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The Koszul Algebra of a Codimension 2 Embedding

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The title refers to the homology of the Koszul complex K on a finite set of generators of an ideal I in a regular (commutative) noetherian ring, which is the ideal of a Cohen-Macaulay subscheme of pure codimension 2. The generic example of this situation is given by the ideal generated by the maximal minors of an $n\times(n+1)$-matrix $X=(X_{ij})$ whose entries are algebraically independent over a field k, R in this case being the polynomial ring $k[\{X_{ij}\}_{\substack{1\leqq i\leqq n\\ 1\leqq j\leqq n+1}}]$. Originally we were interested in the conormal module I/I^2 of this embedding. However the first homology module H_1 of the Koszul complex K is related with I/I^2 through an exact sequence

$$H_1 \xrightarrow{\iota} K_1 \underset{R}{\otimes} R/I \longrightarrow I/I^2 \longrightarrow 0$$

I is called syzygietic if ι is an injection.

Moreover, as A. Simis and W.V. Vasconcelos have recently demonstrated in [14], the higher homology modules of K are related to the higher conormal modules I^k/I^{k+1}.

In this paper we are concerned with the study of the Koszul homology and the conormal modules of a perfect ideal I of grade 2 in a commutative noetherian ring R with unit.

By the theorem of Hilbert-Burch [5], when all finitely generated projectives are stably free, it amounts to consider the following situation:

There exists a non-split injective homomorphism $\varphi: F\to G$ of free R-modules of ranks n and $n+1$ respectively, such that $I=I_n(\varphi)$, where $I_k(\varphi)$ denotes as usual the ideal generated by the $k\times k$-minors of a matrix associated to φ. In particular the perfectness of I is equivalent to the requirement that grade $I_n(\varphi)=2$, which will be our standard assumption for the rest of the paper. We write S for the quotient ring R/I.

The R-resolution of S admits a DG-algebra structure. Our main observation in this paper is, that all the homology modules H_i of the Koszul complex on some generators of I are perfect R-modules of projective dimension 2 and that

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their R-resolutions admit the structure of a DG-module over the DG-algebra resolution of S.

This module structure allows us to compute easily the Poincaré series of the modules H_i, viewed as S-modules. In fact, the computation is based on a result in [2].

Assume for a moment that R is a local Cohen-Macaulay ring. Then the modules H_i are all maximal Cohen-Macaulay modules over S. In particular the natural sequence

$$0 \to H_1 \to K_1 \underset{R}{\otimes} S \to I/I^2 \to 0$$

is exact, so that I is syzygietic and $\operatorname{codepth}_S I/I^2 = 1$.

In fact, we derive the two last mentioned results under much weaker assumptions.

Using the approximation complex of A. Simis and V.W. Vasconcelos [14] we obtain in the above situation under the extra hypothesis that I is prime and

$$\operatorname{grade} I_{n+2-p}(\varphi) \geqq p$$

for $p = 2, \ldots, n$, then

$$pd_R I^d/I^{d+1} \leqq \min(d+2, n+1)$$

(Here and in what follows we set $\operatorname{grade} R = \infty$.)

Thus our investigations can be viewed in the broader context of the study of the associated graded ring with respect to the ideals $I_{n,m}$ and the rings $R/I_{n,m}^d$, where $I_{n,m}$ denotes the ideal of $n \times n$-minors of a generic $n \times m$-matrix, (see [6, 9, 10, 13]).

While we were preparing this paper, we got acquainted with the preprint of Akin, Buchsbaum and Weyman [1], who construct minimal R-free complexes over $R/I_{n,m}^d$ for $n \leqq m$ and all d, and give sufficient conditions for their acyclicity, thus obtaining what can be considered to be the final results on the projective dimensions of these modules.

§ 1. Some Linear Algebra

In this section we describe the resolutions of the homology modules of the Koszul complex on a set of $n+1$ generators of $I = I_n(\varphi)$, where

$$\varphi: F \to G$$

is an injective homomorphism of free R-modules of rank n and $n+1$.

Once for all we choose an isomorphism

$$\varepsilon: \Lambda^{n+1} G \to \Lambda^n F \tag{1.1}$$

For any free R-module M of rank k we denote by

$$\nabla_i: \Lambda^i M \to \operatorname{Hom}(\Lambda^{k-i} M, \Lambda^k M)$$

the canonical isomorphism induced by the multiplication on the exterior algebra ΛM.

φ and ε given rise to a linear form

$$\partial_1 : G \to R,$$

which is the dual of

$$\Lambda^n \varphi \circ \varepsilon : \Lambda^{n+1} G \to \Lambda^n F.$$

To be more precise

$$\partial_1 = V_{n+1}^{-1} \circ \mathrm{Hom}(\Lambda^n \varphi \circ \varepsilon, \Lambda^{n+1} G) \circ V_1 \tag{1.2}$$

Since $\mathrm{Im}\, \partial_1 = I$, the Koszul complex $(\Lambda G, \partial)$ associated to ∂_1 is the one we are interested in.

Here

$$\partial_i : \Lambda^i G \to \Lambda^{i-1} G$$

is given by

$$\partial_i(g_1 \wedge \ldots \wedge g_i) = \sum_{j=1}^{i} (-1)^{j+1} \partial_1(g_j)\, g_2 \wedge \ldots \wedge \hat{g}_j \wedge \ldots \wedge g_i.$$

By abuse of notation we put $H_i = H_i(\Lambda G)$.

Next we define homomorphisms

$$\alpha_i : \Lambda^i G \to \Lambda^{i-1} F$$

for $i = 1, \ldots, n+1$.

$$\alpha_i = V_{n+1-i}^{-1} \circ \mathrm{Hom}(\Lambda^{n+1-i} F, \varepsilon) \circ \mathrm{Hom}(\Lambda^{n+1-i} \varphi, \Lambda^{n+1} G) \circ V_i. \tag{1.4}$$

Note that up to isomorphisms α_i is simply the dual of $\Lambda^{n+1+i} \varphi$.

An easy explicit calculation shows the following identities.

Lemma (1.5). *For $i = 1, \ldots, n+1$ we have*

(a) $\alpha_i \circ \Lambda^i \varphi = 0$

(b) $\partial_i = \Lambda^{i-1} \varphi \circ \alpha_i$.

In particular we obtain complexes

$$C_i : 0 \to \Lambda^{i+1} F \xrightarrow{\Lambda^{i+1} \varphi} \Lambda^{i+1} G \xrightarrow{\alpha_{i+1}} \Lambda^i F \to 0$$

for $i = 0, \ldots, n-1$.

These complexes C_i are isomorphic to the generalized Koszul complexes $K(\Lambda^i f)$, $f = \varphi^*$, of Buchsbaum and Rim [4].

Remember that our basic assumption is that grade $I_n(\varphi) = 2$, then these complexes resolve the modules H_i.

Lemma (1.6). *The complexes C_i are acyclic and*

$$H_i \simeq H_0(C_i) \quad \text{for } i = 0, \ldots, n-1,$$
$$H_i = 0 \quad \text{for } i = n, n+1.$$

Proof. For later purposes we do some more and define a *DG*-algebra structure on C_0 and *DG*-module structures on C_i, viewed as modules over C_0. The module structures for $i=0, \ldots, n-1$ are given by the following homomorphisms (1.7):

$$\mu_1 : F \otimes \Lambda^{i-1} F \xrightarrow{\wedge} \Lambda^i F$$

$$\mu_2 : G \otimes \Lambda^{i-1} F \xrightarrow{1 \otimes \Lambda^{i-1}\varphi} G \otimes \Lambda^{i-1} F \xrightarrow{\wedge} \Lambda^i G$$

$$\mu_3 : G \otimes \Lambda^i F \xrightarrow{\wedge} \Lambda^{i+1} G \xrightarrow{\alpha_{i+1}} \Lambda^i F.$$

Here $\wedge$ denotes the exterior product. It follows that all the homology modules $H_j(C_i)$ are S-modules. Since we assume that grade $I_n(\varphi)=2$, the acyclicity-criterion of Peskine and Szpiro [12], 1.8 asserts the acyclicity of the complexes C_i.

The same argument shows that $H_{n+1}=H_n=0$.

To prove $H_i \simeq H_0(C_i)$, we apply (1.5), (b) and obtain the factorizations

$$\begin{array}{ccccc} \Lambda^{i+1} G & \xrightarrow{\partial_{i+1}} & \Lambda^i G & \xrightarrow{\partial_i} & \Lambda^{i-1} G \\ & {\scriptstyle \alpha_{i+1}} \searrow \quad \nearrow {\scriptstyle \Lambda^i \varphi} \quad {\scriptstyle \alpha_i} \searrow \quad \nearrow {\scriptstyle \Lambda^{i-1}\varphi} & & & \\ & \Lambda^i F & & \Lambda^{i-1} F & \end{array}$$

Since we already know that the complex C_i are acyclic, we find that

$$\operatorname{Ker} \partial_i = \operatorname{Ker} \alpha_i = \operatorname{Im} \Lambda^i \varphi$$

and

$$\operatorname{Im} \partial_{i+1} = (\Lambda^i \varphi)(\operatorname{Im} \alpha_{i+1}),$$

thus

$$\begin{aligned} H_i &\simeq \operatorname{Im} \Lambda^i \varphi / (\Lambda^i \varphi)(\operatorname{Im} \alpha_{i+1}) \\ &\simeq \operatorname{Coker} \alpha_{i+1} = H_0(C_i). \end{aligned}$$

§ 2. The Structure of the Koszul Algebra

In the following theorem we list some results about the structure of the Koszul algebra H on a set of $n+1$ generators of $I_n(\varphi)$. We keep the notations and assumptions of the previous section. A module over a Cohen-Macaulay ring S will be called a maximal Cohen-Macaulay module if depht $M_\mathfrak{p} = \dim S_\mathfrak{p}$ for every prime (or equivalently-maximal) ideal $\mathfrak{p}$ of S.

Theorem (2.1). (a) *The modules H_i are perfect R-modules of projective dimension 2. In particular they are maximal Cohen-Macaulay S-modules if R is a Cohen-Macaulay ring.*

(b) $\operatorname{Ann}_S H_i = 0$

(c) $\operatorname{Ass} H_i = \operatorname{Ass} S$

(d) *All H_i are maximally torsion free over S,* i.e. *every S-regular sequence is H_i-regular and vice-versa.*

(e) *In the Koszul algebra* $H_1^i = H_i$

(f) $H_i \simeq \mathrm{Ext}_R^2(H_{n-1-i}, R)$.

If in particular R *is a local Gorenstein ring, then* H_{n-1} *is a canonical module of* S *and therefore*

$$\mathrm{Hom}_S(H_{n-1-i}, H_{n-1}) \simeq \mathrm{Ext}_R^2(H_{n-1-i}, R).$$

In fact, the multiplication on H *induces isomorphisms*

$$H_i \to \mathrm{Hom}_S(H_{n-1-i}, H_{n-1})$$

(g) *Set* $D_i = \mathrm{Ker}(\Lambda^i H_1 \to H_i)$.

Then we have:

(i) D_i *is the torsion submodule of* $\Lambda^i H_1$, *if* I *is generically a complete intersection.*

(ii) $D_i \neq 0$ *if* $I_{n-1}(\varphi) \neq I_{n-i}(\varphi)$; *in particular this is so for* $i = 2, \ldots, n-1$, *when* R *is local and* $\mu(I) = n+1$.

Proof. (a) has been proved already in (1.6).

(b) Knowing that $H_i = \mathrm{Coker}\ \alpha_i$, we have the assertion by [3].

(c) is an immediate consequence of (b) and the following elementary

Lemma (2.2). *Let* M *and* N *be perfect* R*-modules such that* $\mathrm{Supp}\, M = \mathrm{Supp}\, N$. *Then* $\mathrm{Ass}\, M = \mathrm{Ass}\, N$.

Proof. Since $\mathrm{rad}\,\mathrm{Ann}\, M = \mathrm{rad}\,\mathrm{Ann}\, N$, there is a regular sequence $x_1, \ldots, x_g$ in $\mathrm{Ann}\, M \cap \mathrm{Ann}\, N$ with $g = \mathrm{grade}\,\mathrm{Ann}\, M = \mathrm{grade}\,\mathrm{Ann}\, M$.

The well-known isomorphisms

$$M \simeq \mathrm{Ext}_R^g(\mathrm{Ext}_R^g(M, R), R) \simeq \mathrm{Hom}_R(\mathrm{Hom}_R(M, R/x), R/x)$$

give $\mathrm{Ass}\, M = \mathrm{Supp}\,\mathrm{Hom}_R(M, R/x) \cap \mathrm{Ass}\, R/x$ hence $\mathrm{Ass}\, M = \mathrm{Supp}\, M \cap \mathrm{Ass}\, R/x$, since they also imply that

$$\mathrm{Ann}_R M = \mathrm{Ann}_R \mathrm{Hom}_R(M, R/x).$$

The corresponding formula holding also for N, we obtain $\mathrm{Ass}\, M = \mathrm{Ass}\, N$.

(d) The equality $\mathrm{Ass}\, H_i = \mathrm{Ass}\, S$ gives that $x \in S$ is S-regular if and only if it is H_i-regular. The rest follows by induction from the isomorphism

$$H_i(\Lambda G/\tilde{x}\Lambda G) \simeq H_i(\Lambda G)/x\, H_i(\Lambda G),$$

$\tilde{x}$ being a lifting of x to R. Note that by [12] Chap. II, Cor. 1.7 $\tilde{x}$ is automatically a non-zerodivisor.

(e) We know by (1.6) that $(\Lambda^i \varphi)(\Lambda^i F)$ is the i-th module of cycles of the Koszul complex, hence the assertion follows.

(f) $\mathrm{Ext}_R^2(H_{n-1-i}, R) \simeq \mathrm{Coker}\ (\Lambda^{n-i+1}\varphi)^* \simeq \mathrm{Coker}\,\alpha_i \simeq H_i$. The other statements of (f) follow from the following result proved in [7], 2.4.1, since by (a) all H_i are Cohen-Macaulay modules.

Proposition (2.3). *Let* R *be a local Gorenstein ring,* I *a proper ideal of* R *and* H *the homology of the Koszul complex on a system of generators of* I. *If all the modules*

H_i are Cohen-Macauly, then the multiplication on H induces isomorphisms $H_i \to \mathrm{Hom}(H_{m-i}, H_m)$, where H_m is the last non-vanishing homology module.
(g) (i) If I is generically a complete intersection then $\Lambda^i H_1 \to H_1^i$ is generically an isomorphism and hence $D_i \subseteq \mathrm{Torsion}\, \Lambda^i H_1$. The other inclusion is also valid, since by (b) H_i is torsion free.
(ii) We have a commutative diagramm

$$\begin{array}{ccccc} \Lambda^2 G \otimes \Lambda^i F & \xrightarrow{\beta^i} & \Lambda^i F & \longrightarrow & \Lambda^i H_1 \to 0 \\ \downarrow & & \| & & \downarrow \\ \Lambda^{i+1} G & \xrightarrow[\alpha_{i+1}]{} & \Lambda^i F & \longrightarrow & H_i \to 0 \end{array}$$

where $\beta_i(a \otimes b) = \alpha_1(a) \wedge b$.

It follows that $I_1(\beta_i) = I_{n-1}(\varphi)$, while $I_1(\alpha_{i+1}) = I_{n-i}(\varphi)$. Thus $D_i = 0$ would imply that $I_{n-1}(\varphi) = I_{n-i}(\varphi)$.

§ 3. The Structure of I/I^2 and the Higher Conormal Modules I^d/I^{d+1}

We will demonstrate here that information about the H_i helps to better understand the structure of the conormal modules.

For a prime $\mathfrak{p} \in \mathrm{Spec}\, R$ we denote by $\mu_{\mathfrak{p}}(I)$ the minimal number of generators of $IR_{\mathfrak{p}}$.

We first observe

Lemma (3.1). *Suppose that* $\mathrm{grade}\, I_{n+2-i}(\varphi) \geqq i$ *for some* $i = 2, \ldots, n+1$, *then* $\mu_{\mathfrak{p}}(I) \leqq i-1$ *for all* $\mathfrak{p} \in \mathrm{Spec}\, R$ *such that depth* $R_{\mathfrak{p}} \leqq i-1$.

Proof. The condition $\mathrm{grade}\, I_{n+2-i}(\varphi) \geqq i$ insures that for any prime $\mathfrak{p}$ with $\mathrm{depth} R_{\mathfrak{p}} \leqq i-1$ a basis element of $\Lambda^{n+2-i} F_{\mathfrak{p}}$ is mapped onto a basis element of $\Lambda^{n+2-i} G_{\mathfrak{p}}$. Therefore a direct summand $H_{\mathfrak{p}} \subset F_{\mathfrak{p}}$ of rank $n+2-i$ is mapped onto a direct summand $\varphi_{\mathfrak{p}}(H_{\mathfrak{p}}) \subset G_{\mathfrak{p}}$ of rank $n+2-i$. This shows that $\mu_{\mathfrak{p}}(I) \leqq n+1-(n+2-i) = i-1$.

Consider the map of complexes $\psi\colon C_1 \to G \otimes C_0$ defined by setting:

$$\psi_0\colon F \to G \otimes R,\ \psi_0 = -\varphi$$
$$\psi_1\colon \Lambda^2 G \to G \otimes G,\ \psi_1(g_1 \wedge g_2) = g_1 \otimes g_2 - g_2 \otimes g_1$$
$$\psi_2\colon \Lambda^2 F \to G \otimes F,\ \psi_2(f_1 \wedge f_2) = \varphi(f_1) \otimes f_2 - \varphi(f_2) \otimes f_1.$$

Writing C for the mapping cone of ψ we have:

Theorem (3.2). (a) *The following conditions on φ are equivalent:*

(i) $\mathrm{grade}\, I_{n-1}(\varphi) \geqq 3$
(ii) *I is generically a complete intersection.*
(iii) *I is syzygietic.*
(iv) *The complex C is a R-free resolution of I/I^2. When these conditions are satisfied, one has*

$$2 \leqq pd\, I/I^2 \leqq 3,$$

the lower bound being attained if and only if I is locally a complete intersection, if and only if $I_{n-1}(\varphi)=R$.

(b) *If* grade $I/I^2 \geqq 4$, *then* I/I^2 *is torsion free.*

(c) *If R is regular and I is prime then the following conditions are equivalent:*

(i) $I^2=I^{(2)}$ *the second symbolic power of* I.

(ii) S *is a complete intersection in codimension* 1.

Remarks (3.3). The equivalence of (ii) and (iii) in statement (a) has been noticed independently by Simis and Vasconcelos [14], Theorem 2.2 and the remark following it.

Statement (c) is very similar to a theorem proved by Kunz [11] for almost complete intersections.

Proof. (a) (i)⇒(ii) follows from the lemma, since by the perfectness of I, we have depth $R_{\mathfrak{p}}=2$ for all $\mathfrak{p}\in \operatorname{Ass} R/I$.

Conversely, assume that grade $I_{n-1}(\varphi)<3$. Then grade $I_{n-1}(\varphi)=2$, since $I\subset I_{n-1}(\varphi)$. Let x_1, x_2 be an R-regular sequence in I, and $\mathfrak{p}$ an ideal of Ass $R/(x_1, x_2)$ containing $I_{n-1}(\varphi)$. From (2.2) we see that $\mathfrak{p}\in \operatorname{Ass} R/I$. Localizing at $\mathfrak{p}$ we see that $I_{n-1}(\varphi)_{\mathfrak{p}}\neq R_{\mathfrak{p}}$, hence I is not a complete intersection. This shows (ii) ⇒ (i).

If (ii) is satisfied, then the canonical map ι: $H_1\to S\otimes K_1$ localizes to an isomorphism for every $\mathfrak{p}\in \operatorname{Ass} S$. But since by (2.1) c) Ass $H_1=$ Ass S, it follows that for every $\mathfrak{p}\in \operatorname{Ass}(\operatorname{Ker} i)\subset \operatorname{Ass} H_1$ one has $(\operatorname{Ker}\iota)_{\mathfrak{p}}=0$, hence ι is injective.

Consider the mapping cone exact sequence associated with ψ. Since ψ_0 induces ι in zero-dimensional homology, it degenerates into relations $H_i(C)=0$ for $i\geqq 2$ and an exact sequence

$$0\to H_1(C)\to H_1\xrightarrow{\iota} S\underset{R}{\otimes} K_1\to H_0(C)\to 0.$$

We see that $H_0(C)=I/I^2$, and that the assumption that ι is injective is equivalent to $H_1(C)=0$, hence (iii)⇔(iv). In order to establish the last statement of (a), note that by the lemma $I_{n-1}(\varphi)=R$ if and only if I is locally a complete intersection, and in this case clearly $pd_R I/I^2=2$. Now if $I_{n-1}(\varphi)\neq R$, we may suppose that R is local and that $I_{n-1}(\varphi)$ is contained in the maximal ideal of R. Then R is not a complete intersection. We may also assume that $\varphi(F)\subset mG$ and that rank $F\geqq 2$. If k denotes the residue class field R/m, we see that the mapping

$$\Lambda^2 F\otimes k\xrightarrow[\Lambda^2\varphi\otimes k\oplus\psi_2\otimes k]{} \Lambda^2 G\otimes k\oplus(F\otimes G)\otimes k$$

is zero and thus

$$\operatorname{Tor}_3^R(I/I^2, k)\simeq \Lambda^2 F\otimes k\neq 0.$$

(b) Let $\mathfrak{p}\in\operatorname{Spec} R$, $\mathfrak{p}\supseteq I$. If depth $R_{\mathfrak{p}}\geqq 4$, then depth $H_{1_{\mathfrak{p}}}=$ depth $S_{\mathfrak{p}}\geqq 2$, hence from the exact sequence

$$0\to H_{1_{\mathfrak{p}}}\to S_{\mathfrak{p}}^{n+1}\to (I/I^2)_{\mathfrak{p}}\to 0$$

we obtain depth $(I/I^2)_{\mathfrak{p}}\geqq 1$.

If depth $R_{\mathfrak{p}}<4$, then by the lemma $IR_{\mathfrak{p}}$ is a complete intersection and therefore $(I/I^2)_{\mathfrak{p}}$ is free.

(c) (ii) $\Rightarrow$ (i) has just been noted. To prove (i) $\Rightarrow$ (ii) it suffices to assume that R is a regular local ring of dimension 3, in which case it follows from [8], 3.5.

For the study of the higher conormal modules we will assume that R is a Cohen-Macaulay ring, since in this situation we can apply a theorem of Simis and Vasconcelos. To formulate the theorem, we need some notations:

In the sequence

$$H_1 \xrightarrow{\iota} K_1 \underset{R}{\otimes} S \to I/I^2 \to 0$$

let $H = \iota(H_1)$.

If B_d denotes the d-th homogeneous part of the symmetric algebra $\mathrm{Sym}(K_1 \underset{R}{\otimes} S)$, then $\mathrm{Sym}_d(I/I^2) = B_d/HB_{d-1}$.

For our purposes we need a slight modification of Theorem 3.2 in [14].

Theorem (3.4) (Simis-Vasconcelos). *Let R be a Cohen-Macaulay ring and let I be a Cohen-Macaulay ideal of height l generated by $l+r$ elements. Assume that for each prime $\mathfrak{p}$ containing I and such that $l+1 \leqq \text{height } \mathfrak{p} \leqq l+r-1$ one has $\mu_{\mathfrak{p}}(I) \leqq \text{height } \mathfrak{p}$. If all the homology modules H_i of the Koszul complex on a set of generators of I are Cohen-Macaulay modules and $H_i = H_1^i$ for all $i \geqq 1$, then there is an exact complex $\mathcal{N}_d$:*

$$0 \to H_d \to H_{d-1} \otimes B_1 \to \ldots \to H_1 \otimes B_{d-1} \to B_d \to B_d/HB_{d-1} \to 0.$$

The assumption $H_i = H_1^i$ allows to consider $\mathcal{N}_d$ as an epimorphic image of the complex $\mathcal{L}_d$ in [14]. *In their proof of the exactness of $\mathcal{N}_d$ the requirement $\mu_{\mathfrak{p}}(I) \leqq \text{height } \mathfrak{p}$ (instead of $\mu_{\mathfrak{p}}(I) \leqq \text{height } \mathfrak{p} - 1$) is sufficient.*

Applying Lemma (3.1) and Th. (2.1), (a), (e) we now obtain a result which, in the generic case, implies Hochster's theorem [9], 2.2.

Theorem (3.5). (a) *Suppose that R is a Cohen-Macaulay ring and that*

$$\text{height } I_{n+2-i}(\varphi) \geqq i$$

for $i = 2, \ldots, n+2$, then the above complex $\mathcal{N}_d$ (for $I = I_n(\varphi)$) is exact for all d.
(b) *If moreover I is a prime ideal generated by $n+1$ elements and*

$$\text{height } I_{n+2-i}(\varphi) \geqq i+1 \quad \cdot$$

for $i = 3, \ldots, n+1$, then

i) $I^d = I^{(d)}$ *for all d*

ii) $pd_R(I^d/I^{d+1}) \leqq \min(d+2, n+1)$,

with equality if in addition R is Gorenstein and $I_{n-d}(\varphi) \neq R$.

Proof (b), i) follows from [14], Cor. 3.3. There it is also shown that under the above assumptions $\mathrm{Sym}_d(I/I^2) = I^d/I^{d+1}$. Therefore using that $\mathcal{N}_d$ is exact and that $pd_R H_i = 2$ by Th. 2.1, (a), we find the inequality ii).

In order to prove equality if R is Gorenstein, it suffices to assume that R is local and to show that the codepth is 1 for the cokernels of the following

mappings:

$$0 \to H_d \xrightarrow{\partial_d^d} H_{d-1} \otimes B, \quad \text{when } d \leqq n-1$$

and

$$0 \to H_{n-1} \otimes B_{d-n+1} \xrightarrow{\partial_{n-1}^d} H_{n-2} \otimes B_{d-n+2}, \quad \text{when } d > n-1.$$

Since we assume that R is Gorenstein, H_{n-1} is the canonical module of S and it suffices to show that

$$\operatorname{Ext}_S^1(\operatorname{Coker} \partial_d^d, H_{n-1}) \neq 0 \quad \text{for } d \leqq n-1$$

and

$$\operatorname{Ext}_S^1(\operatorname{Coker} \partial_{n-1}^d, H_{n-1}) \neq 0 \quad \text{for } d > n-1.$$

In fact we prove that the mappings $\operatorname{Hom}_S(\partial_i^d, H_{n-1})$ are not surjective for any $1 \leqq i \leqq n-1$ and $d \geqq i$.

In our particular case ∂^d can be described as follows

$$\partial_i^d \colon H_i \otimes B_{d-i} \to H_{i-1} \otimes B_{d-i+1}$$

$$\partial_i^d(x_1 \wedge \ldots \wedge x_i \otimes b) = \sum_{k=1}^{i} (-1)^{k+1} x_1 \wedge \ldots \wedge \hat{x}_k \wedge \ldots \wedge x_i \otimes \iota(x_k)\, b,$$

where ι is the natural mapping

$$\iota \colon H_1 \to B_1, \ \iota(x) = \varphi(x) \otimes 1.$$

We now apply (2.1), (f) and obtain a commutative diagramm

$$\begin{array}{ccc} \operatorname{Hom}(H_{i-1} \otimes B_{d-i+1}, H_{n-1}) & \xrightarrow{\operatorname{Hom}(\partial_i^d, H_{n-1})} & \operatorname{Hom}(H_i \otimes B_{d-i}, H_{n-1}) \\ \downarrow \wr & & \downarrow \wr \\ H_{n-i} \otimes B_{d-i+1} & \xrightarrow[d_i]{} & H_{n-i-1} \otimes B_{d-i}. \end{array}$$

Choosing a system of generators $x_1, \ldots, x_{n-1}$ of H_1 and a basis $b_1, \ldots, b_{n+1}$ of B_1, the mapping d_i is the following:

First of all if $b \in B_1$, $b = \sum_{i=1}^{n+1} s_i b_i$, then b defines a homomorphism

$$b^* \colon B_t \to B_{t-1} \quad \text{for } t \geqq 1$$

$$b^*(b_1^{\nu_1} \ldots b_{n+1}^{\nu_{n+1}}) = \sum_{i=1}^{n+1} s_i b_1^{\nu_1} \ldots b_i^{\nu_i - 1} \ldots b_{n+1}^{\nu_{n+1}},$$

where $b_1^{\nu_1} \ldots b_i^{\nu_i} \ldots b_{n+1}^{\nu_{n+1}} = 0$ if $\nu_i < 0$ for some i. Now

$$d_i(x_{j_1} \wedge \ldots \wedge x_{j_{n-i}} \otimes b) =$$
$$= \sum_{K=1}^{n-i} (-1)^{k+1} x_{j_1} \wedge \ldots \wedge \hat{x}_{j_k} \wedge \ldots \wedge x_{j_{n-i}} \otimes \iota(x_{j_k})^* b.$$

This formula shows that the image of d_i lies in $\mathfrak{m}H_{n-i-1}\otimes B_{d-i}$, $\mathfrak{m}$ being the maximal ideal of R. Hence $\mathrm{Hom}(\partial_i^d, H_{n-1})$ cannot be surjective.

§ 4. The Poincaré Series of I/I^2 and of the Modules H_i

In the previous sections we have computed R-free resolutions of the modules H_i and of I/I^2. These modules considered as S-modules have infinite projective dimension. We will show that in the local case one can compute the ranks of the free S-modules in a minimal free S-resolution of these modules. The result is best expressed in terms of Poincaré series:

$$P_R^M(t)=\sum_{i\geqq 0}\dim_k \mathrm{Tor}_i^R(k, M)\,t^i.$$

Here (R, m, k) denotes a noetherian local ring and M a finitely generated R-module.

Our tool in computing the Poincaré series $P_S^{H_i}(t)$ will be the following result proved in [2], (10), (12)

Proposition (4.1). *Let S be a homomorphic image of a local ring R and M be a finitely generated S-module. Then there is a spectral sequence with*

$$E^2_{p,q}=\mathrm{Tor}^{\mathrm{Tor}^R(S,k)}_{p,q}(\mathrm{Tor}^R(M, k), k)\Rightarrow \mathrm{Tor}^S_{p+q}(M, k).$$

Moreover this sequence degenerates at E^2 if the minimal R-free resolution P of S admits a structure of a DG-algebra and the minimal R-free resolution $\bar{P}$ of M admits a structure of a DG P-module.

In (1.7) we have given C_0 a DG-algebra structure and the complexes C_i a DG C_0-module structure. Thus the above spectral sequence degenerates in our case and we only need to know the dimensions of $E^2_{p,q}$.

Writing $V(i)$ for the graded module $\mathrm{Tor}^R(H_i, k)$ over the graded algebra $U=\mathrm{Tor}^R(S, k)$, one has for $1\leqq i\leqq n-1$:

(4.2)
$$\begin{aligned}&\text{(i)}\quad U_1^2=0\\ &\text{(ii)}\quad U_1V(i)_0=0\\ &\text{(iii)}\quad U_2V(i)_0=V(i)_2.\end{aligned}$$

This follows immediately from (1.7). Applying (4.1) we obtain

Theorem (4.3). *Suppose R is local and $\mu(I)=n+1>2$. Then*

(a) $$P_S^{H_i}=\binom{n}{i}+\frac{\binom{n+1}{i+1}(t+it^2)}{1-t-nt^2}\quad \textit{for } i\geqq 1,$$

(b) *If* grade $I_{n-1}(\varphi)\geqq 3$ (cf. (3.2), a)) *then*

$$P_S^{I/I^2}=(n+1)+nt+\frac{\binom{n+1}{2}(t^2+t^3)}{1-t-nt^2}.$$

Proof. (a) Using (4.1) we conclude that $\dim_k \operatorname{Tor}_n^S(H_i, k) = \sum_{p+q=n} \dim_k \operatorname{Tor}_{p,q}^U(V(i), k)$ with U and $V(i)$ as in (4.2). Writing

$$P_U^{V(i)}(t, u) = \sum_{p,q} \dim_k \operatorname{Tor}_{p,q}^U(V(i), k)\, t^p u^q$$

for the "Poincaré series in 2 variables" of the graded U-modules $V(i)$, the above relation becomes

$$P_S^{H_i} = P_U^{V(i)}(t, t).$$

Let V denote an arbitrary finitely generated graded U-module and $\pi\colon F \to V$ be a minimal epimorphism with F free over U.

Since $U_+^2 = 0$, where $U_+ = \operatorname{Ker}(U \to k)$, one sees that as an U-module $\operatorname{Ker}\pi$ is isomorphic to a direct sum of copies of trivial V-modules k (with degree shifts). Hence

$$P_U^V(t, u) = H_{V/U_+V}(u) + t[H_{V/U_+V}(u)\, H_U(u) - H_V(u)]\, P_U^k(t, u).$$

Here $H_W(u) = \sum_q \dim_k W_q u^q$ denotes the Hilbert series of a graded k-vector space. In the case $V = V(i)$ this formula becomes (using (1.6) and (4.2))

$$P_U^{V(i)}(t, u) = \binom{n}{i} + \binom{n+1}{i+1} u +$$

$$+ \left[(n+1)\binom{n}{i} + (n+i+1)\binom{n+1}{i+1} u + n\binom{n+1}{i+1} u^2\right] \frac{t}{1-(n+1)tu-ntu^2}.$$

If we put $t = u$, the result follows.

(b) Our assumptions insure that the sequence

$$0 \to H_1 \to K_1 \underset{P}{\otimes} S \to I/I^2 \to 0$$

is exact (cf. (3.2), a)). Therefore (b) follows from (a).

Supplement

We take the opportunity to add another example of ideals, where the homology modules of the Koszul complex are Cohen-Macaulay modules.

Proposition. *Let R be a local Gorenstein ring and I an ideal with $\mu(I) \leqq$ height $I + 2$, and such that $S = R/I$ is a Cohen-Macalay ring. Then the homology modules of the Koszul complex on a minimal set of generators for I are Cohen-Macaulay modules.*

Proof. Since R is assumed to be a Gorenstein ring, the last non-vanishing homology of the Koszul complex K on a minimal set of generators for I is the canonical module of S, and therefore a Cohen-Macaulay module. Thus we have only to prove that in the case $\mu(I) =$ height $I + 2$ the first homology H_1 of K is a Cohen-Macaulay module. Let $x \in S$ be an S-regular element and $\tilde{x} \in R$ be an R-

regular element, which is a lifting of x. From the short exact sequence

$$0 \to K \xrightarrow{\tilde{x}} K \to \bar{K} \to 0$$

we obtain the long exact sequence

$$0 \to H_2 \xrightarrow{x} H_2 \to H_2(\bar{K}) \to H_1 \xrightarrow{x} H_1 \to H_1(\bar{K}) \to$$
$$\to H_0 \xrightarrow{x} H_0 \to H_0(\bar{K}) \to 0.$$

In the above sequence the mapping $H_2 \to H_2(\bar{K})$ is surjective, and the mapping $H_0 \to H_0$ is injective. Therefore x is H_1-regular and $H_1/xH_1 \simeq H_1(\bar{K})$. By induction on the dimension of S, the assertion follows.

If in addition to the assumptions of the proposition I is generically a complete intersection, one concludes that $\operatorname{codepht}_S I/I^2 \leqq 1$. This inequality may be strict or not. One has codepth $I/I^2 = 1$, if I is perfect of height 2 (generated by 4 elements), and codepth $I/I^2 = 0$, if I is a perfect Gorenstein ideal of height 3 (generated by 5 elements), see [8], 2.8.

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