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Inequalities Involving Maps of Finite Sets

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§ 1. Introduction

In [1] Ahlswede and Daykin obtained various inequalities involving maps of finite sets. The purpose of this note is to supplement some of their results but, before we can describe our results, we need the following notation and definitions.

Let R_+ denote the non-negative real numbers, S a finite set and $|S|$ the cardinality of S. For a function $\phi\colon S\times S\to S$ and $A, B\subseteq S$ we write $\phi(A,B)$ for $\{\phi(a,b)\colon a\in A,\ b\in B\}$ and when $\alpha\colon S\to R_+$ and $A\subseteq S$ we put $\alpha(A)$ for $\sum_{a\in A}\alpha(a)$. A family L of subsets of S is a ring if $A\cap B$ and $A\vartriangle B$ belong to L whenever A and B do. Finally $(\alpha,\beta,\gamma,\delta)$, where $\alpha,\beta,\gamma,\delta\colon S\to R_+$, is compatible with (ϕ,ψ), where $\phi,\psi\colon S\times S\to S$, if

$$\alpha(a)\,\beta(b)\leqq\gamma(\phi(a,b))\,\delta(\psi(a,b)) \quad \text{for all} \quad a,b\in S.$$

The main theorem of the paper is:

Theorem 1. *Let L be a ring of subsets of the finite set S and let $\alpha,\beta,\gamma,\delta\colon L\to R_+$ with γ and at least one of α and β monotone increasing functions on L. If $(\vartriangle,\cap)$ is compatible with $(\alpha,\beta,\gamma,\delta)$, then*

$$a(A)\,\beta(B)\leqq\gamma(A\vartriangle B)\,\delta(A\wedge B) \quad \textit{for all} \quad A,B\subseteq L.$$

Ahlswede and Daykin's Theorem 4 is the corresponding result when the assumption that γ and one of α and β are monotone increasing is replaced by the assumption that δ is monotone increasing; in fact their theorem was phrased in terms of a lattice of subsets but a ring of subsets seems a more natural setting for the theorem. As we shall see, our method of proof of Theorem 1 encompasses their theorem as well. An immediate consequence of these two results is:

Theorem 2. *Let L be a ring of subsets of the finite set S and let $(\alpha,\beta,\alpha,\beta)$ be compatible with $(\vartriangle,\cap)$ where $\alpha,\beta\colon L\to R_+$ are such that at least one of α and β is*

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monotone increasing on L, then

$$a(A)\,\beta(B) \leqq \alpha(A \vartriangle B)\,\beta(A \wedge B) \quad \text{for all} \quad A, B \subseteq L.$$

In [1] it was conjectured that Theorem 2 holds without any restrictions involving monotonicity but we show that this conjecture is false for $n \geqq 2$ and that it is even false in the particular case $\alpha = \beta$ when $n \geqq 3$. Finally we prove that the conjecture is true when $\alpha = \beta$ and $n = 2$.

§ 2. Proof of Theorem 1

We require the following result [3, Theorem 3]; see also [2].

Theorem 3. *If $M_1, \ldots, M_m$ are m distinct sets and $I = \{1, \ldots, m\}$, there is a permutation μ: $I \to I$ such that the m differences $M_i \setminus M_{\mu(i)}$ with $i \in I$ are all distinct.*

We firstly prove:

Theorem 4. *Let S be a finite set and A, B subsets of the power set of S then there is an injection H: $A \times B \to (A \vartriangle B) \times (A \wedge B)$ such that $pr_1 H(a,b) = a \vartriangle b$ and $a \cap b \subseteq pr_2 H(a,b) \subseteq a$ where pr_i denotes the projection into the ith coordinate set.*

Proof. Let π be the partition of $A \times B$ determined by the function F: $A \times B \to (A \vartriangle B) \times (A \wedge B)$ given by $F(a,b) = (a \vartriangle b, a \cap b)$. For $X \in \pi$ take a representative (a,b) of X and put $X_1 = \{x \setminus (a \cap b)\colon (x,y) \in X\}$, $X_2 = \{y \setminus (a \cap b)\colon (x,y) \in X\} = \{(a \vartriangle b) \setminus x\colon x \in X_1\}$.

If (x, y_1) and (x, y_2) are in X then $x \vartriangle y_1 = x \vartriangle y_2$ and $y_1 = y_2$; furthermore $a \cap b \subseteq x$ when $(x,y) \in X$ so $|X_1| = |X|$. Now $X_1 \wedge X_2 = \{x \cap u^c : x, u \in X_1\}$ where u^c denotes the complement of u in $a \vartriangle b$, i.e. $X_1 \wedge X_2 = X_1 \setminus X_1$. Hence, by Theorem 3, there is a permutation μ of X_1 such that $x \setminus \mu(x)$ are distinct. Define H_X: $X \to (A \vartriangle B) \times (A \wedge B)$ by $H_X(x,y) = (x \vartriangle y,\ x \setminus \mu(x \setminus (x \cap y)))$ and then H: $A \times B \to (A \vartriangle B) \times (A \wedge B)$ by $H(a,b) = H_X(a,b)$ where X is the member of π with representative (a,b). Clearly $pr_1 H(a,b) = a \vartriangle b$ and $a \cap b \subseteq pr_2 H(a,b) \subseteq a$.

Assume $H(a,b) = H(c,d)$ then $a \vartriangle b = c \vartriangle d$ and $a \setminus \mu_1(a \setminus (a \cap b)) = c \setminus \mu_2(c \setminus (c \cap d))$ for suitable μ_1 and μ_2.

(i) If (a,b) and (c,d) belong to different members of π then $a \cap b \neq c \cap d$. Without loss of generality we can assume that there is a $P \in a \cap b$ with $P \notin c \cap d$ then $P \in c \setminus \mu_2(c \setminus (c \cap d))$ and so $P \in c \vartriangle d$. But $P \notin a \vartriangle b = c \vartriangle d$ and we have a contradiction.

(ii) If (a,b) and (c,d) belong to the same member of π then $\mu_1 = \mu_2$ and $a \setminus \mu_1(a \setminus (a \cap b)) = c \setminus \mu_1(c \setminus (c \cap d))$. Hence $a = c$ and then $a \vartriangle b = c \vartriangle d$ implies $b = d$.

Thus H is an injection and the theorem is proved.

The Ahlswede-Daykin Theorem 4 clearly follows immediately from the above theorem. To see that Theorem 1 does, we may assume without loss of generality that γ and β are monotone increasing. Let $A, B \subseteq L$ and H be an injection given by Theorem 4; for $(a,b) \in A \times B$ put $b' = b \cup pr_2 H(a,b)$ then $b \subseteq b' \subseteq a \cup b$, $a \cap b' = pr_2 H(a,b)$ and $a \vartriangle b' \subseteq a \vartriangle b$. Since β and γ are monotone increasing we have

$$\alpha(a)\,\beta(b) \leqq \alpha(a)\,\beta(b') \leqq \gamma(a \,\triangle\, b')\,\delta(a \cap b')$$
$$\leqq \gamma(a \,\triangle\, b)\,\delta(a \cap b') = \gamma(pr_1 H(a,b))\,\delta(pr_2 H(a,b))$$

and Theorem 1 follows since H is an injection.

Note that Theorem 1 implies that the condition $\gamma(0) \geqq \gamma(1)$ in Lemma 3(d) of [1] is unnecessary and that this amended version of Lemma 3(d) also includes Lemma 3(c).

§ 3. Ahlswede and Daykin's Conjecture 2

The conjecture is:

Conjecture. Let α, β: $\{0,1\}^n \to R_+$. If $(\alpha, \beta, \alpha, \beta)$ is compatible with $(\triangle, \cap)$, then $\alpha(A)\,\beta(B) \leqq \alpha(A \,\triangle\, B)\,\beta(A \wedge B)$ for all $A, B \subseteq \{0,1\}^n$.

The following example shows that the conjecture is false when $n \geqq 2$.

Example. For $n=2$ take α and β given by the following table

	11	10	01	00
α	1	3	2	6
β	2	3	4	12

It is easy to check that $(\alpha, \beta, \alpha, \beta)$ is compatible with $(\triangle, \cap)$; however when $A = \{10,00\}$, $B = \{01,11\}$ we have $\alpha(A)\,\beta(B) = 54 > 45 = \alpha(A \,\triangle\, B)\,\beta(A \wedge B)$. The case when $n > 2$ clearly follows immediately from this example.

Furthermore the conjecture is even false in the special case $\alpha = \beta$ when $n = 3$ as the following example shows.

Example. When $n=3$ define α by $\alpha(111) = 1/6$, $\alpha(110) = 1$,

$$\alpha(101) = \alpha(011) = \alpha(001) = 1/2, \quad \alpha(100) = 3 = \alpha(010), \quad \alpha(000) = 9.$$

It is routine to verify that $(\alpha, \alpha, \alpha, \alpha)$ is compatible with $(\triangle, \cap)$; however when $A = \{011, 010\}$ and $B = \{100, 101\}$ we have

$$\alpha(A)\,\alpha(B) = 49/4 > 133/12 = \alpha(A \,\triangle\, B)\,\alpha(A \wedge B).$$

The case when $n > 3$ obviously follows immediately from this example.

Finally we prove:

Theorem 5. *Let* α: $\{0,1\}^2 \to R_+$. *If* $(\alpha, \alpha, \alpha, \alpha)$ *is compatible with* $(\triangle, \cap)$ *then* $\alpha(A)\,\alpha(B) \leqq \alpha(A \,\triangle\, B)\,\alpha(A \wedge B)$ *for all* $A, B \subseteq \{0,1\}^2$.

Remark. Note that the theorem is true when $|A| = 1$ or $|B| = 1$ by Lemma 1 of [1].

Proof. Assume there are subsets A and B of $\{0,1\}^2$ and α: $\{0,1\}^2 \to R_+$ such that $(\alpha, \alpha, \alpha, \alpha)$ is compatible with $(\triangle, \cap)$ and

$$\alpha(A)\,\alpha(B) > \alpha(A \,\triangle\, B)\,\alpha(A \wedge B) \qquad (\dagger)$$

By the remark we must have $|A| \geqq 2$ and $|B| \geqq 2$.

(i) If $11 \in A$ then $B \subseteq A \wedge B$ whilst, if $00 \in A$, $B \subseteq A \triangle B$ so $00 \in A$ implies $11 \notin B$ and $11 \in A$ implies $00 \notin B$.

(ii) Suppose $\alpha(10) = 0 = \alpha(01)$; put $A' = A \setminus \{01, 10\}$, $B' = B \setminus \{01, 10\}$ so that $\alpha(A)\,\alpha(B) = \alpha(A')\,\alpha(B') \leqq \alpha(A' \triangle B')\,\alpha(A' \wedge B') \leqq \alpha(A \triangle B)\,\alpha(A \wedge B)$ using the Remark and (i). Thus we may assume without loss of generality that $\alpha(10) = \max\{\alpha(10), \alpha(01)\} > 0$. Hence $\alpha(10)\,\alpha(11) \leqq \alpha(01)\,\alpha(10)$ gives $\alpha(11) \leqq \alpha(01)$ and $\alpha(10)\,\alpha(10) \leqq \alpha(00)\,\alpha(10)$ gives $\alpha(10) \leqq \alpha(00)$ and we therefore have

$$\alpha(11) \leqq \alpha(01) \leqq \alpha(10) \leqq \alpha(00) \qquad (*)$$

(iii) Suppose $|A| \geqq 3$ then $00 \in A \triangle B$ and $00 \in A \wedge B$; if $A = \{00, 01, 10\}$ then $11 \notin B$ by (i) and $B \subseteq A \wedge B$ whilst if $11 \in A$ we again have $B \subseteq A \wedge B$. Hence (†) gives $\alpha(A) > \alpha(A \triangle B)$ and $00 \notin B$. Since $|A \triangle B| \geqq |A|$ and $00 \in A \triangle B$, (*) gives $\{00, 10\} \subseteq A$ so $11 \notin B$ and $B = \{01, 10\}$ in which case $\alpha(A) \leqq \alpha(A \triangle B)$ and we have a contradiction. Thus $|A| \not\geqq 3$ and we can assume $|A| = 2 = |B|$.

If $11 \in A$ then $B \subseteq A \wedge B$ so $\alpha(A) > \alpha(A \triangle B)$. Since $|A \triangle B| \geqq |A|$, (*) tells us that $00 \notin A \triangle B$ so A and B are disjoint. By (i) $00 \notin B$ so $A = \{11, 00\}$ and $B = \{01, 10\}$ in which case $B \subseteq A \triangle B$ and $\alpha(A) \leqq \alpha(A \wedge B)$ which contradicts (†).

Thus $11 \notin A$ and similarly $11 \notin B$. Hence $00 \in A \triangle B$ and $00 \in A \wedge B$. Now $|A \wedge B| = 1$ only if $\{A, B\} = \{\{00, 10\}, \{00, 01\}\}$ in which case $\alpha(A \triangle B)\,\alpha(A \wedge B) - \alpha(A)\,\alpha(B) = \alpha(00)\,\alpha(11) - \alpha(10)\,\alpha(01) \geqq 0$.

Thus $|A \wedge B| \geqq 2$. If $00 \in A$ then $A \subseteq A \wedge B$ and $B \subseteq A \triangle B$ so we need only consider the case $A = B = \{01, 10\}$; but by a well-known result [4, p 350, 3.9.22] $\alpha(10)\,\alpha(01) \leqq \alpha(11)\,\alpha(00)$ implies $\alpha(10) + \alpha(01) \leqq \alpha(00) + \alpha(11)$ so we also have a contradiction to (†) in this case and the proof is complete.

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