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Titel: Framed Links for Pfeiffer Identities.

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Jahr: 1980

PURL: https://resolver.sub.uni-goettingen.de/purl?266833020_0175|log19

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Math. Z. 175, 125–137 (1980)

Mathematische Zeitschrift
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Framed Links for Peiffer Identities

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1. Introduction

An *aspherical* CW complex K is one whose higher dimensional homotopy groups are trivial. In [15], Whitehead questioned whether asphericity is a hereditary property for 2-dimensional CW complexes, i.e., whether every subcomplex of an aspherical 2-dimensional CW complex is itself aspherical. Despite considerable investigation of aspherical 2-dimensional CW complexes since then (for example, [1–10]), this question remains unresolved, in general.

A 2-dimensional complex K is aspherical if and only if its second homotopy group $\pi_2(K)$ is trivial, i.e., every spherical map $f\colon S^2 \to K$ admits a null homotopy $H\colon B^3 \to K$. It seems reasonable to investigate an aspherical 2-dimensional complex K and its subcomplexes from the viewpoint of the complexity of the null homotopies required for spherical maps into K. This is the purpose of this paper.

In Sects. 2 and 3, we show how to measure the complexity of a null homotopy H by the *framed link* Λ_H in B^3 obtained as the closure of the inverse images under H of the open 2-cells of K. In Sect. 4, we analyze the special situation of a 2-dimensional complex K which admits for each spherical map $f\colon S^2 \to K$ a null homotopy $H\colon B^3 \to K$ whose framed link Λ_H is geometrically split. In Sect. 5, the following is established.

Theorem 1. *A group presentation* $(X:R)$ *is aspherical (in the sense of Lyndon and Schupp* [9]*) and satisfies the Relator Conditions* (6) *if and only if every spherical map into the cellular model* K *of* $(X:R)$ *admits a null homotopy whose framed link is geometrically split.*

In Sect. 6, we show how to begin with an abstract framed link and realize it as the framed link Λ_H of a null homotopy $H\colon B^3 \to K$ of a spherical map $f\colon S^2 \to K$ into some 2-dimensional complex K. In several instances, we apply this construction to a non-geometrically split link to produce a presentation $(X:R)$ which is not aspherical (in the sense of [9]), yet whose cellular model K is aspherical. Moreover, these examples show that the asphericity of a group presentation, while a hereditary property, is not a combinatorial invariant of the presentation. In fact, we have

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Theorem 2. *Every group presentation is combinatorially equivalent (in the sense of* [14]) *to one which is not aspherical.*

This work began with an attempt to deduce Papakyriakopoulos' result [11] on the asphericity of a knot complement from just these facts: (i) asphericity of a group presentation is a hereditary property, and (ii) a knot complement has as a strong deformation retract the cellular model of a subpresentation of a presentation that is combinatorially equivalent to the trivial presentation $(X:X)$, which is definitely aspherical. Theorem 2 frustrates this approach.

To a reader of [9], it should be clear that Statement (5) of Sect. 4 and Theorem 1 of Sect. 5 of this paper serve as a substitute for [9, III. Prop. 10.1]. I understand that I. Chiswell, D. Collins, and J. Huebschmann have joint work that offers further alternatives to the treatment of asphericity of group presentations.

I thank J. Brandenburg and M.N. Dyer for numerous conversations that helped in the formulation of this work, and D. Collins for suggesting the terminology for Peiffer transformations employed in Sect. 5.

2. Nothing New Here

Let $(X:R)$ be any group presentation. Let $F(X)$ denote the free group on the set X of generators, and let N denote the normal closure in $F(X)$ of the set R of relators. The presentation $(X:R)$ has a *cellular model* $K=c^0 \cup c_x^1 \cup c_r^2$ $(x \in X, r \in R)$ whose cells are oriented by characteristic maps ϕ_x: $B^1 \to K$ and ϕ_r: $B^2 \to K$ such that there is an identification $\pi_1(K^1) \equiv F(X)$, under which the cellular path ϕ_x represents x, and the attaching map $\dot{\phi}_r = \phi_r | S^1$ represents r.

Let $F(X:R)$ denote the free group on the set $F(X) \times R$. There is an action of $F(X)$ on $F(X:R)$ given by $x(w, r) = (x w x^{-1}, r)$, and there is the homotopy action of $\pi_1(K^1)$ on the relative homotopy group $\pi_2(K^2, K^1)$. Let η: $F(X:R) \to \pi_2(K^2, K^1)$ denote the unique homomorphism that respects these actions and also carries $(1, r)$ to $[\phi_r: (B^2, S^1) \to (K^2, K^1)]$, for all $r \in R$.

(1) η: $F(X:R) \to \pi_2(K^2, K^1)$ *is surjective.*

To illustrate this fact, consider in the 2-ball B^2 a sequence $B_1, \ldots, B_n$ of disjoint discs centered on the axis $0 \times B^1$ and compatibly oriented with B^2. Let $\ell_1, \ldots, \ell_n$ denote line segments joining these discs to the basepoint $* = (1, 0)$ in the boundary S^1 of B^2. Associated with a sequence $\omega = ((w_1, r_1)^{\varepsilon_1}, \ldots, (w_n, r_n)^{\varepsilon_n})$ in the generators of $F(X:R)$ and their inverses (hereafter, ω is called a *word* in $F(X:R)$ and often abbreviated by $\prod (w_i, r_i)^{\varepsilon_i}$) is a map f_ω: $(B^2, S^1) \to (K^2, K^1)$ which carries the discs $B_1, \ldots, B_n$ via the signed characteristic maps $\varepsilon_1 \phi_{r_1}, \ldots, \varepsilon_n \phi_{r_n}$ and sends their complement into K^1, with the arcs $\ell_1, \ldots, \ell_n$ mapped as representative loops for the elements $w_1, \ldots, w_n$ in $F(X) \equiv \pi_1(K^1)$. If the word ω represents (i.e., freely reduces to) the group element $W \in F(X:R)$, then f_ω represents the homotopy class $\eta(W) \in \pi_2(K^2, K^1)$. Simplicial techniques show that any map $(B^2, S^1) \to (K^2, K^1)$ is homotopic to one of the form f_ω for some word ω in $F(X:R)$, thus η is surjective.

The homomorphism ∂: $F(X:R) \to F(X)$, $\partial(w, r) = w r w^{-1}$, and the boundary operator ∂: $\pi_2(K^2, K^1) \to \pi_1(K^1) \equiv F(X)$ are both $F(X)$-homomorphisms, where their range is given the action of conjugation, and they are compatible with the homomorphism $\eta: F(X:R) \to \pi_2(K^2, K^1)$.

Notice that $\mathrm{Im}(\partial: F(X:R) \to F(X))$ is the normal closure N in $F(X)$ of the set R of relators, and $E = \mathrm{Ker}(\partial: F(X:R) \to F(X))$, called the *group of identities* for the presentation $(X:R)$, contains $\mathrm{Ker}\,\eta$. The subgroup $P \leqq E$ of *Peiffer identities* for the presentation $(X:R)$ is the normal closure in $F(X:R)$ of the set of basic Peiffer elements

$$\{(w,r)(v,s)(w,r)^{-1}(wrw^{-1}v,s)^{-1} : w, v \in F(X), r, s \in R\}.$$

(This is the notation employed by Ratcliffe [13] for these special identities studied by Peiffer [12].)

There is this convenient algebraic characterization of Peiffer identities ([10, Theorem 3.1]):

(2) *A word* $\omega = \prod (w_i, r_i)^{\varepsilon_i}$ *in* $F(X:R)$ *represents a Peiffer identity* $W \in P$ *if and only if* $\partial(W) = \prod w_i r_i^{\varepsilon_i} w_i^{-1} = 1$ *in* $F(X)$ *and there is a pairing* (i,j) *of the indices of* ω *such that* (a) $r_i = r_j$, (b) $\varepsilon_i = -\varepsilon_j$, *and* (c) $w_i \in w_j N$.

Whenever the word ω represents an identity element $W \in E$, the map f_ω: $(B^2, S^1) \to (K^2, K^1)$ can be required to carry S^1 to $c^0 \in K$. In this case, we call f_ω a *spherical map*, since it easily converts to a map $S^2 \to K$.

There is also this homotopy characterization of the Peiffer identities:

(3) *A word* ω *in* $F(X:R)$ *represents a Peiffer identity* $W \in P$ *if and only if the map* $f_\omega: (B^2, S^1) \to (K^2, K^1)$ *is null homotopic. In short,* $P = \mathrm{Ker}(\eta: F(X:R) \to \pi_2(K^2, K^1))$.

Here is the geometry that underlies this assertion. By simplicial techniques, any homotopy H: $(B^2, S^1) \times I \to (K^2, K^1)$ of the map f_ω can be deformed to become one for which the closure of each inverse image $H^{-1}(c_r^2)$ of an attached 2-cell c_r^2 in K is a *framed link* Λ_r in this sense: each component L of Λ_r is either an embedded solid cylinder $D^2 \times B^1$ whose end discs $D^2 \times \pm 1$ lie in $B^2 \times \{0, 1\}$ or is an embedded solid torus $D^2 \times S^1$, in either case, on whose oriented cross-sectional discs $D^2 \times t$ the homotopy H acts like the characteristic map ϕ_r for the 2-cell c_r^2. The embedding of the segment $* \times B^1$ on the cylinder, or the loop $* \times S^1$ of the torus, gives an *index curve* γ_L on the cylindrical or toroidal component L that records its twisting. When the homotopy H is in such a form, we call the union $\bigcup \{\Lambda_r : r \in R\}$ the *framed link* Λ_H *of the homotopy* H.

If f_ω is a spherical map and H is a null homotopy for f_ω, the ends of a cylindrical component L of Λ_r must be two discs in just the floor $B^2 \times 0$ of $B^2 \times I$, and so they represent two factors $(w_i, r_i)^{\varepsilon_i}$ and $(w_j, r_j)^{\varepsilon_j}$ of the word ω for which $r_i = r = r_j$ and $\varepsilon_i = -\varepsilon_j$. Furthermore, the index curve γ_L on L and the arcs ℓ_i and ℓ_j in the floor $B^2 \times 0$ constitute a null homotopic loop in $B^2 \times I$. Since H is constant on γ_L and f_ω represents w_i and w_j on ℓ_i and ℓ_j, it follows that $w_i N = w_j N$ in $\pi_1(K) \equiv F(X)/N$. So the algebraic characterization (2) shows that ω represents a Peiffer identity $W \in P$. Thus, P contains $\mathrm{Ker}\,\eta$.

For the converse portion of (3), notice that each basic Peiffer element comes from a word $\omega = ((w,r), (v,s), (w,r)^{-1}, (wrw^{-1}v, s)^{-1})$ for which the map f_ω admits a null homotopy H with the framed link Λ_H in Fig. 1. In this figure, the index

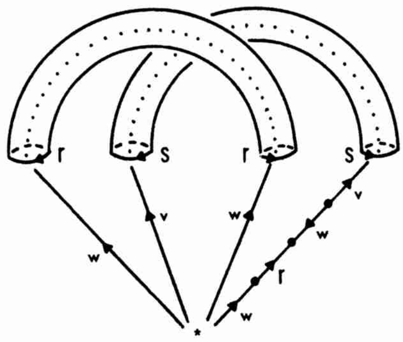


Fig. 1

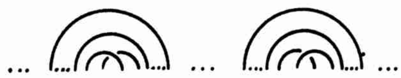
curves are indicated by the dotted lines that appear on the front of the framing cylinders. In subsequent figures, more involved framed links are depicted by just their central links, with the understanding that were the missing framing to be supplied, the index curves would appear on the front of all the link components. In this way, twists in a link component appear as self-crossings of the central link.

Statements (1) and (3) yield Whitehead's basic description [15] of the relative homotopy group $\pi_2(K^2, K^1)$ as a free crossed module.

3. Framed Links

We pursue further the geometry of the framed link Λ_H of a null homotopy H for a spherical map f_ω, where ω is a word that represents a Peiffer identity $W \in P$ for the presentation $(X:R)$.

For any word v that is an unreduced product of conjugates of basic Peiffer elements and their inverses, there is a null homotopy for f_v with framed link of this form:



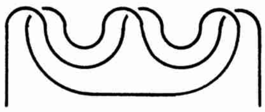
and any reduction of v (by some cancellation in the free group $F(X:R)$) to the word ω provides a homotopy from f_ω to f_v with a framed link of this form:

Stacked together, these provide a null homotopy H, in *cancellation form*, for f_ω. The framed link Λ_H may involve both toroidal and cylindrical components that are knotted, linked, and twisted.

For example, for any presentation $(X:R)$, the word

$$\nu=[(w,r)(wr,r)(w,r)^{-1}(wr^2,r)^{-1}][(wr^2,r)(w,r)(wr^2,r)^{-1}(wr,r)^{-1}]$$
$$\cdot[(wr,r)(wr^2,r)(wr,r)^{-1}(wr^3,r)^{-1}]$$

reduces to the word $\omega=(w,r)(wr^3,r)^{-1}$. There results a null homotopy for f_ω with a trefoil knot in its framed link:



Further, the word

$$\mu=[(w,r)(w,r)(w,r)^{-1}(wr,r)^{-1}][(wr,r)(wr,r)(wr,r)^{-1}(wr^2,r)^{-1}]$$
$$\cdot[(wr^2,r)(wr^2,r)(wr^2,r)(wr^3,r)^{-1}]$$

reduces to the same word $\omega=(w,r)(wr^3,r)^{-1}$. This time the resulting null homotopy for f_ω has the unknot with three twists in its framed link:

So different products ν that reduce to ω can lead to null homotopies for f_ω with quite different links. Moreover, different reductions to ω of the same product ν can produce different links.

The product

$$\nu=[(w,r)(w,r)(w,r)^{-1}(wr,r)^{-1}][(wr,r)(w,r)(r,r)^{-1}(w,r)^{-1}]$$
$$\cdot[(w,r)(w,r)(w,r)^{-1}(wr,r)^{-1}]$$

reduces to the word $\omega=(w,r)(wr,r)^{-1}$ in two different manners to give null homotopies for f_ω with these framed links

 and 

The single twist on the cylindrical component of the first homotopy is removed in the second homotopy, at the expense of introducing a toroidal component that links non-trivially with the cylindrical component.

It is possible to reverse this process of constructing a null homotopy for f_ω from a product ν that reduces to the word ω. The framed link Λ_H in any null homotopy H for f_ω not only gives a pairing as in the proof of (3) that shows that the word ω represents a Peiffer identity, but in fact, when molded to be in cancellation form, Λ_H provides a specific reduction to ω of a product ν of conjugates of basic Peiffer elements and their inverses. This gives a geometric proof of (3), independent of (2).

Hereafter, when ω is the reduced word in $F(X:R)$ that represents the Peiffer identity $W \in P$, we view the framed link Λ_H of a null homotopy H for f_ω as a good representative of the complexity of W, and we call Λ_H *a framed link for the Peiffer identity W.*

4. Geometrically Split Null Homotopies

The asphericity of the 2-dimensional complex K modeled on the presentation $(X:R)$ is equivalent to the triviality of the homotopy group $\pi_2(K)$, which we view as $\mathrm{Ker}(\partial: \pi_2(K^2, K^1) \to \pi_1(K^1))$. Because $\partial: F(X:R) \to F(X)$ coincides with $\partial\eta: F(X:R) \to \pi_2(K^2, K^1) \to \pi_1(K^1)$, K is aspherical if and only if the group of identities $E = \mathrm{Ker}\,\partial$ for $(X:R)$ coincides with the subgroup of Peiffer identities $P = \mathrm{Ker}\,\eta$ for $(X:R)$. This is precisely the requirement that for each word ω in $F(X:R)$ that represents an identity element $W \in E$, the spherical map f_ω admits a null homotopy H. As noted in the previous sections, H has a framed link Λ_H that may involve both toroidal and cylindrical components that are possibly knotted, linked, and twisted.

Each subcomplex M of K is the model of some subpresentation $(X':R')$ of $(X:R)$. Form, as in Section 2, the homomorphism $\partial': F(X':R') \to F(X')$, and the group of identities $E' = \mathrm{Ker}\,\partial'$ and the subgroup $P' \leqq E'$ of Peiffer identities for $(X':R')$. Then $F(X':R') \cap E = E'$ and $F(X':R') \cap P \supset P'$. Clearly, we have

(4) *The following are equivalent for a subcomplex M of an aspherical 2-dimensional complex K:*

(a) *M is aspherical,*

(b) $F(X':R') \cap P = P'$,

(c) *for each word ω in $F(X':R')$ that represents an identity element $W \in E'$, the spherical map f_ω in M admits a null homotopy H in K whose framed link Λ_H is free of toroidal links Λ_r for $r \notin R'$.*

The simplest way to insure that the condition (4c) holds for each subcomplex M of K is to consider a complex K which admits a geometrically split null homotopy for each of its spherical maps. There are the following two approaches to this geometric splitting.

A null homotopy $H: (B^2, S^1) \times I \to (K^2, K^1)$ of a spherical map f_ω is *split by spheres* if there exists a family $\{S_L\}$ of disjoint embedded 2-spheres S_L in $B^2 \times I$ indexed by the cylindrical components L of the framed link Λ_H such that: (i) S_L meets Λ_H in precisely the two end discs of L in $B^2 \times 0$, and (ii) S_L meets the boundary of $B^2 \times I$ in just a disc d_L in $B^2 \times 0$ that contains $S_L \cap \Lambda_H$. Then each sphere S_L bounds a 3-ball B_L in $B^2 \times I$ that contains the entire cylindrical component L, and possibly some totoidal components, both of which may be knotted and linked together.

A null homotopy H is *split by discs* if there exists a family $\{D_L\}$ of disjoint embedded 2-discs D_L in $B^2 \times I$ indexed by the cylindrical components L of the framed link Λ_H such that the boundary curve of D_L is the union of the index curve

γ_L on L with a simple arc J_L in $B^2 \times 0$, and D_L has no other intersection with Λ_H or the boundary of $B^2 \times I$. This is a much more restrictive situation. The existence of a disc D_L implies that the cylindrical component L is unknotted, and the existence of the family $\{D_L\}$ implies that the cylindrical components do not link with themselves nor with the toroidal components.

(5) *A spherical map f_ω for the word $\omega = \prod (w_i, r_i)^{\varepsilon_i}$ admits a null homotopy H that is split by discs if and only if there is a pairing (i, j) of the indices of ω and a family $\{J_{i,j}\}$ of disjoint simple arcs in $B^2 - \bigcup_{i=1}^{n} \mathring{B}_i$ such that* (i) $r_i = r_j$, (ii) $\varepsilon_i = -\varepsilon_j$, *and* (iii) *$J_{i,j}$ joins the basepoints of B_i and B_j, and $f_\omega(J_{i,j})$ represents $1 \in \pi_1(K^1)$.*

Proof. When H is split by the discs $\{D_L\}$, then each simple curve $J_L = D_L \cap (B^2 \times 0)$ is homotopic over D_L to the index curve γ_L on L. H carries this homotopy in D_L to a null homotopy of $f_\omega(J_L)$ in K^1. Since the discs $\{D_L\}$ are disjoint, $\{J_L\}$ is a suitable family of disjoint simple arcs. Conversely, given a suitable pairing (i, j) and corresponding family $\{J_{i,j}\}$ of disjoint simple arcs, there is a null homotopy H for f_ω whose levels show each pair of discs B_i and B_j moving toward each other along their simple arc $J_{i,j}$ to eventually cancel out. This null homotopy is split by a family of discs associated with the given family $\{J_{i,j}\}$.

Despite the differences in the definitions of splitting by spheres and splitting by discs, we have the following result.

Theorem 3. *The following are equivalent for any (necessarily aspherical) 2-dimensional complex K:*

(i) *Every spherical map in K admits a null homotopy which is split by spheres.*

(ii) *Every spherical map in K admits a null homotopy which is split by discs. And in either case, the null homotopy can be made free of all toroidal components.*

For convenience, we first establish the following

Lemma. *Let H be a null homotopy with framed link Λ_H, and let B be an embedded 3-ball in $B^2 \times I$ with boundary sphere S.*

(i) *If S misses Λ_H, then H can be modified on B relative to S to eliminate link components in B.*

(ii) *If S meets Λ_H in just two cross-sectional discs $D_\pm$ of some component L of Λ_r and if $r \in F(X)$ is not a proper power, then H can be modified on B relative to S so that $B \cap \Lambda_H$ becomes just an unknotted cylinder with some twists.*

Proof. (i) Since by hypothesis, H carries S into the 1-dimensional, hence aspherical, skeleton K^1 of K, $H|S$ extends over B into K^1.

(ii) The loop $\alpha = \phi_r$ that represents $r \in F(X)$ lifts through H to give the boundary loops $\tilde{\alpha}_\pm$ of the discs $D_\pm$ on S. Let $\tilde{\beta}$ be any simple arc in S that meets $D_\pm$ in just the basepoints of $\tilde{\alpha}_\pm$. The product path $\tilde{\alpha}_- \tilde{\beta} \tilde{\alpha}_+^{-1} \tilde{\beta}^{-1}$ is null homotopic in $S - (D_- \cup D_+)$, and so its image under H is a null homotopic path $\alpha \beta \alpha^{-1} \beta^{-1}$ in K^1. It follows that α and β represent commuting elements r and w in the free group $F(X)$, which must be powers of the same element. But because r is not a proper, we conclude $w = r^k$ and $\beta \simeq \alpha^k$, for some k.

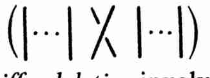
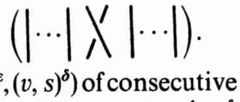
Then $H|S$ extends over an embedded cylinder $D^2 \times B^1$ in B with ends $D^2 \times \pm 1 \equiv D_\pm$ by $H|(D^2 \times t) = \phi_r$. We use the embedding that makes the spiral path that twists k-times around the boundary of the cylinder $D^2 \times B^1$ lie parallel to the simple arc $\tilde{\beta}$ in S. Then the homotopy $\beta \simeq \alpha^k$ in K^1 provides a further extension of $H|S$ over an embedded disc in B that meets S in the simple arc $\tilde{\beta}$ and meets the embedded cylinder $D^2 \times B^1$ in the spiral path. The remainder of B is an open ball whose (singular) boundary is mapped into K^1. So again, the asphericity of K^1 provides the required final extension of $H|S$ over all of B.

Proof of Theorem 3. If a null homotopy H is split by the discs $\{D_L\}$, then H is split by the spheres $\{S_L\}$ that bound regular neighborhoods in $B^2 \times I$ of the unions $\{D_L \cup L\}$. Conversely, if a null homotopy H is split by the spheres $\{S_L\}$, then by the lemma, H can be modified on the balls $\{B_L\}$ they bound so that each B_L becomes free of toroidal components and meets Λ_H in just an unknotted cylindrical component L. Since L simply twists in B_L, there is an embedded disc D_L in B_L spanned by the index curve γ_L on L and a simple arc J_L in the disc $d_L = S_L \cap B^2 \times 0$. The family of discs $\{D_L\}$ splits H in the defined manner.

Finally, H can be further modified on a ball B in the complement of the floor $B^2 \times 0$ and the balls $\{B_L\}$ to eliminate all remaining toroidal components.

We say that K *admits geometrically split null homotopies* if the equivalent conditions of Theorem 3 hold. Then by (4), every subcomplex M of K is aspherical.

5. Aspherical Presentations

A *Peiffer exchange* involves the replacement of a pair $((w, r)^\varepsilon, (v, s)^\delta)$ of consecutive entries of a word ω in $F(X{:}R)$ by either $((wr^\varepsilon w^{-1}v, s)^\delta, (w, r)^\varepsilon)$ or $((v, s)^\delta, (vs^{-\delta}v^{-1}w, r)^\varepsilon)$. If ν is the new word that results from the Peiffer exchange, there is a homotopy H from f_ω to f_ν with framed link Λ_H that involves a single overcrossing  or a single undercrossing .

A *Peiffer deletion* involves a deletion of a pair $((w, r)^\varepsilon, (v, s)^\delta)$ of consecutive entries of a word ω in $F(X{:}R)$, provided that the boundary condition $wr^\varepsilon w^{-1}vs^\delta v^{-1} = 1$ holds in $F(X)$. This transformation has no homotopy interpretation unless we assume the following

(6) *Relator Conditions: No relator of R is a proper power, nor a conjugate of another relator or its inverse.*

These conditions are necessary for the presentation $(X{:}R)$ to have an aspherical model K ([4]). In the presence of (6), the boundary condition $wr^\varepsilon w^{-1}vs^\delta v^{-1} = 1$ implies that $r = s$, $\varepsilon = -\delta$, and $w = vr^k$ for some integer k (since $v^{-1}w$ commutes with r, which is not a proper power). Then, if ν is the result of the Peiffer deletion of the pair $((w, r)^\varepsilon, (v, s)^\delta)$ in the word ω, there is a homotopy H from f_ω to f_ν with framed link Λ_H that involves a cylindrical component with k twists that caps off the deleted pair. Alternately, the Peiffer deletion factors into a sequence of Peiffer exchanges, followed by a strict cancellation in which a consecutive pair $((w, r)^\varepsilon, (w, r)^{-\varepsilon})$ is deleted. (Recall your most recent encounter with a twisted garden hose.) Strict cancellation corresponds to a homotopy with a framed link that involves a cap with no twists .

A group presentation $(X:R)$ is *aspherical* (in these sense of Lyndon and Schupp [9, III 10]) if every word ω in $F(X:R)$ that represents an identity element $W \in E$ reduces to the empty word by a sequence of Peiffer exchanges and Peiffer deletions. One relator and staggered presentations, planar presentations, and certain presentations with small cancellation are examples of aspherical presentations offered by [9].

Theorem 1. *The presentation* $(X:R)$ *is aspherical and satisfies the Relator Conditions* (6) *if and only if the cellular model K of* $(X:R)$ *admits a geometrically split null homotopy for each spherical map into K.*

Proof. If a word ω reduces to the empty word by a sequence of Peiffer exchanges and deletions, then there corresponds a null homotopy H of f_ω built of a stack of overcrossing, undercrossing, and cancellation cap homotopies. This null homotopy H has a framed link Λ_H with only cylindrical components that are somewhat braided. So H is split by the discs $\{D_L\}$, where D_L is created by an arc whose ends slide down the two halves of the index curve Λ_L on the link component L, starting at the center of the cap homotopy portion of L. Conversely, if for a word ω, the map f_ω admits a null homotopy H that is split by discs $\{D_L\}$, these discs show how H can be molded to look like a stack of overcrossing, undercrossing, and cancellation cap homotopies. These determine a sequence of Peiffer transformations that reduce ω to the empty word.

6. Construction of Examples

Let Λ be an abstract framed link in $B^2 \times I$ whose cylindrical components meet the boundary in just the floor $B^2 \times 0$. Assume that the components of Λ have been assigned relators from a presentation $(X:R)$, and let Λ_r involve all components associated with $r \in R$.

It is quite easy to deduce conditions on the set of relators R that guarantee the presence of a Peiffer identity $W \in P$ in $F(X:R)$ with this framed link Λ. Simply arrange Λ in $B^2 \times I$ so that there exists levels $\{0 = t_0 < t_1 < \ldots < t_n = 1\}$ between which the link Λ involves just a single cap $\frown$, overcrossing X, undercrossing X, or cup $\smile$. Then work downward. Let ω_n be the empty word, and suppose that the level t_k has already suggested a word ω_k in $F(X:R)$. If the portion of the link Λ between the levels t_{k-1} and t_k involves:

(i) a cap that introduces a component of Λ_r, then insert in ω_k a pair $((w,r)^\varepsilon, (w,r)^{-\varepsilon})$ at the appropriate place to form ω_{k-1},

(ii) an overcrossing involving $((w,r)^\varepsilon, (v,s)^\delta)$ in ω_k, then replace them by $((v,s)^\delta, (vs^{-\delta}v^{-1}w, r)^\varepsilon)$ to form ω_{k-1},

(iii) an undercrossing involving $((w,r)^\varepsilon, (v,s)^\delta)$ in ω_k, then replace them by $((wr^\varepsilon w^{-1}v, s)^\delta, (w,r)^\varepsilon)$ to form ω_{k-1}, and

(iv) a cup involving $((w,r)^\varepsilon, (v,r)^{-\varepsilon})$ in ω_k, then delete them to form ω_{k-1} and record the cancellation condition: $w = v$ in $F(X)$.

If all the cancellation conditions are satisfied by the presentation $(X:R)$, then the word $\omega = \omega_0$ will necessarily represent a Peiffer identity $W \in P$ for $(X:R)$ with framed link Λ.

Fig. 2

For example, the non-geometrically split link $\Lambda=\Lambda_r\cup\Lambda_t$ in Fig. 2 leads level by level to the following words (in which we abbreviate conjugation wrw^{-1} by r^w)

$$(1,r)^{-1}\quad(1,r)\quad(w,r)^{-1}\quad(w,r)$$
$$(1,r)^{-1}\quad(w,r)^{-1}\quad(r^w,r)\quad(w,r)$$
$$(r^{-1}w,r)^{-1}\quad(1,r)^{-1}\quad(r^{r^w}w,r)\quad(r^w,r)$$
$$(r^{-1}w,r)^{-1}\quad(1,r)^{-1}\quad(1,t)\quad(1,t)^{-1}\quad(r^{r^w}w,r)\quad(r^w,r)$$
$$(r^{-1}w,r)^{-1}\quad(r^{-1},t)\quad(1,r)^{-1}\quad(t^{-1}r^{r^w}w,r)\quad(1,t)^{-1}\quad(r^w,r)$$
$$(r^{-1}w,r)^{-1}\quad(r^{-1},t)\quad(1,t)^{-1}\quad(r^w,r)$$

and to the cancellation condition $t=r^{r^w}w=wrw^{-1}rwr^{-1}$. For any presentation $(X:R)$ in which this condition is satisfied, the last word $\omega=(r^{-1}w,r)^{-1}(r^{-1}\,t)\cdot(1,t)^{-1}(r^w,r)$ represents a Peiffer identity $W\in P$ with the given framed link Λ.

The simplest example of such a presentation is $(x,y:x,yxy^{-1}xyx^{-1})$. Notice that the cellular model of this presentation is aspherical, in fact, contractible, and has all of its subcomplexes aspherical. But this presentation is not aspherical in the sense of [9], since the word

$$\omega=(x^{-1}y,x)^{-1}(x^{-1},yxy^{-1}xyx^{-1})(1,yxy^{-1}xyx^{-1})^{-1}(yxy^{-1},x)$$

gives a spherical map f_ω for which there is no family of simple arcs as in (5). In fact, on the cluster C of discs B_i and arcs l_i, $1\leqq i\leqq 4$, in B^2, the map f_ω has the cellular assignments indicated in Fig. 3 and any path in $B^2-\bigcup_{i=1}^{4}\mathring{B}_i$ that joins basepoints of paired discs deforms relative its endpoints to a cellular path in the 1-skeleton of this cluster C. But no such path can represent the trivial element $1\in F(x,y)$, as is easily seen by consideration of the quotient space of C presented in Fig. 4. (Incidently, this quotient space of C describes a map $S^2\to K$ which is combinatorial in the sense of [7].) By (5) and the proof of Theorem 1, the Peiffer identity ω cannot reduce to the empty word by Peiffer exchanges and deletions. (Collins informs me that this example is equivalent to one constructed by Chiswell.) We conclude from this example that

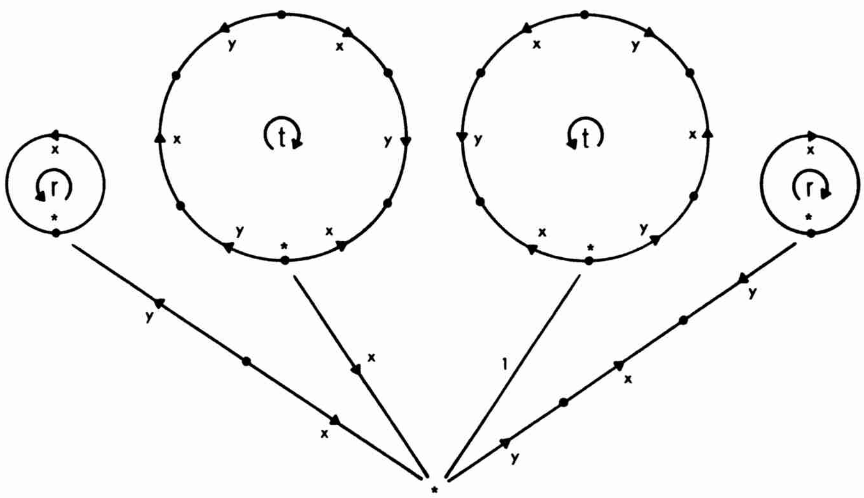


Fig. 3

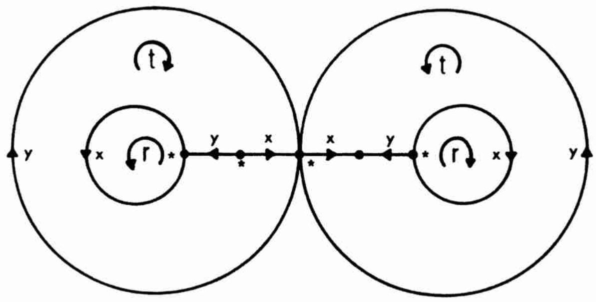


Fig. 4

(7) *Not every aspherical 2-dimensional complex admits geometrically split null homotopies for all its spherical maps; not every aspherical 2-dimensional complex is modeled on a presentation that is aspherical in the sense of* [9].

Furthermore, the presentation $(x, y: x, yxy^{-1}xyx^{-1})$ results from the aspherical presentation $(x, y: x, y)$ by the combinatorial operation (in the sense of [14]) of replacing the relator y by its product with the conjugate $xy^{-1}xyx^{-1}$ of the other relator x. So we also see that

(8) *The asphericity of a group presentation in the sense of* [9] *is not a combinatorial invariant; the existence of geometrically split null homotopies for a 2-dimensional complex is not an invariant of homotopy-type.*

Since there are combinatorial operations that expand any presentation to include $(x, y: x, yxy^{-1}xyx^{-1})$, Theorem 2 of the introduction follows.

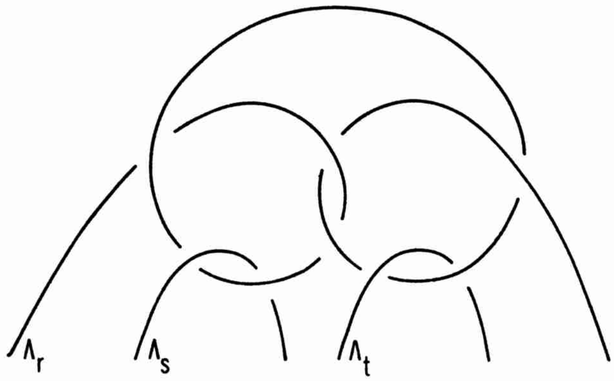


Fig. 5

For a second example to illustrate the construction technique of this section, consider the framed link $\Lambda = \Lambda_r \cup \Lambda_s \cup \Lambda_t$ in Fig. 5. Here the analysis leads to the Peiffer identity

$$\omega = (xy, r)(1, s)(sxs^{-1}, s)^{-1}(1, t)(zx^{-1}z^{-1}xzxz^{-1}, t)^{-1}(z, r)^{-1}$$

in the presentation $(x, y, z : r, s, t)$ where $r = x$, $s = x^{-1}y^{-1}zx^{-1}z^{-1}yxy^{-1}$, and $t = yx^{-1}y^{-1}zzxz^{-1}$. This presentation is not aspherical in the sence of [9], because this word ω does not reduce to the empty word by Peiffer transformations. There are simply no simple paths in the domain of f_ω as required by (5). To see this, consider the quotient space in Fig. 6 for the cluster of discs B_i and arcs l_i, $1 \leqq i \leqq 6$, in the domain B^2 of f_ω.

As in the first example, the non-aspherical presentation of Example 2 results from some sequence of combinatorial operations (in the sense of [14]) on the

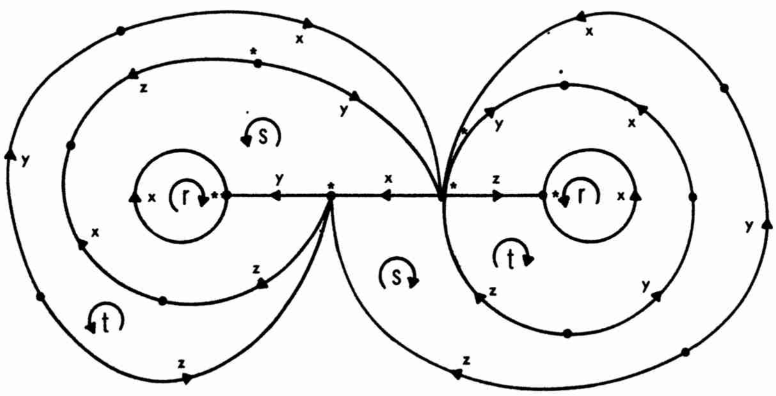


Fig. 6

trivial presentation $(x, y, z : x, y, z)$. If one prefers an example which presents a non-trivial group, it is possible to begin with such a presentation with an aspherical model (say, a deficiency-one knot presentation) and perform the combinatorial operations suggested by the cancellation conditions to obtain a non-aspherical presentation with a non-contractible, though aspherical, model.

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Received February 27, 1980