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Kontakt/Contact

Digizeitschriften e.V.
SUB Göttingen
Platz der Göttinger Sieben 1
37073 Göttingen

info@digizeitschriften.de

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On Deformations of Kähler Spaces. I

Jürgen Bingener

Fakultät für Mathematik der Universität, Universitätsstraße 31, D-8400 Regensburg, Federal Republic of Germany

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Introduction

Among the compact complex manifolds, the Kähler manifolds enjoy a number of remarkable properties: For these manifolds we have the powerful Hodge theory with its numerous consequences at our disposal. Kodaira and Spencer showed that the class of Kähler manifolds is stable under small deformations. Moreover, the Douady spaces of Kähler manifolds have the important property that their connected components are compact, see [10, 11].

Kähler spaces, i.e. Kähler manifolds with singularities, were first introduced by Grauert in [12]. These studies were continued by Moishezon in [20]. Since one is frequently forced to consider "relative" situations, it became necessary to introduce Kähler mappings, see Fujiki [10]. We recall the definition:

Let $f\colon X \to S$ be a holomorphic map. A *Kähler metric* φ *for* f is given by a collection (φ_i) of real differentiable functions on an open covering (U_i) of X such that the differences $\varphi_i - \varphi_j$ on the intersections U_{ij} are locally the real part of holomorphic functions, and the Levi forms $\mathrm{L}(\varphi_i) = i\partial\bar{\partial}(\varphi_i)$ are strictly positive on the complement of the zero section of the tangent bundle

$T(X/S)|U_i$. If $\varphi_i - \varphi_j$ is only locally the real part of a differentiable function being holomorphic on the fibres of f, we call φ a *weak Kähler metric for f*.

We say that f is *Kähler* (resp. *weakly Kähler*), if such a metric exists. In the case that $S = \mathrm{Spec}(\mathbb{C})$ is a point, we obtain the notion of a *Kähler space*. Obviously the fibres of weakly Kähler maps are Kähler spaces. For example projective maps are Kähler.

Let $f: X \to S$ be a proper holomorphic map. In this paper we give criteria for f to be Kähler resp. weakly Kähler resp. projective. The first result is the following proposition ((6.1)): *If $s \in S$ is a point such that all infinitesimal neighbourhoods X_n of the fibre X_s are Kähler spaces, then f is Kähler in s.* An example due to Deligne presented in [4] shows that it is not sufficient here to assume the fibre X_s to be Kähler, even if f is smooth. Thus, in order to investigate deformations of Kähler spaces, we have to restrict ourselves in most cases to weakly Kähler maps. In this context we show the following proposition (compare (6.2)):

Let $s \in S$ be a point with infinitesimal neighbourhoods S_n. If for any n there exists a weakly real analytic Kähler metric for $f_n := f \times_S S_n$, then f is weakly Kähler in s.

Examples of Moishezon from [20] show that local deformations of compact Kähler spaces are in general not Kähler, see also [4] § 3. In particular there is in general a non-trivial obstruction for lifting a weakly Kähler metric for f_n to a weakly Kähler metric for f_{n+1}. But, if f is flat and if $H^2(X_s, \mathbb{R}) \to H^2(X_s, \mathcal{O}_{X_s})$ is surjective, these obstructions vanish, and we obtain (compare (6.3)):

Let $f: X \to S$ be a proper and flat holomorphic map and let $s \in S$ be a point such that X_s is Kähler. If the natural homomorphism $H^2(X_s, \mathbb{R}) \to H^2(X_s, \mathcal{O}_{X_s})$ is surjective, f is weakly Kähler in s.

This theorem, which has been announced without proof in a somewhat more special form by Moishezon in [20], generalizes the above mentioned result of Kodaira-Spencer. In this context we remark that the relative analytic spaces are the natural frame for investigating weakly Kähler maps.

Let now f be smooth, and let V be the set of points $s \in S$ such that the spectral sequence $E_1^{p,q}(s) = H^q(X_s, \Omega^p_{X_s}) \Rightarrow H^{p+q}(X_s, \mathbb{C})$ degenerates at the E_1-level and the Hodge decomposition $H^n(X, \mathbb{C}) = \coprod_{p+q=n} H^q(X_s, \Omega^p_{X_s})$ as well as the Hodge symmetry $\overline{H^q(X_s, \Omega^p_{X_s})} = H^p(X_s, \Omega^q_{X_s})$ holds. We prove in (6.5), that *V is the complement of a real analytic subset of S.*

Of course it is of interest to find criteria guaranteeing that a weakly Kähler proper map $f: X \to S$ is Kähler. Here we show the following proposition (compare (5.9)): *Suppose that there is a weak real analytic Kähler metric for f, whose period class is horizontal. Then f is Kähler locally on S, if moreover $R^1 f_*(\mathcal{O}_X)$ vanishes.*

Finally we mention some notations and conventions. Real analytic spaces in the sense of [2] (1.1) are called here *real spaces* for short. For a real space Y let $\mathcal{O}_Y$ denote the structure sheaf and $\mathcal{A}_Y := \mathcal{O}_Y \otimes_{\mathbb{R}} \mathbb{C}$ its complexification, so that $Y_{\mathbb{C}} := (Y, \mathcal{A}_Y)$ is a $\mathbb{C}$-ringed space. We denote by Y_d the *differentiable space* corresponding to Y. If $V \subseteq \mathbb{R}^n$ is an open set and $\mathcal{J} \subseteq \mathcal{O}_V$ a coherent ideal with