1) **Theorem.** *Let $f: X \to S$ be a proper holomorphic map, $s \in S$ a point and X_n the n-th infinitesimal neighborhood of X_s in X. Then the following conditions are equivalent:*

(1) *f is Kähler in s.*

(2) *The complex spaces X_n are Kähler for all n.*

Proof. Clearly (2) is a consequence of (1). Suppose conversely that (2) holds. By (4.12) (1) the images of the maps $\Gamma(X_m, \mathscr{H}_{X_m}) \to \Gamma(X_n, \mathscr{H}_{X_n})$, $m \geqq n$, are constant for $m \gg n$, say equal to $M_n \subseteq \Gamma(X_n, \mathscr{H}_{X_n})$. Obviously the homomorphisms $M_{n+1} \to M_n$ are surjective, and we have $\varprojlim M_n = \varprojlim \Gamma(X_n, \mathscr{H}_{X_n})$. Moreover $f_*(\mathscr{H}_X)_s \to M_n$ is surjective for all n by (4.13). For a given n we can find an integer $m \geqq n$ such that the image of $\Gamma(X_m, \mathscr{H}_{X_m}) \to \Gamma(X_n, \mathscr{H}_{X_n})$ is M_n. By assumption, there exists a Kähler metric ϕ_m on X_m. Then the image ϕ_n of ϕ_m in $\Gamma(X_n, \mathscr{H}_{X_n})$ is a Kähler metric on X_n lying in M_n. After shrinking S as a neighborhood of s if necessary, there is an element ϕ from $\Gamma(X, \mathscr{H}_X)$ such that $\phi | X_n = \phi_n$. Then ϕ is a Kähler metric for f in s by (4.3).

We have a similar result for weakly Kähler maps:

(6.2) **Theorem.** *Let S be a real space, $f: X \to S$ a proper and flat complex space over S and $s \in S$ be a point with infinitesimal neighborhoods S_n, and let $f_n: X_n \to S_n$ denote the maps obtained by base change. Suppose that there is a real analytic weakly Kähler metric for f_n for every n. Then f is weakly Kähler in s.*

The proof is analogous to the proof of (6.1) using (5.4) instead of (4.12) and (4.13). – (6.2) implies of course a similar statement for maps of complex spaces.

In [18] Kodaira and Spencer proved that local deformations of compact Kähler manifolds are Kähler. The following theorem, which has been already announced without proof by Moishezon in [20] in a somewhat more special form, generalizes this result to the singular case. Compare also [17], p. 180.

(6.3) **Theorem.** *Let S be a real space, $f\colon X\to S$ a proper and flat complex space over S, and let ϕ_0 be a real analytic Kähler metric on the fibre X_s of a point $s\in S$. Then, if the natural map $\mathrm{H}^2(X_s,\mathbb{R})\to \mathrm{H}^2(X_s,\mathcal{O}_{X_s})$ is surjective, there exists – after shrinking S as a neighborhood of s if necessary – a weak real analytic Kähler metric ϕ for f with $\phi|X_s=\phi_0$.*

Proof. Let S_n denote the n-th infinitesimal neighborhood of s in S and put $X_n := X\times_S S_n$. By (5.6) and our assumption the obstruction groups $\mathrm{O}(\mathfrak{m}_s^n/\mathfrak{m}_s^{n+1})$ vanish for all n. Using (5.6) again, we see that ϕ_0 lifts to an element $(\phi_n)_{n\in\mathbb{N}}$ from $\varprojlim_n f_*(\mathcal{H}_{X_n/S_n})$. The assertion now follows from (5.4) and (4.3).

(6.4) Let S be a real space and $f\colon X\to S$ be a proper and smooth complex space over S. We call f *pseudo-Kähler*, if for any hermitian coherent $\mathcal{A}_S$-module $\mathcal{M}$ the following conditions are satisfied:

(1) The spectral sequence

$$\mathrm{E}_1^{p,q}(\mathcal{M})=\mathrm{R}^q\,\mathrm{f}_*(\Omega^q_{X/S}\otimes\mathcal{M})\Rightarrow \mathrm{R}^{p+q}f_*(f^{-1}(\mathcal{M}))$$

degenerates at the E_1-level, The $\mathrm{E}_1^{p,q}(\mathcal{A}_S)$ are locally free and the maps $\mathrm{E}_1^{p,q}(\mathcal{A}_S)\otimes\mathcal{M}\to\mathrm{E}_1^{p,q}(\mathcal{M})$ are bijective.

(2) If $\mathrm{F}^{\cdot}(\mathcal{M})=\mathrm{F}_n^{\cdot}(\mathcal{M})$ denotes the corresponding descending filtration on $\mathrm{R}^n f_*(f^{-1}(\mathcal{M}))$, we have $\mathrm{F}^p(\mathcal{M})\amalg\overline{\mathrm{F}^{n-p+1}(\mathcal{M})}=\mathrm{R}^n f_*(f^{-1}(\mathcal{M}))$ for all p.

The map f is pseudo-Kähler if and only if the hermitian double complex $K^{\cdot,\cdot}:=f_*(\mathcal{A}_{X/S}^{\cdot,\cdot})$ is pseudo-Kähler in the sense of (2.16). Indeed, $K^{\cdot,\cdot}$ is locally bounded and has $\mathcal{A}_S$-flat components, and $\mathcal{A}_{X/S}^{\cdot}\otimes\mathcal{M}$ resp. $\mathcal{A}_{X/S}^{p,\cdot}\otimes\mathcal{M}$ is an acyclic resolution of $f^{-1}(\mathcal{M})$ resp. $\Omega^p_{X/S}\otimes\mathcal{M}$.

We put $\mathrm{H}^{p,q}(\mathcal{M}):=\mathrm{F}^p(\mathcal{M})\cap\overline{\mathrm{F}^q(\mathcal{M})}$ for (p,q) with $p+q=n$. (2.7) and the previous remark imply that for a pseudo-Kählerian map f the natural homomorphisms

$$\coprod_{p'\geqq p}\mathrm{H}^{p',n-p'}(\mathcal{M})\to\mathrm{F}^p(\mathcal{M})$$

are bijective and that the modules $\mathrm{H}^{p,q}(\mathcal{A}_S)$ are locally free with

$$\mathrm{H}^{p,q}(\mathcal{A}_S)\otimes_{\mathcal{A}_S}\mathcal{M}=\mathrm{H}^{p,q}(\mathcal{M}).$$

Obviously the fibres of a pseudo-Kählerian map are pseudo-Kählerian complex manifolds. By the classical Hodge theory compact Kähler manifolds are pseudo-Kähler in the above sense. This holds in fact more general for any complex manifold, which is a surjective image of a compact Kähler manifold, see [11]. – From (2.17) we obtain the following result.

(6.5) **Theorem.** *Let S be a real space and $f\colon X\to S$ a proper and smooth complex space over S, and let V be the set of points $s\in S$ such that X_s is pseudo-Kähler. Then:*

(1) *V is Zariski open in S.*

(2) *$f|V\colon X|V\to V$ is pseudo-Kähler.*

It follows from (6.5) that our notion of a pseudo-Kähler map is stable under arbitrary base change.