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Uniqueness of Representation Spaces*

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For a Banach lattice V, Schaefer ([3] or [2, p. 173]) defined a locally compact space X to be a *representation space* for V if the space $C_\infty(X)$ of continuous functions on X having compact support can be identified with a dense ideal in V. In this paper we show that if X and Y are representation spaces for V there are dense, open subspaces of X and Y which are themselves representation spaces for V and which are homeomorphic to each other. This homeomorphism preserves the features of V in complete analogy to Schaefer's description for strong representation spaces (X a disjoint union of open, compact sets), given in [2, p. 173].

If L is an ideal in a Riesz space W, we will use the notation L^+ for the set of all non-zero lattice homomorphisms on L. We recall from [2, p. 161] that a *valuation* on W is a non-negative extended real-valued function φ on W satisfying the properties

$$\varphi(v+w)=\varphi(v)+\varphi(w) \qquad (v,w\geqq 0);$$
$$\varphi(v\wedge w)=\varphi(v)\wedge\varphi(w) \qquad (v,w\geqq 0);$$
$$\varphi(t\,w)=|t|\,\varphi(|w|) \qquad (t \text{ real}).$$

Any z in L^+ defines a valuation z^v on L given by $z^v(w)=z(|w|)$, which by the theorem of [2, p. 163] can be extended uniquely to a valuation on W (denoted by z^v). In fact, $z^v(v)=\sup\{z(w): 0\leqq w\leqq |v|, w\in L\}$. If z^v is finite on an ideal M containing L, it determines a (unique) extension in M^+ of z, given by $w\mapsto z^v(w^+)-z^v(w^-)$, which we will again denote by z.

We will need the following lemma on convergence of valuations.

Lemma 1. *Let L be a (non-trivial) ideal in Riesz space W and let net $\{z_j\}$ converge to z in L^+ with the weak topology $\sigma(L^+, L)$. Then $\{z_j^v(w)\}$ converges to $z^v(w)$ for every w in W.*

Proof. There is an $u>0$ in L such that $z(u)>0$. Given $w\geqq 0$ in W, $\{z_j(w\wedge n\,u)\}$ converges to $z(w\wedge n\,u)$ for all n. If $z^v(w)$ is infinite then, given $t>0$, N can be chosen so that $z(w\wedge Nu)=z^v(w)\wedge N z(u)>t$. Eventually, $z_j(w\wedge Nu)>t$ so that

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$z_j^v(w) > t$. On the other hand, if $z^v(w) < +\infty$, N can be chosen so that $Nz(u) > z(w)$. But for large j, $Nz_j(u) > z(w)$, implying that $\{z_j^v(w)\}$ converges to $z(w)$.

If Banach lattice V has a representation space X, it can be represented as a space $V(X)$ of continuous extended real-valued functions on X, each finite on a dense set, containing $C_\infty(X)$ as a dense ideal. For convenience, we review this representation: Where I denotes the copy of $C_\infty(X)$ in V, each point in X can be viewed as a member of I^+, extending to a valuation x^v on V. The mapping $(\Phi v)(x) = x^v(v^+) - x^v(v^-)$ maps V to a space $V(X)$ of extended real-valued functions on X, and maps I lattice isomorphically onto $C_\infty(X)$. Denoting Φv by $\hat{v}$, we suppose $\hat{v}(x) = +\infty$ on an open set O in X, for $v > 0$ in V. There is a function $\hat{u}$ in $C_\infty(X)$ such that $\hat{u}$ vanishes off O. Clearly, $n\,\hat{u}(x) \leqq \hat{v}(x)$ for all x in X and $n = 1, 2, \ldots$, so that $x(v \wedge nu) = x(nu)$. Since X separates I, we obtain $nu \leqq v$ for all n. By the Archimedean property $u = 0$, a contradiction. Thus each $\hat{v}$ is finite on a dense set. By Lemma 1, each $\hat{v}$ is clearly continuous. It now follows that $V(X)$ is a lattice under the pointwise operations (on points of finiteness), and Φ is a lattice homomorphism. If $\hat{v} = 0$ then $|\hat{v}| \wedge |\hat{u}| = 0$ for all u in I, so that v is orthogonal to I, since Φ is one-to-one on I. By the denseness of I in V, $v = 0$. Thus Φ is one-to-one on V.

We will need a characterization (stated below) of a Banach lattice having a representation space which was established in [1].

A collection $\{e_\alpha\}$ of positive elements in a Banach lattice V is a *topological order partition* (t.o.p.) of V if the following conditions are satisfied:

(1) the lattice ideal I generated by $\{e_\alpha\}$ is dense in V;

(2) for each index α there is an index β so that for any index γ,

$$e_\gamma \wedge n\, e_\alpha \leqq t\, e_\beta \qquad (n = 1, 2, \ldots)$$

for some real t depending on γ;

(3) there is a continuous real-valued function $\mathscr{E}$ on the set I^+ with the weak topology $\sigma(I^+, I)$ such that $\mathscr{E}(z) \geqq z(e_\alpha)$ for all α and $\mathscr{E}(t\,z) = t\mathscr{E}(z)$ for $t > 0$.

We will write $\beta > \alpha$ to denote the relationship of condition (2), and we note that $\beta > \alpha$ implies that for each u in I, there is a $t > 0$ such that $u \wedge n\, e_\alpha \leqq t\, e_\beta$ for all n. For convenience, we will assume that a t.o.p. contains all suprema of its finite subcollections.

Theorem [1]. *There exists a representation space X for a Banach lattice V if and only if there exists a t.o.p. $\{e_\alpha\}$ of V. Moreover, $C_\infty(X)$ is the image in the representation $V(X)$ of the ideal generated by $\{e_\alpha\}$.*

The following lemma on extensions of lattice homomorphisms will be used throughout the paper.

Lemma 2. *Let φ be a valuation on V such that $0 < \varphi(e_\alpha)$ and $\varphi(e_\beta) < +\infty$ for indices α and β satisfying $\alpha < \beta$. Then φ is a lattice homomorphism on I.*

Proof. We need only show that φ is finite on I. By condition (2) of the definition, for every $u \geqq 0$ in I there is a number $t > 0$ such that $u \wedge n\, e_\alpha \leqq t\, e_\beta$ ($n = 1, 2, \ldots$). Thus for all n, $\varphi(u) \wedge n\,\varphi(e_\alpha) \leqq t\,\varphi(e_\beta) < +\infty$. Since $\varphi(e_\alpha) > 0$, it follows that $\varphi(u)$ is finite.

Proposition. *Let $\{e_\alpha\}$ and $\{v_\lambda\}$ be two t.o.p.'s of Banach lattice V. Then $\{e_\alpha \wedge v_\lambda\}_{\alpha,\lambda}$ is a t.o.p. of V.*

Proof. Let I and J be the dense ideal generated by $\{e_\alpha\}$ and $\{v_\lambda\}$, respectively, and let $\mathscr{E}$ and $\mathscr{V}$ be the corresponding functions on I^+ and J^+ in the definition. Clearly, $I \cap J$ is the ideal generated by $\{e_\alpha \wedge v_\lambda\}_{\alpha,\lambda}$ and is dense, since $\overline{I \cap J} = \bar{I} \cap \bar{J}$. It is routine to verify that condition (2) of the definition is satisfied. We will verify condition (3). First, we define the extended real-valued function $\mathscr{E}'$ on $(I \cap J)^+$ by setting $\mathscr{E}'(z) = \mathscr{E}(z)$ if z extends to a lattice homomorphism on I and $\mathscr{E}'(z) = +\infty$ otherwise. We note that $\mathscr{E}'(z) \geqq z^v(e_\alpha)$ for all α and $\mathscr{E}'(tz) = t\mathscr{E}'(z)$ for $t > 0$. To see that $\mathscr{E}'$ is continuous in the weak topology $\sigma((I \cap J)^+, I \cap J)$, let net $\{z_j\}$ converge to z in $(I \cap J)^+$. Now if $\mathscr{E}'(z) = +\infty$, then $z^v(e_\alpha) = +\infty$ for some α. In this case, $\mathscr{E}'(z_j) \geqq z_j^v(e_\alpha)$, which converges to $+\infty$ by Lemma 1. On the other hand, if $\mathscr{E}'(z) < +\infty$, then $z(e_\alpha) > 0$ for some α and (of course) for $\beta > \alpha$, $z(e_\beta) < +\infty$. Since $\{z_j^v\}$ converges to z^v on V, there is an index j_0 such that $0 < z_j^v(e_\alpha)$ and $z_j^v(e_\beta) < +\infty$ for $j \geqq j_0$. By Lemma 2, z_j is in I^+ for $j \geqq j_0$ so that $\mathscr{E}'(z_j) = \mathscr{E}(z_j)$ converges to $\mathscr{E}(z) = \mathscr{E}'(z)$. Let $\mathscr{V}'$ be the continuous extended real-valued function on $(I \cap J)^+$ corresponding to $\mathscr{V}$. To verify that the function $\mathscr{F}$ defined on $(I \cap J)^+$ by $\mathscr{F}(z) = \mathscr{E}'(z) \wedge \mathscr{V}'(z)$ satisfies condition (3), we will show that it is real-valued. Given z in $(I \cap J)^+$, suppose $\mathscr{V}'(z) = +\infty$. Then $z^v(v_\delta) = +\infty$ and $z(e_\alpha \wedge v_\lambda) > 0$ for some δ, α and λ. For $\beta > \alpha$, $z^v(e_\beta) \wedge z^v(v_\delta) = z(e_\beta \wedge v_\delta)$ is finite, so that $z^v(e_\beta) < +\infty$ (and $z^v(e_\alpha) > 0$). By Lemma 2, z is in I^+, implying $\mathscr{F}(z) = \mathscr{E}'(z) < +\infty$. The other properties of $\mathscr{F}$ can be easily shown.

Theorem. (1) *Let $(V, \|\cdot\|)$ be a Banach lattice with representation space X. There is a minimal weakly compact set M of positive Radon measures on X such that for each v in V,*

$$\|v\| = \sup_{\mu \in M} \int_X |\hat{v}| \, d\mu,$$

where $\hat{v}$ is the representation of v as a continuous extended real-valued function on X (finite on a dense set).

(2) *If the pair (Y, N) represents $(V, \|\cdot\|)$ as does (X, M) above, then (Y, N) and (X, M) are equivalent in the following sense: There exists a homeomorphism of a dense open subspace $X_0 \subseteq X$ onto a like subspace $Y_0 \subseteq Y$, an isomorphism of vector lattices $C_\infty(Y_0) \to C_\infty(X_0)$ whose adjoint carries $M_0 := M|X_0$ to $N_0 := N|Y_0$, and the pair (X_0, M_0) is another representation for $(V, \|\cdot\|)$.*

Proof. Let Φ be the isomorphism of V onto its representation as functions on X and let I be the ideal $\Phi^{-1} C_\infty(X)$ in V.

For (1), we note that Φ^{-1} from $C_\infty(X)$ with the order topology into $(V, \|\cdot\|)$ is continuous with dense image, so that its adjoint $(\Phi^{-1})'$ is a bijection of the continuous dual V' of V onto a weakly dense subspace of the space $\mathscr{M}(X)$ of Radon measures on X. By Bauer's theorem [2, p. 87] there is a unique minimal $\sigma(V', V)$-compact subset P of $\{\varphi \in V' : \varphi \geqq 0, \|\varphi\| \leqq 1\}$ such that for each v in V,

$$\|v\| = \sup\{\varphi(|v|) : \varphi \in P\}.$$

Letting $\hat{v}$ denote $\Phi(v)$ and M denote $(\Phi^{-1})'P$, we obtain that $\|u\| = \sup\{\int_X |\hat{u}|\,d\mu\colon \mu\in M\}$ for all u in I. Given v in V, there is a positive increasing sequence $\{u_n\}$ in I converging in norm to $|v|$; thus $\{\hat{u}_n\}$ converges pointwise on X to $|\hat{v}|$. For each μ in M, by the Monotone Convergence Theorem, $\int_X \hat{u}_n\,d\mu$ converges to $\int_X |\hat{v}|\,d\mu$. Thus

$$\|v\| = \lim_{n\to\infty} \|u_n\| = \sup\{\int_X |\hat{v}|\,d\mu\colon \mu\in M\}.$$

We remark that if

$$\|v\| = \sup\{\int_X |\hat{v}|\,dv\colon v\in H\} = \sup\{|v(\Phi v)|\colon v\in H\}$$

for a minimal weakly compact collection H, then $v\circ\Phi$ is in V' for each v in H, so that the weak continuity of Φ' on H implies $\Phi'H$ contains P. By the uniqueness of P in Bauer's theorem, $H=(\Phi^{-1})'P=M$.

For (2), we let Ψ be the isomorphism of V onto its representation as functions on Y and we let J be the ideal $\Psi^{-1}C_\infty(Y)$ in V. We denote by X_* the collection of lattice homomorphisms in X which do not vanish on $I\cap J$. Clearly, X_* is open in X and $C_\infty(X_*)$ is contained in $C_\infty(X)$. We will first show that $\Phi^{-1}C_\infty(X_*)$ is dense in V. Given $u>0$ in $I\cap J$ and $\varepsilon>0$, the function $\hat{u}$ has compact support in X. Let χ be a function in $C_\infty(X)$ which is one on the (closed) support of $\hat{u}$ and let g be a continuous function on X, $0\leqq g\leqq 1$, which is one on $\{x\in X\colon \hat{u}(x)\geqq\varepsilon/\|\chi\|\}$ and zero outside $\{x\in X\colon \hat{u}(x)\geqq\varepsilon/(2\|\chi\|)\}$. Then the support of $\hat{u}g$ is in this latter set, which is compact in X_*, so that $\hat{u}g$ is in $C_\infty(X_*)$. Furthermore, $|\hat{u}(x) - \hat{u}(x)g(x)| < (\varepsilon/\|\chi\|)\chi(x)$ for all x in X, implying $\|u-\Phi^{-1}(ug)\|<\varepsilon$. Since $I\cap J$ is dense in V, it follows that $\Phi^{-1}C_\infty(X_*)$ is dense in V. Thus if $\hat{v}$ vanishes on X_* it is orthogonal to a dense ideal and therefore zero; hence, X_* is dense in X. For later use, we show that $\Phi^{-1}C_\infty(X_*)$ is contained in $I\cap J$. Given $\hat{w}$ in $C_\infty(X_*)$ with compact support K, for each x in K there is a member u of $I\cap J$ such that $x(u)>0$. The open supports of finitely many of these functions $\hat{u}$ cover K, so that w is dominated by a multiple of their (finite) supremum, as desired. By the theorem above, I and J are ideals generated by t.o.p.'s; thus, by the Proposition, $I\cap J$ is the ideal generated by a t.o.p. of V. There is a function $\mathscr{F}\colon (I\cap J)^+\to\mathbb{R}$ (chosen as in the proof of the Proposition above), a corresponding representation space $Z=\{z\in(I\cap J)^+\colon \mathscr{F}(z)=1\}$ and an isomorphism Γ of V onto its representation as functions on Z for which $I\cap J=\Gamma^{-1}C_\infty(Z)$. Since each x in X_* restricts to a non-trivial lattice homomorphism on $I\cap J$ (with $\mathscr{F}(x)>0$), we can define a mapping $\tau_1\colon X_*\to Z$ by setting $\tau_1(x)=x/\mathscr{F}(x)$. Since $\mathscr{F}$ is continuous, it follows that τ_1 is continuous. Let x_1 and x_2 be distinct points in X_* and let u be a member of $\Phi^{-1}C_\infty(X_*)$ for which $\hat{u}(x_1)=1$ and $\hat{u}(x_2)=0$. Since (as noted above) u is in $I\cap J$, we obtain $(\tau_1 x_1)(u)\neq 0$ and $(\tau_1 x_2)(u)=0$, showing that τ_1 is one-to-one. Viewing I as the ideal of a t.o.p. of V, one interprets X as $\{x\in I^+\colon \mathscr{E}(z)=1\}$ for an appropriate continuous function $\mathscr{E}\colon I^+\to\mathbb{R}$ (corresponding to the t.o.p.). In this formulation, one can verify that

$\tau_1^{-1}(z)=z/\mathscr{E}(z)$. It now follows from Lemma 1 and the continuity of $\mathscr{E}$ that τ_1 is a homeomorphism onto $\tau_1 X_*$. To see that $\tau_1 X_*$ is open in Z, consider z in $\tau_1 X_*$. By definition (in the proof of the Proposition above) $\mathscr{E}'(z)$ is finite, and by the continuity of $\mathscr{E}'$ on Z there is a neighborhood U of z in Z on which $\mathscr{E}'$ is finite. Thus all members of U are extendable to I^+; i.e., U is contained in $\tau_1 X_*$. Clearly $\tau_1 X_*$ is dense in Z, since $(\Gamma v)(\tau_1 X_*)=0$ implies $\hat{v}=0$ on X_*. We note that for v in V and x in X_*,

$$\hat{v}(x)=\mathscr{F}(x)(\Gamma v)(\tau_1 x).$$

Thus the support of $\hat{v}$ in X_* is the same as the support of $(\Gamma v)\circ\tau_1$. It follows that $\Gamma^{-1} C_\infty(\tau_1 X_*)=\Phi^{-1} C_\infty(X_*)$ is dense in V. By symmetry, for $Y_* =\{y\in Y: y(I\cap J)\neq 0\}$, there is a corresponding homeomorphism $\tau_2: Y_*\to Z$ with dense, open image, and $\Gamma^{-1} C_\infty(\tau_2 Y_*)=\Psi^{-1} C_\infty(Y_*)$ is dense in V. We define Z_0 to be $\tau_1 X_*\cap\tau_2 Y_*$. Clearly, Z_0 is dense and open in Z and $C_\infty(\tau_1 X_*)\cap C_\infty(\tau_2 Y_*)$ is contained in $C_\infty(Z_0)$. Equivalently, $\Phi^{-1} C_\infty(X_*)\cap\psi^{-1} C_\infty(Y_*)$ is contained in $\Gamma^{-1} C_\infty(Z_0)$, so that $\Gamma^{-1} C_\infty(Z_0)$ is dense in V. The mapping $\tau=\tau_2^{-1}\tau_1$ is a homeomorphism of $X_0:=\tau_1^{-1} Z_0$ onto $Y_0:=\tau_2^{-1} Z_0$, and these spaces are open in X and Y, respectively. By the indented formula above, $\Phi^{-1} C_\infty(X_0)$ is $\Gamma^{-1} C_\infty(Z_0)$, and hence is dense in V. Similarly, $\Psi^{-1} C_\infty(Y_0)$ is $\Gamma^{-1} C_\infty(Z_0)$, so that X_0 and Y_0 (representation spaces) are dense in X and Y, respectively. Thus (a) is established. By part (1) of this theorem, there is a unique minimal weakly compact set M_0 of Radon measures on X_0 characterizing the norm of V. For (c), we need only note that for each μ in M and v in V, $\int_X |\hat{v}|\, d\mu=\int_{X_0} |(\hat{v}|X_0)|\, d\mu$, since it is true for all v in $\Phi^{-1} C_\infty(X_0)$. Then $\|v\|=\sup\{\int_{X_0} |\hat{v}|\, d\mu: \mu\in M\}$, so that $M_0=M|X_0$ by the uniqueness of M_0. For (b), it is clear that $k:=\Phi\Psi^{-1}$ is a lattice isomorphism of $C_\infty(Y_0)$ onto $C_\infty(X_0)$, and that $k' M_0=(\Phi\Psi)'(\Phi^{-1}))' P=(\Psi^{-1})' P=N|X_0$, where Φ and Ψ are appropriately restricted and P is the Bauer set mentioned in the proof of part (1).

We remark that one can prove the following Stone-Weierstrass-type result, in analogy to [2, p. 177]: Let H be a vector sublattice of a Banach space V, whose closure contains a t.o.p. of V. If H separates the nonzero lattice homomorphisms on the ideal generated in V by the t.o.p., then H is dense in V.

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