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Some Pontrjagin Rings II

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1. Introduction

The symbol X is reserved for an H-space whose underlying space has the homotopy type of a connected CW-complex of finite type or is the p-localization of such a space, where p is a fixed prime. Also it is assumed always that $H_*(X, Z)$ has no p-torsion and that $H_*(X, Z_{(0)})$ is an associative, commutative, graded algebra where the multiplication is induced by the H-structure and $Z_{(0)}$ is the field of rational numbers. Examples of X such as BU, BSp and for odd primes BO, etc. and certain connected coverings of these, for example $BU\langle 6, \ldots, \infty\rangle$ for $p=3$ [12], were discussed in [7]. Here we extend the class to include ΩY where Y is a simply connected mod p finite H-space [9, 10 where if $p=2$, $H_*(Y, Z/2Z)$ is assumed to be associative]. This paper is not what was intended as the first sequel to Pontrjagin Rings I. The main result below confirms a conjecture which has been in circulation since the advent of localization in H-space theory in the early 1970's. The promised discussion of K-theory Pontrjagin ring structures appears briefly in the final section and will be continued later.

The symbol A will always denote a connected, bi-associative, bi-commutative, graded Hopf algebra of finite type over $Z_{(p)}$ where each A_n is a free $Z_{(p)}$-module. Here $Z_{(p)}$ is the ring of integers localized at the prime ideal (p). Both $H^*(X, Z_{(p)})$ and $H_*(X, Z_{(p)})$ are examples of such Hopf algebras. A Hopf algebra A is *quasi-monogenic* if the algebra $A \otimes Z_{(0)}$ is generated by a single positive dimensional element. As $A \otimes Z_{(0)}$ inherits a Hopf algebra structure from A, it is isomorphic to an exterior algebra on a single odd dimensional generator or to a polynomial algebra over $Z_{(0)}$ on one even dimensional generator. We do not need to consider the most general type of quasi-monogenic Hopf algebras; those which arise in our Pontrjagin rings are of one of the following two types:

(a) $A \cong E(x)$, an exterior algebra over $Z_{(p)}$ on the odd dimensional generator x,

(b) $A \cong \Gamma(x, r)$ where x is of positive dimension $2n$ and $0 \leqq r \leqq \infty$.

To define $\Gamma(x, r)$, we consider it as the subring of $Z_{(0)}[x]$ generated as an algebra over $Z_{(p)}$ by classes $1, x, (p!)^{-1} x^p, \ldots, (p^r!)^{-1} x^{p^r}$. Equivalently $\Gamma(x, r)$ is

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the subalgebra of $Z_{(0)}[x]$ generated by classes $1, x, (p^{v(0)})^{-1}x^p, \ldots, (p^{v(r-1)})^{-1}x^{p^r}$ where $v(t)=1+p+\ldots+p^t$. Routine computations verify that if m has p-adic expansion $a_0+a_1p+\ldots+a_rp^r+\ldots+a_sp^s$ and $x\in A_{2n}$, then an additive generator for A_{2nm} is $(p^\alpha)^{-1}x^m$ where

$$\alpha=a_1+a_2v(1)+\ldots+a_{r-1}v(r-2)+(a_r+pa_{r+1}+\ldots+p^{s-r}a_s)\,v(r-1).$$

If $r=0$, $\Gamma(x,r)=Z_{(p)}[x]$. When $r=\infty$, $\Gamma(x,r)=\Gamma(x)$, the divided polynomial algebra on x. $\Gamma(x,r)$ has a unique Hopf algebra structure of the required type where the comultiplication ψ is well determined as $\psi(x)=x\otimes 1+1\otimes x$ must hold.

$\Gamma(x,r)\otimes Z/pZ$ inherits a Hopf algebra structure from $\Gamma(x,r)$ and as an algebra is isomorphic to a tensor product of r polynomial algebras generated by $\overline{(p^i!)^{-1}x^{p^i}}$ truncated at height p and a polynomial algebra generated by $\overline{(p^r!)^{-1}x^{p^r}}$, where the bar denotes reduction modulo p. The submodule of primitive elements is generated by $\bar{x}$ and $\overline{((p^r!)^{-1}x^{p^r})^{p^i}}$ for $i>0$.

Theorem 1.1. (a) *$A=H_*(X,Z_{(p)})$ is isomorphic as an algebra to a tensor product of quasi-monogenic Hopf algebras of types $E(x)$ and $\Gamma(x,r)$.*

(b) *If $\Gamma(x,r)$ occurs as a factor in* (a) *where x has dimension $2n$ and n is not binary, then $r=0$.*

We recall from the language of Sect. 3 of [7] that n is binary if it has p-adic expansion of the form $1+p^{\alpha_2}+p^{\alpha_3}+\ldots+p^{\alpha_s}$ with $1\leqq\alpha_2<\alpha_3<\ldots<\alpha_s$. The element x in $\Gamma(x,r)$ is called a quasi-generator. Further information on the action of the Steenrod algebra on the mod p reductions of the quasi-generators is implicit in Sect. 2.

If $X=\Omega Y$ with Y a simply connected simple Lie group and p is odd, then $A\cong\Gamma(x,r)\otimes$ Polynomial algebra where x has dimension 2. It is an open conjecture whether or not this is true for Y a simply connected mod p finite H-space. Also if Y is Lie, then $r=0$ or 1 with the single exception that if Y is E_8 and $p=3$, then $r=2$. Again it is not known what restrictions can be placed upon r for finite H-spaces. Theorem 1.1(b) implies for well known reasons that if Y is a simply connected finite H-space and p is odd, then even dimensional indecomposables in $H^*(Y,Z/pZ)$ lie in dimensions $2m$ with m binary, provided of course that one knows beforehand that $H_*(\Omega Y,Z)$ has no p-torsion. Examples of X where the decomposition of $H_*(X,Z_{(p)})$ has a factor $\Gamma(x,r)$ with $r\neq 0$ and $\dim x>2$ can be found among the connective coverings of BU, such as $BU\langle 6,\ldots,\infty\rangle$ at $p=3$ mentioned earlier [12].

For many X, particularly when $X=\Omega Y$ as above, it is quite easy to determine the ring $H_*(X,Z/pZ)$. One consequence of Theorem 1.1 is that this determines $H_*(X,Z_{(p)})$.

Corollary 1.2. *The ring $H_*(X,Z/pZ)$ determines the ring $H_*(X,Z_{(p)})$.*

The proof of the corollary is easily deduced from the theorem. Since $H_*(X,Z_{(p)})$ has no p-torsion, $\dim_{Z/pZ}H_i(Y,Z/pZ)=\dim_{Z_{(0)}}H_i(Y,Z_{(0)})$ for all i and from this one knows how many factors $E(x)$ and $\Gamma(x,r)$ are present and the different dimensions of x. The integers r are well determined since

$$\dim_{Z/pZ}QH_i(X,Z/pZ)=\dim_{Z/pZ}\{QH_i(X,Z_{(p)})\otimes Z/pZ\}.$$

The need for Corollary 1.2 may be clarified by the following example taken from [4]. Let $p=2$ and $A=\Gamma(x_2,2)\otimes\Gamma(y_2,0)$ as a Hopf algebra where subscripts denote dimension. Let $B=Z_{(2)}[\xi_2,\eta_2,\xi_4,\zeta_8]/J$ where J is the ideal generated by $2\xi_4-\xi_2^2$ and $2\zeta_8-\xi_4^2-\eta_2^4$. B is a bi-associative, bi-commutative Hopf algebra with ξ_2 and η_2 both primitive. Then the rings $A\otimes Z/2Z$ and $B\otimes Z/2Z$ are isomorphic but A is not isomorphic B. Of course B is not isomorphic to a tensor product of quasi-monogenic algebras. In fact it is not difficult to see that B cannot arise as a Pontrjagin ring. If one computes B^*, which would have to be $H^*(X,Z_{(2)})$, one finds that the ring structure contradicts the Tumbling Lemma 1.3 of [5]. However the author has not been able to find a general proof of Theorem 1.1 based on this lemma.

The following corollary is again an immediate consequence of statements above.

Corollary 1.3. *The Hopf algebra structures of $H_*(X,Z/pZ)$ and $H_*(X,Z_{(p)})$ are determined up to isomorphism by the ring structure of $H_*(X,Z/pZ)$ and the action of the comultiplication on any choice of quasi-generators of $H_*(X,Z_{(p)})$.*

Finally we mention a result on $K_*(X,Z_{(p)})$ to be developed in a separate paper, where $K_*(\ )$ denotes $Z/2Z$-graded unitary K-homology theory. As a $Z/2Z$-graded Hopf algebra $K_*(X,Z_{(p)})\cong E(PK_1(X,Z_{(p)}))\otimes B$ where $E(PK_1(X,Z_{(p)}))$ is the exterior algebra generated by the primitive classes lying in $K_1(X,Z_{(p)})$ and $B=K_*(X,Z_{(p)})/I$ with I the ideal generated by $K_1(X,Z_{(p)})$. If $H_{2i+1}(X,Z_{(0)})=0$ for all i, as in all the examples mentioned above, slightly abusing notations we have $K_0(X,Z_{(p)})=B$ and $K_1(X,Z_{(p)})=0$. Now given $H_*(X,Z/pZ)$, we have shown that the ring $H_*(X,Z_{(p)})$ is determined. In Sect. 3 we will give conditions which ensure that $H_*(X,Z_{(p)})$ determines the ring $K_*(X,Z_{(p)})$. If p is odd and $X=\Omega Y$ as above, these conditions are sufficiently general to include all known examples. To put it a little more precisely, the standard spectral sequence $H_*(X,Z_{(p)})\Rightarrow K_*(X,Z_{(p)})$ collapses and we investigate the multiplicative extension problem. The general result at the heart of this is as follows.

Theorem 1.4. *The nilradical of $B\otimes Z/pZ$ is zero (that is, $\bar{x}^k=0$ for $k>0$ implies that $\bar{x}=0$).*

A simple example illustrates this. Let X be the Eilenberg-MacLane space $K(Z,2)$. It is well known that $H_*(X,Z)\cong\Gamma(x)$ where x is a 2-dimensional class. Also $K_0(X,Z)\cong\Delta(x)$ where $x\in\tilde{K}_2(X,Z)\cong H_2(X,Z)$ and $\Delta(x)\subset Z_0[x]$ is the subring generated over Z by classes 1 and x_r, $0\leqq r\leqq\infty$ with $x_r=\dfrac{x(x-1)\ldots(x-r)}{(r+1)!}$, [2]. Now $H_*(X,Z/pZ)$ is a tensor product of polynomial algebras all truncated at height p on generators $\overline{(p^r!)^{-1}x^{p^r}}$. In $K_0(X,Z/pZ)\cong K_0(X,Z)\otimes Z/pZ$ one finds that

$$\bar{x}=\bar{x}^p=\bar{x}^{p^2}=\ldots,\ \bar{x}_1=\bar{x}_1^p=\bar{x}_1^{p^2}=\ldots,\quad \text{etc.}$$

For a general X with $H_{2i+1}(X,Z/pZ)=0$ for all i, $H_*(X,Z/pZ)$ deviates from being a polynomial algebra by truncations, using Borel's Theorem. Theorem 1.4 shows that in K-theory the deviation from being a free commutative algebra is always more complicated.

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2. The Proof of Theorem 1.1(a)

Theorem 1.1 of [6] implies Theorem 2.1 below.

Theorem 2.1. *$A\cong\{\otimes B(2s+1)\}\otimes\{\otimes C(2r)\}$ as a Hopf algebra where s runs over all non negative integers, $B(2s+1)$ is a sub-Hopf algebra of A which is an exterior algebra generated by classes of dimension $2s+1$, r runs over all positive integers prime to p and $C(2r)$ is a sub-Hopf algebra of A which as an algebra is generated by classes with dimensions $2rp^{\alpha}$, $\alpha\geqq 0$.*

We apply this theorem with $A=H_*(X,Z_{(p)})$ and can now outline the proof of Theorem 1.1(a). We concentrate on the sub-Hopf algebras $C(2r)$. We will assume that each $C(2r)$ is isomorphic as an algebra to a tensor product of quasi-monogenic sub-algebras of A in dimensions less than $2rp^{\gamma}$ for some γ. It is clear that we may assume that $\gamma>1$ by Borel's Theorem applied to $C(2r)\otimes Z/pZ$. Corollary 2.5 below implies that each quasi-monogenic algebra is of the form $\Gamma(x,t)$ through the appropriate dimensional range. Proposition 2.8 below shows that $t=0$ unless the dimension of x is $2r$. We then show that γ can be increased to $\gamma+1$ and so by induction each $C(2r)$ is isomorphic to a tensor product of algebras $\Gamma(x,t)$.

Complex $Z/2Z$ graded K cohomology theory is used in the proof. In this paper the more elaborate notations of [7] will not be employed being reserved for finer results. In the notation of Theorem 2.1 with $A=H_*(X,Z_{(p)})$, we have nothing to say about $\otimes B(2r+1)$. As taking quotients by the ideal generated by odd dimensional elements in both cohomology and K-theory causes no problems, we will assume for notational simplicity that $H_{2i+1}(X,Z_{(p)})=0$ for all i. Then $K(X,Z_{(p)})$ is a filtered ring given the CW-filtration with associated graded ring $\oplus H^{2i}(X,Z_{(p)})$, where $K(X,Z_{(p)})_{2i}/K(X,Z_{(p)})_{2i+1}\cong H^{2i}(X,Z_{(p)})$. Also $K(X,Z_{(p)})$ is canonically isomorphic to a direct sum $\oplus K^{\alpha}(X,Z_{(p)})$, $1\leqq\alpha\leqq p-1$ and $K^{\beta}(X,Z_{(p)})$. $K^{\gamma}(X,Z_{(p)})\subset K^{[\beta+\gamma]}(X,Z_{(p)})$ where $[\beta+\gamma]+s(p-1)=\beta+\gamma$. The filtration induced on $K^{\alpha}(X,Z_{(p)})$ has as associated graded group $\oplus H^{2\alpha+2i(p-1)}(X,Z_{(p)})$, $i\geqq 0$. The following standard properties of the Adams operators ψ^p: $K(X,Z_{(p)})\to K(X,Z_{(p)})$ will be used, sometimes without further comments, see [1 and 3].

(a) ψ^p is a natural filtration preserving ring homomorphism;

(b) $\psi^p(x)=x^p \bmod p$ for all x;

(c) $\psi^p(K^{\alpha}(X,Z_{(p)}))\subset K^{\alpha}(X,Z_{(p)})$;

(d) If $\xi\in K^{\alpha}(X,Z_{(p)})$ has exact filtration $2n$ where $\alpha=n \bmod(p-1)$, then there exist $\xi_i\in K^{\alpha}(X,Z_{(p)})$ of filtration $2n+2i(p-1)$ such that

$$\psi^p\xi=p^n\xi+p^{n-1}\xi_1+\ldots+p\xi_{n-1}+\xi_n.$$

Further if ξ has image $x\in H^{2n}(X,Z_{(p)})$ and ξ_i has image $x_i\in H^{2n+2i(p-1)}(X,Z_{(p)})$, then $P^i\bar{x}=\bar{x}_i$ where $P^i\colon H^{2n}(X,Z/pZ)\to H^{2n+2i(p-1)}(X,Z/pZ)$ is the cyclic reduced power or Sq^{2i}.

The homomorphisms induced by an H-space p-th power map with respect to a fixed though arbitrary order of association are denoted by $T\colon A\to A$, $T^*\colon A^*\to A^*$ where $A^*=H^*(X,Z_{(p)})$ and $t\colon K(X,Z_{(p)})\to K(X,Z_{(p)})$. Then T^* is the graded homomorphism associated with t, slightly abusing notations. Greek letters ξ,η,ζ, etc. will be used to denote elements of $K(X,Z_{(p)})$ always assumed to lie in a single $K^q(X,Z_{(p)})$. If ξ has exact filtration $2n$ and image $x\in H^{2n}(X,Z_{(p)})$, we write $\xi\to x$. A bar over x will denote reduction mod p so that $\bar{x}\in H^{2n}(X,Z/pZ)$ and we write $x\to\bar{x}$. If both $\xi\to x$ and $x\to\bar{x}$, we write $\xi\to\bar{x}$. Where just $\bar{x}$ or x has been defined, the undefined ξ and x in $x\to\bar{x}$, $\xi\to\bar{x}$ and $\xi\to x$ will denote any choices as above.

Recall from Proposition 2.1 of [6] that for $x\in A^*$, $T^*(x)=px+py+z^p$ for some y and z where y is decomposable.

Proposition 2.2. *Let $x\in H^{2np}(X,Z_{(p)})$, $T^*(x)=px+py+z^p$ and $\bar{x}^p\neq 0$. Then there exists $w\in H^{2np^2}(X,Z_{(p)})$ such that $T^*(w)=x^p+y^p \bmod p$.*

Proof. Let $\xi\to x$. By properties (b) and (d) of the Adams operators $\psi^p(\xi)=p^{np}\xi+\ldots+p^2\xi_{np-2}+p\xi'_{np-1}+\xi^p$ where we have written ξ'_{np-1} rather than ξ_{np-1} because we will modify it. We work throughout the proof in the free $Z_{(p)}$-module, $K(X,Z_{(p)})/K(X,Z_{(p)})_{2np^2+2}$.

If $\xi'_{np-1}=p\xi''_{np-1}+\xi_{np}$ where $\xi_{np}\in K(X,Z_{(p)})_{2np^2}$, we set $\xi_{np-1}=p\xi''_{np-1}$ and obtain the expression $\psi^p(\xi)=p\xi_{np-1}+\xi^p+p\xi_{np} \bmod p^2$.

Otherwise $P^{np-1}\bar{x}=\bar{x}'_{np-1}\neq 0$ where $\xi\to\bar{x}$ and $\xi'_{np-1}\to\bar{x}'_{np-1}$. The Adem relation $P^1P^{np-1}=0$ implies that $P^1\bar{x}'_{np-1}=0$. Therefore

$$\psi^p(\xi'_{np-1})=p^{np^2-p+1}\xi'_{np-1}+p^{np^2-p+1}\xi'_{np}$$

where $\xi'_{np}\in K(X,Z_{(p)})_{2np^2}$. Routine computations confirm that

$$\psi^p(\xi'_{np-1}+(1-p^{p-1})^{-1}\xi'_{np})=p^{np^2-p+1}(\xi'_{np-1}+(1-p^{p-1})^{-1}\xi'_{np}).$$

We set $\xi_{np-1}=\xi'_{np-1}+(1-p^{p-1})^{-1}\xi'_{np}$ and $\xi_{np}=-(1-p^{p-1})^{-1}\xi'_{np}$ and thus obtain as before $\psi^p(\xi)=p\xi_{np-1}+\xi^p+p\xi_{np} \bmod p^2$ but where now $\psi^p(\xi_{np-1})=p^{np^2-p+1}\xi_{np-1}$.

Let $\eta\to y$ and $\zeta\to z$ and as $T^*(x)=px+py+z^p$, we must have $t(\xi)=p\xi+p\eta+\zeta^p+\chi$ where $\chi\in K(X,Z_{(p)})_{2np+2}$. We consider the equation

$$\psi^p(t(\xi))=pt(\xi_{np-1})+t(\xi^p)+pt(\xi_{np}) \bmod p^2.$$

The left hand side equals $p\xi^p+p\eta^p+\zeta^{p^2} \bmod p^2$. As $t(\xi^p)=\zeta^{p^2} \bmod p$, we then have

$$t(\xi_{np-1})+t(\xi_{np})=\xi^p+\eta^p \bmod p.$$

When $\xi_{np-1}=0 \bmod p$ as in the first situation discussed above, this implies that $t(\xi_{np})=\xi^p+\eta^p \bmod p$ and lifting to the associated graded ring we obtain $T^*(w)=x^p+y^p \bmod p$ where $\xi_{np}\to w$. In the second case we must consider $t(\xi_{np-1})$ in more detail.

In the Hopf algebra A^* (or A), we will write $\psi^{(p)}: A \to A \otimes A \otimes \ldots \otimes A$, p factors, for the iterated comultiplication $(\psi \otimes 1 \otimes \ldots \otimes 1) \ldots (\psi \otimes 1)\psi$, which must not be confused with ψ^p, the Adams operator. Given $x \in \bar{A}^*$, we then have $\psi^{(p)}(x) = x \otimes 1 \otimes \ldots \otimes 1 + \ldots + 1 \otimes \ldots \otimes 1 \otimes x + \sum x_{i_1} \otimes x_{i_2} \otimes \ldots \otimes x_{i_p} + z_1 \otimes z_1 \otimes \ldots \otimes z_1$, where in the summation not all of the $\bar{x}_{i_j}$ are equal in any non zero $\bar{x}_{i_1} \otimes \bar{x}_{i_2} \otimes \ldots \otimes \bar{x}_{i_p}$ all permutations of the subscripts associated with distinct $\bar{x}_{i_j}$ will occur and so the number will be a multiple of p. For later use we introduce the notation

$$\psi^{(p)}(x) = U(x) + S(x) \tag{2.1}$$

where $S(x) = z_1 \otimes z_1 \otimes \ldots \otimes z_1$. Of course $S(x)$ is not well defined but $S(x) \bmod p$ is unique and this is all that is needed. We shall not prove this here as a proof is implicit in later work and it suffices to take a particular choice of $S(x)$.

We return to $\bar{x} \in H^{2np}(X, Z/pZ)$. Then in the explicit notations introduced above

$$\begin{aligned}\psi^{(p)} P^{np-1} \bar{x} &= P^{np-1} \bar{x} \otimes 1 \otimes \ldots \otimes 1 + \ldots + 1 \otimes \ldots \otimes 1 \otimes P^{np-1} \bar{x} \\ &+ P^{np-1}(\textstyle\sum \bar{x}_{i_1} \otimes \bar{x}_{i_2} \otimes \ldots \otimes \bar{x}_{i_p}) + \{P^{n-1} \bar{z}_1 \otimes \bar{z}_1^p \otimes \ldots \otimes \bar{z}_1^p \\ &+ \ldots + \bar{z}_1^p \otimes \bar{z}_1^p \otimes \ldots \otimes P^{n-1} \bar{z}_1\}.\end{aligned}$$

To obtain the last term in parenthesis, we have used the Cartan formula and unstable properties of the cyclic reduced powers. Now $\xi_{np-1} \to P^{np-1} \bar{x}$ and so if

$$\begin{aligned}\xi_{np-1} \to x_{np-1},\ \psi^{(p)}(x_{np-1}) &= x_{np-1} \otimes 1 \otimes \ldots \otimes 1 + \ldots + 1 \otimes \ldots \otimes 1 \otimes x_{np-1} \\ &+ P^{np-1}(\textstyle\sum x_{i_1} \otimes \ldots \otimes x_{i_p}) + \{P^{n-1} z_1 \otimes z_1^p \otimes \ldots \otimes z_1^p \\ &+ \ldots + z_1^p \otimes \ldots \otimes z_1^p \otimes P^{n-1} z_1\} + p \textstyle\sum y_{i_1} \otimes y_{i_2} \otimes \ldots \otimes y_{i_p}.\end{aligned}$$

Here by an expression like $P^{n-1} z_1$ is meant an element whose reduction mod p is $P^{n-1} \bar{z}_1$ and $\sum y_{i_1} \otimes y_{i_2} \otimes \ldots \otimes y_{i_p}$ is a symmetric expression in which for dimensional reasons not all the y_{i_j} in any summand are equal. Therefore

$$T^*(x_{np-1}) = p x_{np-1} + p P^{np-1} w + p z_1^{p(p-1)} P^{n-1} z_1 \bmod p^2$$

and

$$t(\xi_{np-1}) = p \xi_{np-1} + p\phi + p \zeta_1^{p(p-1)} \theta + \sigma \bmod p^2,$$

where

$$pw = \textstyle\sum x_{i_1} x_{i_2} \ldots x_{i_p},\ \phi \to P^{np-1} \bar{w},\ \zeta_1 \to z_1,\ \theta \to P^{n-1} \bar{z}$$

and $\sigma \in K(X, Z_{(p)})_{2np^2}$. As $t(\xi_{np-1}) = \sigma \bmod p$, it is σ which must be investigated. We consider the equation

$$\psi^p(t(\xi_{np-1})) = p^{np^2-p+1} t(\xi_{np-1}) \bmod p^{np^2-p+2},$$

using $\psi^p(\xi_{np-1}) = p^{np^2-p+1} \xi_{np-1}$. The left hand expression is $\psi^p(p\xi_{np-1} + p\phi + p\zeta_1^{p(p-1)}\theta + \sigma)$ and $\psi^p(p\xi_{np-1})$ and $\psi^p(\sigma)$ give zero contribution. As $\phi \to P^{np-1} \bar{w}$ and $P^1 P^{np-1} = 0$, $\psi^p(\phi) = 0 \bmod p^{np^2-p+1}$ and again $\psi^p(p\phi)$ gives zero contribution. As $\theta \to P^{n-1} \bar{z}_1$ and $P^1 P^{n-1} \bar{z}_1 = n P^n \bar{z}_1 = n \bar{z}_1^p$, $\psi^p(p\zeta_1^{p(p-1)}\theta)$

$=np^{np^2-p+1}\zeta_1^{p^2} \bmod p^{np^2-p+2}$. Equating this to the right hand side of the equation gives

$$np^{np^2-p+1}\zeta_1^{p^2}=p^{np^2-p+1}\sigma \bmod p^{np^2-p+2} \text{ or } \sigma=n\zeta_1^{p^2} \bmod p.$$

But $z_1^p=z^p$ and so $\sigma=n\zeta^{p^2} \bmod p$. Returning to the equation $t(\xi_{np-1})+t(\xi_{np})=\xi^p+\eta^p \bmod p$, we deduce that $t\xi_{np}=\xi^p+\eta^p-n\zeta^{p^2} \bmod p$ and so $T^*(x_{np})=x^p+y^p-nz^{p^2} \bmod p$, where $\xi_{np}\to x_{np}$. Let $w=x_{np}+nx^p$. Then $T^*(w)=x^p+y^p \bmod p$ as required. This completes the proof of Proposition 2.2.

Proposition 2.2 is the key result in the proof of Theorem 2.3.

Theorem 2.3. *Let C and D be sub-Hopf algebras of $A=H_*(X,Z_{(p)})$ such that the tensor product of the inclusions i: $C\otimes D\to A$ is a Hopf algebra morphism and an isomorphism in dimensions $\leqq 2np$. Let $\bar{u}\in C\otimes Z/pZ$ be a non zero primitive and $\bar{v}\in D\otimes Z/pZ$, both of dimension $2np$. Then $(\bar{u}+\bar{v})^p\neq 0$ in $A\otimes Z/pZ$.*

Before commencing the proof, we note two corollaries.

Corollary 2.4. *Let $\bar{u}\in A\otimes Z/pZ$ be a non zero primitive of dimension $2np$. Then $\bar{u}^p\neq 0$.*

Corollary 2.5. *Let $\bar{u}\in A\otimes Z/pZ$ have finite height (that is $\bar{u}^k=0$ for some k). Then $\bar{u}^p=0$.*

Corollary 2.4 follows immediately from the theorem setting $D=Z_{(p)}$ so that $\bar{v}=0$. To prove Corollary 2.5 one notes that if $\bar{u}$ is an element of least possible dimension with $\bar{u}^{p^k}=0$ but $\bar{u}^{p^{k-1}}\neq 0$ and $k>2$, then $\bar{u}^{p^{k-1}}$ is primitive. This contradicts Corollary 2.4.

Corollary 2.5 was first proved as Theorem 4.2 of [8]. In fact in [8] it is suggested that Theorem 1.1(a) is a simple consequence of other results proved there, but as that author himself later pointed out, this is not obvious.

We now begin the proof of Theorem 2.3 and use the notation of (2.1).

Lemma 2.6. *Let $\bar{x}\in A^*\otimes Z/pZ$ and $\bar{x}^p=0$. Then*

$$S(p^{-1}x^p)=x\otimes x\otimes\ldots\otimes x \bmod p \quad \text{and} \quad U(p^{-1}x^p)=\sum y_{i_1}\otimes y_{i_2}\otimes\ldots\otimes y_{i_p} \bmod p$$

where in each term of the summation some $\bar{y}_{i_j}$ is decomposable and $x\to\bar{x}$.

To prove the lemma one chooses a Borel basis for $A^*\otimes Z/pZ$ which is then lifted to a multiplicative basis for A^*. After writing out $\psi^{(p)}(x)$ in terms of this multiplicative basis, the result follows by routine computations.

In the notation of the statement of Theorem 2.3, let $u\in C$, $v\in D$ where $u\to\bar{u}$, $v\to\bar{v}$ and choose an $x\in C^*$ such that $\langle x,u\rangle=1$. Then $\langle x,v\rangle=0$. We suppose first that $\bar{x}^p=0$. Then

$$\begin{aligned}\langle p^{-1}x^p,u^p\rangle&=\langle\psi^{(p)}(p^{-1}x^p),u\otimes u\otimes\ldots\otimes u\rangle\\&=\langle S(p^{-1}x^p)+U(p^{-1}x^p),u\otimes u\otimes\ldots\otimes u\rangle\\&=\langle S(p^{-1}x^p),u\otimes u\otimes\ldots\otimes u\rangle \bmod p\end{aligned}$$

using Lemma 2.6 and the fact that primitive elements annihilate decomposable elements mod p,

$$= \langle x, u \rangle^p \bmod p$$

again using Lemma 2.6

$$= 1 \bmod p.$$

Also

$$\langle p^{-1} x^p, v^p \rangle = \langle \psi^{(p)}(p^{-1} x^p), v \otimes v \otimes \ldots \otimes v \rangle$$
$$= 0$$

as $x \in C^*$.

Thus $\langle p^{-1} x^p, u^p + v^p \rangle = 1 \bmod p$ and $(\bar{u} + \bar{v})^p \neq 0$.

Hence if $x \to \bar{x}$, we can assume that $\bar{x}^p \neq 0$ and so by Proposition 2.2, there exists w such that $T^*(w) = x^p + y^p \bmod p$ with y decomposable. Calculations similar to those given above yield that $\langle x^p, u^p \rangle = p \bmod p^2$, $\langle x^p, v^p \rangle = 0$ and also $\langle y^p, v^p \rangle = 0$ as $y \in C^*$. In addition

$$\langle y^p, u^p \rangle = \langle y \otimes y \otimes \ldots \otimes y, \psi^{(p)}(u^p) \rangle$$
$$= \langle y \otimes y \otimes \ldots \otimes y, pu \otimes u \otimes \ldots \otimes u \rangle \bmod p^2$$

as $\bar{u}$ is primitive

$$= 0 \bmod p^2.$$

as y is decomposable.

Lemma 2.7. *Let $S(v) = r \otimes r \otimes \ldots \otimes r$, where $\bar{v}^p$ is primitive. Then $\bar{r}^p = 0$.*

The proof of Lemma 2.7 is a routine exercise starting with a Borel basis for $A \otimes Z/pZ$.

We can now complete the proof of Theorem 2.3. If $(\bar{u} + \bar{v})^p = 0$, as $\bar{u}$ is primitive, $\bar{v}^p$ is primitive and so by Lemma 2.7, $T(v) = 0 \bmod p$ and so $T(v^p) = 0 \bmod p^2$. Now if $p^{-1}(u^p + v^p)$ exists in A, $1 = \langle x^p + y^p, p^{-1}(u^p + v^p) \rangle \bmod p$ by the calculations given before Lemma 2.7. Hence

$$1 = \langle T^*(w), p^{-1}(u^p + v^p) \rangle \bmod p$$
$$= \langle w, p^{-1} T(u^p + v^p) \rangle \bmod p$$
$$= 0 \bmod p.$$

This contradiction implies that $(\bar{u} + \bar{v})^p \neq 0$, as required.

We return to the consideration of $C(2r)$ for a fixed integer r. Let $B(\gamma)$ be the sub-Hopf algebra of $C(2r)$ generated by elements with degrees $< 2rp^\gamma$ and suppose that $B(\gamma)$ decomposes as an algebra as a tensor produce of quasi-monogenic algebras. We may assume that $\gamma \geqq 2$ by lifting a Borel basis of $C(2r) \otimes Z/pZ$ to $C(2r)$. It follows from Corollary 2.5 that $B(\gamma)$ is isomorphic to a tensor product of algebras of type $\Gamma(x, s)$. We write $B(\gamma)$ as $\{\otimes \Gamma(x_i, s_i)\} \otimes \{\otimes \Gamma(y_j, t_j)\}$ where each x_i, $1 \leqq i \leqq \alpha$, has dimension $2r$ and each y_j has dimension $2rp^\beta$ with $\beta > 0$.

Proposition 2.8. *$t_j = 0$ for each j.*

The proof may be clarified if we first give a simple algebraic example for $p = 2$ which illustrates the germ of the problem which must be overcome in the

proof. Suppose that $C(2r)=\Gamma(x_{2r},1)\otimes\Gamma(y_{4r},1)$ where the comultiplication is well defined by setting $\psi(y_{4r})=y_{4r}\otimes 1+x_{2r}\otimes x_{4r}+1\otimes y_{4r}$. This possibility cannot be eliminated from the conclusions of Corollaries 2.4 and 2.5. It does however contradict Theorem 2.3, for $\bar{y}_{4r}=(\bar{y}_{4r}-\overline{2^{-1}x_{2r}^2})+\overline{2^{-1}x_{2r}^2}$ and $\bar{y}_{4r}-\overline{2^{-1}x_{2r}^2}$ is primitive. We set C to be the sub-Hopf algebra generated by $y_{4r}-2^{-1}x_{2r}^2$ and D to be the sub-Hopf algebra $\Gamma(x_{2r},1)$. Then $C\otimes D$ is isomorphic to $C(2r)$ in dimensions $\leqq 4r$ and setting $\bar{u}=\bar{y}_{4r}-\overline{2^{-1}x_{2r}^2}$ and $\bar{v}=\overline{2^{-1}x_{2r}^2}$, we deduce that $\bar{y}_{4r}^2=(\bar{u}+\bar{v})^2$ is non zero in $C(2r)$.

We return to the proof of Proposition 2.8. If $s_i=0$ for $1\leqq i\leqq\alpha$, then any y_j of least possible dimension with $y_j^p=0 \bmod p$ must be primitive. This contradicts Corollary 2.4 and so $B(\gamma)$ is a polynomial algebra.

Thus we can assume that some of the s_i are greater than zero. We shall suppose that Proposition 2.8 is false so that there exists a quasi-generator $y=y_i$ of $B(\gamma)$ such that $\bar{y}^p=0$, using Corollary 2.5. We choose y of least possible dimension $2rp^\delta$ where $\delta+1<\gamma$. In $\Gamma(x_i,s_i)$ with $s_i>0$, write z_i for $p^{-v(s_i-1)}x_i^{p^{s_i}}$. We will show that there exists r with $\bar{r}^p=0$ such that $y+\sum\lambda_i z_i+r \bmod p$ is primitive.

Lemma 2.9. *Let $w\in A$ and suppose that $S(w)=0 \bmod p$. Then there exists a decomposable element r such that $\bar{w}+\bar{r}$ is primitive.*

Proof. We can assume that $\bar{w}$ is indecomposable for otherwise the lemma has no content. In the exact sequence

$$0\to P(K(\xi(A\otimes Z/pZ)))\to P(A\otimes Z/pZ)\to Q(A\otimes Z/pZ)$$

$$\xrightarrow{i} Q(K(\lambda(A\otimes Z/pZ)))\to 0$$

[11, Proposition 4.23], we wish to show that $i(\bar{w})=0$. If $i(\bar{w})\neq 0$, choose $\bar{x}\in P(\xi A^*\otimes Z/pZ)$ with $\langle\bar{x},i(\bar{w})\rangle=1$ and let j: $P(\xi A^*\otimes Z/pZ)\to P(A^*\otimes Z/pZ)$ be the inclusion. Then $\langle\bar{x},i(\bar{w})\rangle=\langle j(\bar{x}),\bar{w}\rangle$. But $j(\bar{x})$ is primitive and decomposable and so equals $\bar{y}^p$ for some $\bar{y}$. However

$$\langle\bar{y}^p,\bar{w}\rangle=\langle\bar{y}\otimes\bar{y}\otimes\ldots\otimes\bar{y},\psi^{(p)}\bar{w}\rangle$$

$$=\langle\bar{y}\otimes\bar{y}\otimes\ldots\otimes\bar{y},\overline{U(w)}\rangle=0.$$

This contradiction completes the proof.

We now choose the quasi-generators y_i of $B(\delta+1)$ so that as many as possible are primitive mod p, this number being the sum of the dimensions of the images of $P(B(\delta+1)\otimes Z/pZ)\to Q(B(\delta+1)\otimes Z/pZ)$ through the appropriate range of dimensions.

Corollary 2.10. *If y_j has dimension less than $2rp^\delta$ and $y_j^{p^s} \bmod p$ is primitive in $B(\delta+1)$, then $\bar{y}_j$ is primitive.*

Proof. Let $S(y_j)=v\otimes v\otimes\ldots\otimes v$. Since $y_j^{p^s}$ is primitive, $T(y_j^{p^s})=0 \bmod p$. Thus $v^{p^{s+1}}=0 \bmod p$ and so $v^p=0 \bmod p$ by Corollary 2.5. By Lemma 2.6, $S(p^{-1}v^p)=v\otimes v\otimes\ldots\otimes v \bmod p$. Thus $S(y_j-p^{-1}v^p)=0 \bmod p$. Lemma 2.9 implies that

there is a decomposable element r such that $y_j - p^{-1}v^p + r$ is primitive mod p. But by choice of the quasi-generators this implies that y_j was primitive.

We consider y of dimension $2rp^\delta$. As $\bar{y}^p = 0$, it is primitive and so arguing as in the proof of Corollary 2.10, there exists an indecomposable $z = \sum \lambda_i z_i$ and a decomposable r such that $y + z + r \bmod p$ is primitive. Without loss of generality, by rechoosing the quasi-generators in dimension $2r$, we can assume that $z = z_1$ lies in a single $\Gamma(x_1, \delta)$. Now $y^p + z^p + r^p \bmod p$ is primitive and as $\bar{y}^p = 0$ and $\bar{z}^p$ is primitive, it follows that $\bar{r}^p$ is primitive. If $\bar{r}^p \neq 0$, it follows from Corollary 2.10 and the choice of quasi-generators that $\bar{r}^p = \sum \lambda_i \bar{y}_i^{p^{\alpha_i}} + \sum \mu_j z_j^{\beta_j}$ where $\alpha_i > 1$ and $\beta_j > 1$. Thus we can replace r by $r - \sum \lambda_i \bar{y}_i^{p^{\alpha_i - 1}} - \sum \mu_j z_j^{\beta_j - 1}$ so that $\bar{r}^p = 0$ and $\bar{y} + \bar{z} + \bar{r}$ is still primitive. Finally we can replace $\bar{y}$ by $\bar{y} + \bar{r}$ and we are now in a position similar to that of the introductory example before the proof where $\bar{y}$ is non primitive with $\bar{y}^p = 0$ and $\bar{y} + \bar{z}$ is primitive.

We wish to apply Theorem 2.3 with $D = \Gamma(x_1, \delta)$ and $C = \{\bigotimes \Gamma(x_i, s_i)\} \otimes \{\bigotimes \Gamma(y_i, 0)\} \otimes \Gamma(y + z, 0) \otimes \{\bigotimes C(2s)\}$ where $s \neq r$, where y_i are the quasi-generators other than y of dimension less than or equal to that of y. Therefore we must ensure that C is a sub-Hopf algebra and this involves modifying the quasi-generators once more by adding to them elements from the ideal generated by $\Gamma(x_1, \delta)$. We define new y_i inductively as dimensions increase and replace the corresponding factor in $C(2r)$ by the new $\Gamma(y_i, 0)$. Assume that the quasi-generators $\{y_j\}$, $j \in J$ with dimensions less than that of a particular y_i and those of the same dimension which occur earlier in some ordering are such that $C' = \{\bigotimes \Gamma(x_i, s_i)\} \otimes \{\bigotimes \Gamma(y_j, 0)\}$ where $j \in J$ is a sub-Hopf algebra. We will show that there exists $t \in \mathscr{J}(\Gamma(x_1, \delta))$ such that when $y_i' = y_i + t$, $\psi(y_i') - y_i' \otimes 1 - 1 \otimes y_i' \in C' \otimes C'$. This will follow if $\psi^{(p)}(y_i') - y_i' \otimes 1 \otimes \ldots \otimes 1 - \ldots - 1 \otimes 1 \otimes \ldots \otimes y_i' \in C' \otimes C' \otimes \ldots \otimes C'$ which in turn is a consequence of $T(y_i') - p y_i' \in C'$. Let $T(y_i) = p y_i + s + p r_1 + r_2^p$ where $s \in C'$ and r_1 and r_2 lie in $\mathscr{J}(\Gamma(x_1, \delta))$. The structure of $\Gamma(x_1, \delta)$ implies that $\bar{r}_2^p = 0$ and so $T(y_i) = p y_i + s + p t_1$ with $t_1 \in \mathscr{J}(\Gamma(x_1, \delta))$. We follow the idea of the proof of Lemma 2.2 of [6]. Suppose that $t_1 \neq 0 \bmod \bar{A}^{u+1}$ but $t_1 = 0 \bmod \bar{A}^u$. Then $T(t_1) = p^u t_1 + p t_2$ with $t_2 \in \mathscr{J}(\Gamma(x_1, \delta))$ and $t_2 = 0 \bmod \bar{A}^{u+1}$. It follows that

$$T(y_i + (1 - p^{r-1})^{-1} t_1)$$
$$= p(y_i + (1 - p^{r-1})^{-1} t_1) + s + (1 - p^{r-1})^{-1} p t_2.$$

After a finite number of steps, we deduce that there exists $t \in \mathscr{J}(\Gamma(x_1, \delta))$ such that $T(y_i + t) = p(y_i + t) + s$, as required. In this way we may assume that $\{\bigotimes \Gamma(x_i, s_i)\} \otimes \{\bigotimes \Gamma(y_i, 0)\}$ is a sub-Hopf algebra. We now repeat the argument for $y + z$. As $\bar{y} + \bar{z}$ is primitive, in this case t will be divisible by p and so to replace y by $y + t$ does not affect the basic properties that $(\bar{y} + \bar{t})^p = 0$ and $\bar{y} + \bar{t} + \bar{z}$ is primitive.

We apply Theorem 2.3 with C and D as above, $u = y + z$ and $v = -z$. We deduce that $\bar{y}^p = (\bar{u} + \bar{v})^p \neq 0$. This contradiction establishes Proposition 2.8.

The Proof of Theorem 1.1(a). We have proved nothing about the quasi-generators $\{y_k\}$ of dimension $2rp^{\nu-1}$. It is now convenient to assume that as many as possible of the y_k are chosen with $\bar{y}_k^p = 0$. With this modification $B'(\nu) = \{\bigotimes \Gamma(x_i, s_i)\} \otimes \{\bigotimes \Gamma(y_j, 0)\} \otimes \{\bigotimes \Gamma(y_k, 1)\}$ where each $\dim y_k = 2rp^{\nu-1}$ is again a

sub-Hopf algebra of $C(2r)$. If $B(\gamma+1)$ is not isomorphic as an algebra to a tensor product of quasi-monogenic algebras, there exists $u\in C(2r)_{2rp^\gamma}$, $u\notin B'(\gamma)$ but $w=pu\in B'(\gamma)$. Clearly w is primitive and decomposable in $B'(\gamma)$ and so $\bar{w}=\sum\lambda_i\bar{y}_i^{p^{\alpha_i}}+\sum\mu_j\bar{z}_j^{p^{\beta_j}}$ where we can assume that each $\alpha_i>1$ by the choice of the $\{y_k\}$ of dimension $2rp^{\gamma-1}$. Thus by Corollary 2.5 some $\beta_j=1$ and by suitable choices of the quasi-generators in dimension $2r$ we can write $\bar{w}=\bar{z}^p+\sum\lambda_i\bar{y}_i^{p^{\alpha_i}}+\sum\mu_k\bar{z}_k^{p^{\gamma_k}}$, where $\alpha_i>1$ and $\gamma_k>1$.

Let $x=z+\sum\lambda_i y_i^{p^{\alpha_i-1}}+\sum\mu_k z_k^{p^{\gamma_s-1}}$. We apply Theorem 2.3 as in the final stages of the proof of Proposition 2.8. Using Corollary 2.10, we can assume that each $\bar{y}_i$ which occurs in the expression for $\bar{w}$ is primitive. We set $D=\Gamma(x_1,\delta)$ where $p^{-v(\delta-1)}x_1^{p^\delta}=z$ and C equal to the tensor product of the remaining $\Gamma(x_i,s_i)$, $\Gamma(y_k,0)$ and $C(2s)$, $s\neq r$. Let $u=\sum\lambda_i y_i^{p^{\alpha_i-1}}+\sum\mu_k z_k^{p^{\gamma_k-1}}$ and $v=z$. We deduce from Theorem 2.3 that $\bar{w}=\bar{x}^p=(\bar{u}+\bar{v})^p\neq 0$ in A. This contradiction implies that $B(\gamma+1)$ is isomorphic to a tensor product of quasi-monogenic algebras and so by induction on γ we have proved Theorem 1.1(a).

3. The Proof of Theorem 1.1(b)

If $p=2$, Theorem 1.1(b) has been proved in Proposition 2.8 and so throughout this section we assume that p is odd. We consider the vector space

$$W=PH^*(X,Z/pZ)/PH^*(X,Z/pZ)\cap\{\bar{H}^*(X,Z/pZ)^{p+1}+\text{Image } T^*\}$$

where $T^*\colon H^*(X,Z/pZ)\to H^*(X,Z/pZ)$ is induced from a p-th power map on the H-space as before. Informally W consists of indecomposable primitives of $A^*\otimes Z/pZ$ and those p-th powers which are not in the image of T^* modulo $\bar{H}^*(X,Z/pZ)^{p+1}$. We write $W=M\oplus N\oplus\xi(N)$ where $\xi(u)=u^p$ and M is the kernel of ξ restricted to $PH^*(X,Z/pZ)$. If $n\in N$, then $n^p\neq 0$ in W. If V is a subspace of W or $b\in W$, we write V_N, $b_{\xi(N)}$, etc., for the component in the relevant summand of $M\oplus N\oplus\xi(N)$.

Proposition 3.1. $N_{2r}=0$ *unless* r *is binary.*

This clearly implies Theorem 1.1(b) by taking duals. The proof of Proposition 3.1 is completed after Lemma 3.6.

Lemma 3.2. (a) W *is an unstable module over* $\mathscr{A}(p)$, *the* mod p *Steenrod algebra.*

(b) $N_{2pr}=0$.

Proof. The proof of (a) is clear. For (b), let $x\in A^*=H^*(X,Z_{(p)})$ represent $\bar{x}\in N_{2pr}$ and $T^*(x)=px+py$ where y is decomposable. Then by Proposition 2.2, there exists $w\in A^*$ with $T^*(w)=x^p+y^p$ mod p and so $\bar{x}^p\in\bar{H}^*(X,Z/pZ)^{p+1}+\text{Image } T^*$.

Proposition 3.3. (a) $N_{2r}=0$ *unless* $r=1$ mod p.

(b) $\xi(N)\subset P^1(W)$, *but* $P^2(W)_{\xi(N)}=0$.

(c) *If* $\bar{y}\in W$ *and* $(P^r\bar{y})_{\xi(N)}\neq 0$, *then either* $r=1$ mod p *or* $r=ps$ *and* $(P^r\bar{y})_{\xi(N)}\in\xi(P^s(N))$.

Proof. If $\bar{x}\in N$, then $\bar{x}^p\in\text{Image } P^1$ by Lemma 3.2(b) and the Adem relation $P^1P^{n-1}=nP^n$ where $\dim\bar{x}=2n$. Thus (a) will follow using the Adem relation

$P^1P^{n-2}=(n-1)P^{n-1}$ once it is proved that $\bar{x}^p\notin\text{Image}\,P^2$. We therefore prove that $P^2(W)_{\xi(N)}=0$ of (b) or equivalently if $\bar{z}\in W$ then $P^2\bar{z}\neq\alpha\bar{x}^p+\bar{y}$ for any $\bar{y}\in M\oplus N\alpha$ a unit and $\bar{x}\in N$ of dimension $2n$. In fact it is sufficient to contradict the assertion that $P^2\bar{z}=\bar{x}^p$ for the element $\bar{y}$ is irrelevant in the argument which follows and its presence would serve only to complicate notations; alternatively one could work with a suitable quotient of $K(X,Z_{(p)})$ so that $\bar{y}$ would not arise.

Therefore suppose that $P^2\bar{z}=\bar{x}^p$ and set $\bar{y}=2^{-1}P^1\bar{z}$ so that $P^1\bar{y}=\bar{x}^p$. For dimensional reasons we can choose primitive x,y and z in A^* with $x\to\bar{x}$, $y\to\bar{y}$ and $z\to\bar{z}$. Let $\xi\to x$, $\eta'\to y$ and $\zeta'\to z$. Since $P^1\bar{y}=\bar{x}^p$, $\psi^p(\eta')=p^{np-(p-1)}\eta'+p^{np-p}\xi^p+p^{np-(p-1)}w \bmod K(X,Z_{(p)})_{2np+2}$ where $w\in K(X,Z_{(p)})_{2np}$. We set $\eta=\eta'+(1-p^{p-1})^{-1}w$ so that $\eta\to y$ and

$$\psi^p(\eta)=p^{np-(p-1)}\eta+p^{np-p}\xi^p \bmod K(X,Z_{(p)})_{2np+2}.$$

Since $T^*(y)=py$ and $T^*(x)=px$, it follows that $t(\eta)=p\eta+w'$ with $w'\in K(X,Z_{(p)})_{2np}$ and $t(\xi^p)=p^p\xi^p$ both modulo $K(X,Z_{(p)})_{2np+2}$. We obtain from $\psi^p(t\eta)=t\psi^p(\eta)$ the equation

$$p^{np-(p-1)}(1-p^{p-1})w'=p^{np-(p-1)}(1-p^{p-1})\xi^p$$

or $w'=\xi^p$. Therefore

$$t(\eta)=p\eta+\xi^p \bmod K(X,Z_{(p)})_{2np+2}. \tag{3.1}$$

Now $T^*(z)=pz$ and so $t(\zeta')=p\zeta'+pr+s^p \bmod K(X,Z_{(p)})_{2np+2}$ where r is decomposable and both r and s^p have exact filtrations greater than that of ζ'. Repeating the type of argument used in Lemma 2.2 of [6] and discussed before Proposition 3.4 of [7], we can find $\zeta=\zeta'+w$ with $\zeta\to y$ and $t(\zeta)=p\zeta+\theta^p \bmod K(X,Z_{(p)})_{2np+2}$, where $\theta^p\in K(X,\zeta_{(p)})_{2np}$. Let $\psi^p(\zeta)=p^{np-2(p-1)}\zeta+p^{np-2p+1}\eta'$ and $\psi^k(\zeta)=k^{np-2(p-1)}\zeta+\eta''$, both modulo $K(X,Z_{(p)})_{2np-2p+4}$. We choose k to represent a generator in the group of units in Z/p^2Z so that $1-k^{p-1}$ and $1-k^{2(p-1)}$ are each divisible by p but not by p^2. Now $\psi^k(\eta')=k^{np-(p-1)}\eta'$ and $\psi^k(\eta'')=k^{np-(p-1)}\eta'' \bmod K(X,Z_{(p)})_{2np-2p+4}$, and so using the fact that $\psi^k\psi^p(\zeta)=\psi^p\psi^k(\zeta)$, we obtain the equation

$$\begin{aligned}p^{np-2(p-1)}\eta''+p^{np-2p+1}k^{np-(p-1)}\eta'&=k^{np-2(p-1)}p^{np-2p+1}\eta'\\&\quad+p^{np-(p-1)}\eta'' \bmod K(X,Z_{(p)})_{2np-2p+4}\end{aligned}$$

which gives

$$p^{np-2(p-1)}(1-p^{p-1})\eta''=p^{np-2p+1}k^{np-2(p-1)}(1-k^{p-1})\eta'.$$

Thus $\eta''=\alpha\eta' \bmod p$ with α a unit and as $P^1\bar{z}=2\bar{y}$, $\eta'=\beta\eta \bmod p$ with β a unit. Hence we can write

$$\begin{aligned}\psi^k(\zeta)&=k^{np-2(p-1)}\zeta+\alpha\beta\eta\\&\quad+\phi \bmod\{pK(X,Z_{(p)})+K(X,Z_{(p)})_{2np+2}\},\end{aligned}$$

where $\phi\in K(X,Z_{(p)})_{2np}$. We obtain from

$$\psi^k(t(\zeta))=t\psi^k(\zeta),\ \psi^k(\theta^p)=k^{np-2(p-1)}\theta^p+\alpha\beta\xi^p+t(\phi) \bmod p$$

and as $\psi^k(\theta^p)=k^{np}\theta^p$, this reduces to $t(-\phi)=\alpha\beta\xi^p \bmod p$. Thus $\bar{x}^p\in$Image T^* which contradicts the fact that $0\neq\bar{x}^p\in W$. Hence $\bar{x}^p\notin$Image P^2.

We now prove (c). Again no generality is lost in assuming that $P^r\bar{y}=\bar{x}^p$ with $\dim\bar{x}=2n$. If $r\neq 0 \bmod p$, then $r=1 \bmod p$ by (b) and the Adem relations. Therefore let $r=ps$ and choose $y\in A^*$ with $y\to\bar{y}$ and $\eta\to y$. The proof of (c) is similar to the proof of Proposition 2.2 but easier and so some details have been left to the reader. As $\bar{y}$ is primitive, $T^*(y)=py+p^2r+ps^p$. We write $\psi^p(\eta)=p^{np-(p-1)r}\eta+\ldots+p^{np-pr+1}\eta'_{r-1}+p^{np-pr}\xi^p$ using the fact that $P^r\bar{y}=\bar{x}^p$ and throughout the proof we work in $K(X,Z_{(p)})/K(X,Z_{(p)})_{2np+2}$. If $\eta'_{r-1}\to\bar{y}_{r-1}$, the Adem relation $P^1P^{r-1}=rP^r=0$ implies that $P^1\bar{y}_{r-1}=0$ and so we can change η'_{r-1} for $\eta_r\to\bar{y}_{r-1}$ such that

$$\psi^p(\eta)=p^{np-pr+1}\eta_{r-1}+p^{np-pr}\xi^p+p^{np-pr+1}\eta_r \bmod p^{np-pr+2}$$

where $\psi^p(\eta_{r-1})=p^{np-(p-1)}\eta_{r-1}$ and $\eta_r\in K(X,Z_{(p)})_{2np}$. We now compute $\psi^p(t(\eta))=t\psi^p(\eta) \bmod p^{np-pr+2}$. The fact that $T^*(y)=0 \bmod p$ makes this computation easier than in the corresponding step in the proof of Proposition 2.2. We deduce by analogous arguments that if $\eta_r\to y_r$, then $T^*(\bar{y}_r)=P^r\bar{y}+P^r(\bar{s}^p)=\bar{x}^p+(P^s\bar{s}^p)^p$. As $\bar{x}^p\neq 0$ in W, $\bar{s}$ must be indecomposable. It its dimension is not congruent to zero mod p, it can be assumed to be primitive and so setting $\bar{s}=-\bar{z}$, we have proved that in W $(P^s\bar{z})^p=\bar{x}^p$. If $\dim\bar{s}=0 \bmod p$, as $\bar{s}^p=0 \bmod p$, Proposition 2.2 implies that there exists w such that $T^*(\bar{w})=\bar{s}^p \bmod\{\bar{H}^*(X,Z/pZ)^{p+1}\}$. Hence

$$\bar{x}^p=T^*(\bar{y}_r)-(P^s\bar{s})^p\in PH^*(X,Z/pZ)\cap\{\bar{H}^*(X,Z/pZ)^{p+1}+\text{Image } T^*\},$$

which is false. Thus (c) is established.

Implicit in the last part of the proof is the following result.

Corollary 3.4. *We can choose N so that $P^{pq}(M\oplus N)_{\xi(N)}=0$ for all $q>0$.*

Proof. The latter part of the proof of Proposition 3.3(c) shows that it is sufficient to show that $M\oplus N$ has a basis $\{\bar{z}_k\}$ for which we can choose representatives $\{z_k\}$ in A^* with $z_k\to\bar{z}_k$ and $T^*(z_k)=pz_k+p^2r+ps^p$ where $\bar{s}^p\in\bar{H}^*(X,Z/pZ)^{p+1}$. We start with any basis $\{\bar{z}_k\}$ for $M\oplus N$ and let $T^*(z_k)=pz_k+p^2r+ps^p$. If there is a primitive $\bar{s}'$ with $\bar{s}'^p-\bar{s}^p\in\bar{H}^*(X,Z/pZ)^{p+1}$, we replace $\bar{z}_k$ by $\bar{z}_k+\bar{s}'^p$ so that $T^*(z_k+s'^p)=p(z_k+s'^p) \bmod p^2$. Otherwise we can assume that s is indecomposable and of a dimension divisible by p. By Proposition 2.2, there exists w such that $T^*(w)=s^p+t^p \bmod p$ where $T^*(s)=ps+pt+r^p$. We now replace z_k by z_k-pw which does not affect $\bar{z}_k$.

The Proof of Proposition 3.1. The proof is similar to that of the particular case given as Proposition 3.13 of [7] which follows from the sequence of results Lemma 3.10, Corollary 3.11 and Lemma 3.12 given there. We work exclusively in W and choose N to satisfy the conclusions of Corollary 3.4. Let $\{\bar{x}_i\}$ be a basis for N so that $\{\bar{x}_i^p\}$ is a basis for $\xi(N)$. If $\bar{w}\in W$, we write $[\bar{w},\bar{x}_i^p]$ for its component in the $\bar{x}_i^p$ direction in $\xi(N)$.

Lemma 3.5. *Suppose that* $q \neq v(u)$ *for any* u, $\bar{z} \in M \oplus N$ *and* $[P^q \bar{z}, \bar{x}_n^p] \neq 0$. *Then there exists* $\bar{x}_m$ *and* $I=(i_1, i_2, \ldots, i_s)$ *such that* $[P^{i_1} P^{i_2} \ldots P^{i_s} \bar{z}, \bar{x}_m^p] \neq 0$ *and* $[P^{q-|I|} \bar{x}_m^p, \bar{x}_n^p] \neq 0$ *where* $|I| = i_1 + i_2 + \ldots + i_s = 1 \bmod p$ *and* $q \neq |I|$.

Proof. The Adem relations, Lemma 2.13(b) and the choice of N imply that $q = 1 \bmod p$. We argue by induction as q increases with $q \neq v(u)$ but $q = 1 \bmod p$. The induction starts with $q = 2p+1$. Let $q = 1 + a_1 p + a_2 p + \ldots + a_s p^s$ be the p-adic expansion and let a_t be the first $a_i \neq 1$. We apply the Adem relation $P^{p^t} P^{q-p^t} = \alpha P^q + \sum \alpha_j P^{q-j} P^j$ where α is a unit and the summation is for $j \leqq p^{t-1}$. Thus

$$\begin{aligned} \alpha[P^q \bar{z}, \bar{x}_n^p] &= [P^{p^t} P^{q-p^t} \bar{z}, \bar{x}_n^p] - \sum \alpha_j [P^{q-j} P^j \bar{z}, \bar{x}_n^p] \\ &= \sum \mu_i [P^{q-pm_i-1} P^{pm_i+1} \bar{z}, \bar{x}_n^p] \\ &\quad + \sum \nu_j [P^{q-pm_j} P^{pm_j} \bar{z}, \bar{x}_n^p], \end{aligned}$$

by Lemma 2.13(c) and the choice of N.

Suppose first that some $[P^{q-pm_j} P^{pm_j} \bar{z}, \bar{x}_n^p] \neq 0$. Then $P^{pm_j} \bar{z} \in M \oplus N$ for dimensional reasons and $q - pm_j \neq v(u)$ for any u since $pm_j \leqq p^{t-1}$. In this case the result follows by the induction hypothesis. Otherwise $[P^{q-pm_i-1} P^{pm_i+1} \bar{z}, \bar{x}_n^p] \neq 0$ for some m_i and the result follows from Corollary 3.4.

If $n = 1 + p^{\alpha_2} + p^{\alpha_3} + \ldots + p^{\alpha_s}$ with $0 < \alpha_2 < \alpha_3 < \ldots < \alpha_s$, we define $\ell(2n) = s$. Let $m = 1 + p^{\beta_2} + p^{\beta_3} + \ldots + p^{\beta_r}$ also be binary. The proof of the following result is routine.

Lemma 3.6. *Suppose that* $[P^{p^q} \bar{x}_1^p, \bar{x}_2^p] \neq 0$, *where* $\dim \bar{x}_1 = 2n$ *and* $\dim \bar{x}_2 = 2m$. *Then* $\ell(2n) = \ell(2m)$. *Further* $\alpha_i = \beta_i$ *for all* i *except one and for that* $\alpha_i + 1 = \beta_i$.

We can now complete the proof of Proposition 3.1. Let $\bar{x}_n \in N_{2n}$ so that $\bar{x}_n^p = P^n \bar{x}_n$ and assume that $n \neq v(u)$ or there is nothing to prove. We assume as inductive hypothesis that N_{2k} is non-zero only for k binary if $k < n$. In the following expressions the integer r is chosen to be as large as possible.

$$\begin{aligned} &[P^I \bar{x}_n, \bar{y}_1^p] \neq 0, \quad [P^{i_1} \bar{y}_1^p, \bar{y}_2^p] \neq 0, \\ &\qquad\qquad [P^{i_2} \bar{y}_2^p, \bar{y}_3^p] \neq 0, \ldots, [P^{i_r} \bar{y}_r^p, \bar{x}_n^p] \neq 0 \end{aligned}$$

where each $i_j > 0$. Such a sequence exists by Lemma 3.5.

Since r is maximal, it follows from Lemma 3.5 and the Adem relations that $|I| = v(k)$ for some k. Also each $i_j = p^{k_j}$. The remainder of the proof is number theoretic. Let $\dim \bar{y}_r = 2(1 + p^{\gamma_2} + p^{\gamma_3} + \ldots + p^{\gamma_s})$ using the inductive hypothesis. Thus

$$\dim \bar{x}_n = 2(1 + p^{\gamma_2} + \ldots + p^{\gamma_s} + p^{u+1} - p^u) \quad \text{where } u = k_r - 1$$

and so

$$\dim \bar{y}_1^p = 2(1 + p^{\gamma_2} + \ldots + p^{\gamma_s} + p^{u+1} - p^u + p^{k+1} - 1).$$

Thus

$$\dim \bar{y}_1 = p^{\gamma_2 - 1} + p^{\gamma_3 - 1} + \ldots + p^{\gamma_s - 1} + p^u - p^{u-1} + p^k.$$

Now $\ell(\dim \bar{y}_r)=\ell(\dim y_1)$ by Lemma 3.6 and the induction hypothesis. Therefore $u-1=\gamma_j-1$ and $u\neq\gamma_{j+1}-1$ for some j or $u-1=k$ and $u\neq\gamma_i-1$ for any i. The first alternative implies that $2^{-1}\dim\bar{x}_n$ is binary as required. Otherwise $u=k+1$. Now working along the sequence from $[P^{i_1}\bar{y}_1^p,\bar{y}_2^p]\neq 0$ to $[P^{i_{r-1}}\bar{y}_{r-1}^p,\bar{y}_r^p]\neq 0$ and using the second part of Lemma 3.6, we deduce that $\gamma_2-1=0$, $\gamma_3-1=\gamma_2,\ldots,\gamma_i-1=\gamma_{i-1}$ and $k+1=\gamma_i$ for some i and so $u=\gamma_i$ and $u+1\neq\gamma_{i+1}$. Again this implies that $2^{-1}\dim\bar{x}_n$ is binary. This completes the proof of Theorem 1.1.

4.

We consider the Hopf algebra $K_*(X,Z_{(p)})$ filtered with the CW-filtration

$$K_*(X,Z_{(p)})_0\subset K_*(X,Z_{(p)})_2\subset\ldots$$
$$\ldots\subset K_*(X,Z_{(p)})_{2n}\subset K_*(X,Z_{(p)})_{2n+2}\subset\ldots.$$

Again it is convenient in the proofs to assume that $H_{2i+1}(X,Z_{(p)})=0$ for all i so that $K_*(X,Z_{(p)})_{2n+2}/K_*(X,Z_{(p)})_{2n}\cong H_{2n+2}(X,Z_{(p)})$. The direct sum decomposition of $K(X,Z_{(p)})=\oplus K^\alpha(X,Z_{(p)})$, $1\leqq\alpha\leqq p-1$ induces a decomposition of $K_*(X,Z_{(p)})$ and when taking representatives of homology elements in K-homology we assume that they lie in the appropriate summands.

Proposition 4.1. *Let $x_n\in H_{2n}(X,Z_{(p)})$, $\bar{x}_n$ be primitive and $x_n^p=px_{np}$. Let $u_i\in K_*(X,Z_{(p)})_{2i}$ represent x_i. Then*

$$u_n^p=pu_{np}+u_{np-(p-1)}\quad \text{where } u_{np-(p-1)}\in K_*(X,Z_{(p)})_{2np-2(p-1)}.$$

Further, if $u_{np-(p-1)}$ represents $x_{np-(p-1)}\in H_{2np-2(p-1)}(X,Z_{(p)})$, then in $QH_*(X,Z/pZ)$, $P^1_*\bar{x}_{np}=\bar{x}_{np-(p-1)}\neq 0$.

We first deduce Theorem 1.4. Write B for the Hopf algebra $K_*(X,Z_{(p)})/I$ where I is the ideal generated by $K_1(X,Z_{(p)})$, as in the introduction.

The Proof of Theorem 1.4. Clearly it is sufficient to show that $\bar{u}^p=0$ in $\bar{K}_*(X,Z/pZ)$ implies that $\bar{u}=0$. Suppose that this is false and let $\bar{u}_n$ be an element of smallest possible filtration with $\bar{u}_n^p=0$. Then $\bar{u}_n$ is primitive. Let $u_n\in K_*(X,Z_{(p)})_{2n}$ represent $\bar{u}_n$ and $x_n\in H_{2n}(X,Z_{(p)})$ where we can assume that $\bar{x}_n$ is a non zero primitive with $\bar{x}_n^p=0$. The required result now follows from Proposition 4.1.

If p is odd and G is a 1-connected simple Lie group, then from Proposition 4.1 one can write down the ring structures $K_*(\Omega G,Z/pZ)$ and $K_*(\Omega G,Z_{(p)})$ given the ring $H_*(\Omega G,Z/pZ)$, unless $G=E_8$ and $p=3$. Therefore we give a stronger version of Proposition 4.1 which enables us to eliminate this exception.

Lemma 4.2. *Let $\bar{s}_n\in PH^{2n}(X,Z/pZ)$ and $\bar{x}_n\in H_{2n}(X,Z/pZ)$ with $(\bar{s}_n,\bar{x}_n)\neq 0$. Assume that $x_n^p=px_{np}$ and $P^1\bar{s}_{np-(p-1)}=\bar{s}_n^p$ with $\bar{s}_{np-(p-1)}\in PH^{2np-2(p-1)}(X,Z/pZ)$. Let $u_i\in K_*(X,Z_{(p)})_{2i}$ be represented by x_i. Then $u_n^p=pu_{np}+u_{np-(p-1)}$ where*

$$u_{np-(p-1)}\in K_*(X,Z_{(p)})_{2np-2(p-1)},\quad P^1_*\bar{x}_{np}=\bar{x}_{np-(p-1)}$$

in

$$QH_*(X, Z/pZ) \quad \textit{and} \quad \langle \bar{s}_{np-(p-1)},\ \bar{x}_{np-(p-1)} \rangle \neq 0.$$

We check that Lemma 4.2 does imply Proposition 4.1. Let x_n be as in the statement. As $\bar{x}_n$ is primitive and $\bar{x}_n^p = 0$, it follows from Theorem 1.1 that $\bar{x}_n$ is the reduction mod p of a linear combination of quasi-generators. Thus $\bar{x}_n$ is indecomposable and has dimension congruent to $2 \bmod 2p$. We choose $\bar{s}_n \in PH^{2n}(X, Z/pZ)$ with $\langle \bar{s}_n, \bar{x}_n \rangle = 1$. As $\bar{x}_n^p = 0$, it follows that $\bar{s}_n^p \neq 0$ and so if $\bar{s}_{np-(p-1)} = P^{n-1}\bar{s}_n$, it is primitive and $P^1 \bar{s}_{np-(p-1)} = \bar{s}_n^p$.

The Proof of Lemma 4.2. In $PH^*(X, Z/pZ)$, we know that $P^1 \bar{s}_{np-(p-1)} = \bar{s}_n^p$. We choose $\xi_i \to \bar{s}_i$, where for dimensional reasons we can assume that $s_{np-(p-1)}$ is primitive. For suitable choice of the ξ_i (compare (3.1)), we can assume that

$$\psi^p(\xi_{np-(p-1)}) = p^{np-(p-1)}\xi_{np-(p-1)} + p^{np-p}\xi_n^p \bmod K(X, Z_{(p)})_{2np+2}$$

and from this it follows that $t(\xi_{np-(p-1)}) = p\xi_{np-(p-1)} + \xi_n^p \bmod K(X, Z_{(p)})_{2np+2}$. Now, provided we use a completed tensor product, $K^*(X, Z_{(p)})$ is the dual Hopf algebra to $K_*(X, Z_{(p)})$. Thus denoting the iterated comultiplication in $K^*(X, Z_{(p)})$ by $\psi^{(p)}$ we have

$$\psi^{(p)}(\xi_{np-(p-1)}) = \xi_{np-(p-1)} \otimes 1 \otimes \ldots \otimes 1 + \ldots + 1 \otimes \ldots \otimes 1 \otimes \xi_{np-(p-1)}$$

$$\frac{1}{(p^p - p)} \sum \frac{p!}{i_1! i_2! \ldots i_p!} \xi_n^{i_1} \otimes \xi_n^{i_2} \otimes \ldots \otimes \xi^{i_p}$$

$$\cdot \bmod K^*(X \times X \ldots \times X, Z_{(p)})_{2np+2},$$

where the summation is for $i_j \geqq 0$ and $i_1 + i_2 + \ldots + i_p = p$. Let $u_n \in K_*(X, Z_{(p)})_{2n}$ represent x_n so that $u_n^p = p u_{np} + u_{np-(p-1)}$ as $x_n^p = 0 \bmod p$, where

$$u_{np-(p-1)} \in K_*(X, Z_{(p)})_{2np-2(p-1)}.$$

Recall that $K_*(X, Z_{(p)})_{2n}$ is annihilated by $K^*_{2n+2}(X, Z_{(p)})$. Thus $\langle \xi_n, u_n \rangle \neq 0 \bmod p$. Also

$$\langle \xi_{np-(p-1)}, u_n^p \rangle = \langle \psi^{(p)}(\xi_{np-(p-1)}), u_n \otimes u_n \otimes \ldots \otimes u_n \rangle$$
$$= \frac{p!}{p^p - p} \langle \xi_n \otimes \xi_n \otimes \ldots \otimes \xi_n, u_n \otimes u_n \otimes \ldots \otimes u_n \rangle \neq 0 \bmod p.$$

Thus $\langle \xi_{np-(p-1)}, u_{np-(p-1)} \rangle \neq 0 \bmod p$ and so $\langle \bar{s}_{np-(p-1)}, \bar{x}_{np-(p-1)} \rangle \neq 0 \bmod p$ in the duality between $PH^*(X, Z/pZ)$ and $QH_*(X, Z/pZ)$. Also

$$0 \neq \langle \bar{s}_n^p, x_{-np} \rangle = \langle P^1 \bar{s}_{np-(p-1)}, \bar{x}_{np} \rangle = \langle \bar{s}_{np-(p-1)}, P^1_* \bar{x}_{np} \rangle.$$

It follows that $P^1_* \bar{x}_{np} = \bar{x}_{np-(p-1)}$, as $\bar{s}_{np-(p-1)}$ was any element with $P^1 \bar{s}_{np-(p-1)} = \bar{s}_n^p$. This completes the proof.

Example. Let $p=3$ and $X=\Omega E_8$, where E_8 is the simply connected simple Lie group. It is well known that $H_*(\Omega E_8, Z/3Z)=Z/3Z[\bar{x}_1,\bar{x}_3]/(\bar{x}_1^3,\bar{x}_3^3)\otimes\bar{P}$ where $\bar{P}$ is a polynomial algebra. Thus $H_*(\Omega E_8, Z(3))=\Gamma(x_1,2)\otimes P$ where P is a polynomial algebra. Also it is known that if $\bar{s}_1\in H^2(\Omega E_8, Z/3Z)$ is a non zero primitive, there exists $\bar{s}_7\in PH^{14}(\Omega E_8, Z/3Z)$ with $P^1\bar{s}_7=\bar{s}_1^9\neq 0$. Clearly $\langle\bar{s}_1^3,\bar{x}_3\rangle\neq 0$ and so by Lemma 4.3, there exists $u_i\in K_*(X, Z_{(p)})_{2i}$ representing x_i such that $u_3^3=3u_9+u_7$ where $\langle\bar{s}_7,\bar{x}_7\rangle\neq 0$. Applying Proposition 4.1 we deduce that if $u_1\in K_*(\Omega E_8, Z_{(3)})u_1^3=3u_3+\alpha u_1$ where $\alpha\neq 0 \bmod 3$ and we can choose u_1 so that $u_3=\dfrac{u_1(u_1-1)(u_1-2)}{3!}$. We deduce that $K_*(\Omega E_8, Z_{(3)})=\Delta(u_1,1)\otimes P$ where P is a polynomial algebra and $\Delta(u_1,1)\subset Z_{(0)}[u_1]$ is the subalgebra generated by 1, u_1 and $\dfrac{u_1(u_1-1)(u_1-2)}{3!}$ over $Z_{(3)}$.

References

1. Adams, J.F.: On Chern characters and the structure of the unitary group. Proc. Cambridge Philos. Soc. **57**, 189–199 (1961)
2. Adams, J.F.: Primitive elements in the K-theory of BSU. Quarterly J. Math. Oxford Ser. 2 **27**, 253–262 (1976)
3. Atiyah, M.F.: K-theory. New York-Amsterdam: Benjamin 1967
4. Duckworth, P.W.: M.Sc. Thesis, Aberdeen 1980
5. Hubbuck, J.R.: Finitely generated cohomology Hopf algebras, Topology **9**, 205–210 (1970)
6. Hubbuck, J.R.: A Hopf algebra decomposition theorem, Bull. Lond. Math., Soc. **13**, 125–128 (1981)
7. Hubbuck, J.R.: Some Pontrjagin rings, I, Proc. Roy. Soc. Edinburgh Sect. A **90**, 237–256 (1981)
8. Kane, R.: On loop spaces without p-torsion. Pacific J. Math. **60**, 189–201 (1975)
9. Lin, J.: Torsion in H-spaces II. Ann. of Math. (2) **107**, 41–88 (1978)
10. Lin, J.: Two torsion and the loop space conjecture. Preprint
11. Milnor, J., Moore, J.: On the structure of Hopf algebras. Ann. of Math. (2) **81**, 211–264 (1965)
12. Singer, W.M.: Connective Fiberings over BU and U. Topology **7**, 271–303 (1968)

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