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Cellular Actions and Groups of Finite Quasi-Projective Dimension

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1. Introduction

1.1. In [2] quasi-projective dimension (denoted qpd) was introduced as a homological invariant for groups. It agrees with cohomological dimension (denoted cd) on all torsion-free groups. According to [1] the inequality $\operatorname{cd} G \leqq \operatorname{gd} G \leqq \operatorname{cd} G + 1$ holds between geometric dimension gd and cohomological dimension for all groups G. Suppose now there is a free cellular G-action on a contractible n-complex. Then $\operatorname{cd} G \leqq \operatorname{gd} G \leqq n$. We present here a sufficient condition for a group to have finite qpd in terms of suitable group actions on acyclic CW-complexes of finite dimension. Moreover, a geometric interpretation of the Identity Property (cf. 1.2 below) is given in this context.

This note is considered a complement to [2], where groups of finite qpd were studied by algebraic means alone. Some of the algebraic results of [2] could have been rederived in the present context. However, in order to keep the exposition short, we have not done so.

Let R be a commutative ring with unit and let G be a group. Recall from [2] that an exact sequence of RG-modules

$$\mathcal{Q}\colon\ 0 \to Q \oplus P_n \to P_{n-1} \to \dots \to P_0 \to A \to 0$$

is called an *RG-quasi-projective resolution of A* if all the P_i, $0 \leqq i \leqq n$ are RG-projective and either $n=0$ and $Q=0$, or $n>0$ and $Q \cong \bigoplus_I RG/G_\alpha$ is a permutation module. We say that $\operatorname{qpd}_R G = k \leqq \infty$ if k is the minimal length of all RG-quasi-projective resolutions of the RG-module R with trivial G-action. We write qpd G for $\operatorname{qpd}_{\mathbb{Z}} G$.

As shown in [2], the subgroups G_α occurring in any RG-quasi-projective resolution of finite length are finite.

1.2. Our results are based on the following definition. We consider a cellular action of a group G on a CW-complex X. Such an action is called *m-free* if it restricts to a free action on the m-skeleton $X^{(m)}$ of X.

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Note that if G acts m-freely on a CW-complex X and e is a k-cell of X, $k>0$, then the stabilizer G_e of e acts m-freely on the smallest subcomplex X_e of X containing e. Since X_e is finite, and G_e acts faithfully by freely permuting the cells of the m-skeleton of X_e, the group G_e is finite. In particular, if G is torsion-free, then any m-free G-action is free.

In order to state the first result, we need to look more closely at the group actions which can occur. Suppose the group G acts cellularly on the CW-complex X, and the element g of G maps the n-cell e to itself ($n>0$). Then g induces a self-homeomorphism $\tilde{g}$ of the interior of e, which we may regard as an open n-disk. We say that g *inverts* e if $\tilde{g}$ is orientation-reversing. We say that G acts *without inversion* on X if no element of G inverts an n-cell of X for any $n \geqq 1$.

Clearly G acts without inversion on X if G acts freely on X, or if G has finite stabilizers and has no 2-torsion.

Theorem 1. *Suppose G is a group which acts $(m-1)$-freely on the R-acyclic m-complex X.*

(a) *If G acts without inversion, then* $\operatorname{qpd}_R G \leqq m$.

(b) *If $R=\mathbb{Z}$, and some m-cell is inverted under the G-action, then m is odd and* $\operatorname{qpd} G \leqq m+1$.

We define the Identity Property for a presentation of a group such that it is equivalent to condition I.1 of Proposition 10.2 in [3, p. 158].

Definition. A presentation $\langle U \mid R \rangle$ of a group G has the *Identity Property* if the following conditions are satisfied.

(i) The relation module is isomorphic to $\bigoplus_{r \in R} C_r$, where C_r is the cyclic $\mathbb{Z}G$-submodule generated by the image in the relation module of the relator r.

(ii) There is an isomorphism $C_r \cong \mathbb{Z}G/G_r$, where G_r is the image in G of the centralizer of r in the free group $F(U)$.

A group G has the Identity Property if some presentation of G has the Identity Property.

It is obvious from this definition that for any group G satisfying the Identity Property, the inequality $\operatorname{qpd} G \leqq 2$ holds.

Theorem 2. *A group G satisfies the Identity Property if and only if there exists a 1-free G-action on some contractible 2-complex.*

Corollary. *The group G has geometric dimension at most 2 if and only if G is torsion-free and has the Identity Property.*

These results confirm that the Identity Property for G is *a priori* stronger than the property $\operatorname{qpd} G \leqq 2$. In the case of torsion-free groups, the converse implication is the Eilenberg-Ganea problem; it amounts to showing that $\operatorname{cd} G = 2$ implies $\operatorname{gd} G = 2$.

Another related property is for G to be "aspherical" in the sense of Lyndon and Schupp [3]. By [3, p. 158, Prop. 10.2], any "aspherical" group satisfies the Identity Property.

1.3. In section 3 we apply Theorem 1 to find a set of new examples of groups of finite qpd. Rather than constructing quasi-projective resolutions of finite length, we verify that the groups considered act $(n-1)$-freely on some Euclidian or hyperbolic n-space. The criterion easily applies to discrete groups of motions of some planar tesselation. The group actions on hyperbolic n-space $(n>2)$ described in [5] are not $(n-1)$-free for obvious reasons, but we have not been able to decide whether some restriction to a subgroup (with non-trivial torsion) yields a new example of an $(n-1)$-free action.

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2. Proofs

Proof of Theorem 1. (a) The cellular chain complex of X gives rise to a sequence of $\mathbb{Z}G$-modules

$$\mathscr{C}\colon\ 0 \to M \to C_{n-1} \to \ldots \to C_0 \to \mathbb{Z} \to 0$$

in which the C_i are free, and M is a permutation module. Since X is R-acyclic, the sequence $R \underset{\mathbb{Z}}{\otimes} \mathscr{C}$ is exact, and so is an RG-quasi-projective resolution for R.

(b) Suppose now that $R=\mathbb{Z}$, and G does not act without inversion on X. We claim:

(1) n is odd;

(2) any non-trivial element of the stabilizer in G of the n-cell e of X inverts e;

(3) the stabilizer in G of any n-cell of X has order at most 2.

Some element g of G inverts some n-cell e of X. Let S denote the cyclic subgroup generated by g. Then, regarded as a $\mathbb{Z}S$-module, M has a direct summand $\tilde{\mathbb{Z}}$, the $\mathbb{Z}S$-module with underlying abelian group $\mathbb{Z}$ and non-trivial S-action. Now consider $\mathscr{C}$ as a sequence of $\mathbb{Z}S$-modules. Using Schanuel's Lemma, and comparing M with the n'th kernel in a $\mathbb{Z}S$-free resolution of the form

$$\ldots \to \mathbb{Z}S \xrightarrow{(1-g)} \mathbb{Z}S \to \mathbb{Z}S \xrightarrow{(1-g)} \mathbb{Z}S \to \mathbb{Z} \to 0,$$

we conclude that n is odd, establishing (1).

To prove (2), suppose some h in G fixes an n-cell e of X, but does not invert e. Let T denote the cyclic subgroup of G generated by h. Then there is a $\mathbb{Z}T$-module isomorphism $M \cong M_1 \oplus \mathbb{Z}$, where T acts trivially on $\mathbb{Z}$. Since n is odd and T is cyclic, we have

$$0 = H_{n+1}(T;\mathbb{Z}) \cong \operatorname{Tor}_1^{\mathbb{Z}T}(M;\mathbb{Z}) \cong \operatorname{Tor}_1^{\mathbb{Z}T}(M_1;\mathbb{Z}) \oplus H_1(T;\mathbb{Z}).$$

Hence $T \cong H_1(T;\mathbb{Z})$ is trivial, which establishes (2). Claim (3) follows immediately from (2).

It now follows that M has a decomposition as a $\mathbb{Z}G$-module in the form

$$M \cong F \oplus (\bigoplus_I \mathbb{Z}G \otimes_{S_\alpha} \tilde{\mathbb{Z}}_\alpha)$$

where F is $\mathbb{Z}G$-free, each S_α is a subgroup of order 2 in G, and $\tilde{\mathbb{Z}}_\alpha$ is the abelian group $\mathbb{Z}$ with non-trivial S_α-action.

We can now extend $\mathscr{C}$ to a $\mathbb{Z}G$-quasi-projective resolution of $\mathbb{Z}$ of length $n+1$ by splicing on the exact sequence

$$0 \to \bigoplus_I \mathbb{Z}G \otimes_{S_\alpha} \mathbb{Z} \to F \oplus (\bigoplus_I \mathbb{Z}G) \to M \to 0.$$

This completes the proof of Theorem 1.

Proof of Theorem 2. Suppose G has a presentation $\langle U|R\rangle$ which satisfies the Identity Property. We construct a CW-complex X which may be thought of as a geometrical realization of the combinatorial Cayley Complex [3, p. 123]. We first form the 2-complex Y associated to the presentation $\langle U|R\rangle$. Thus $Y^{(1)}$ is a wedge of circles with $\pi_1(Y^{(1)}) = F(U)$, and we make the following convention for the attaching maps of 2-cells in Y. If $r \in R$ has the form s^m, where $m \geqq 1$ and s is not a proper power in $F(U)$, choose a map a', $S^1 \to Y^{(1)}$ in the homotopy class s, and define $a: S^1 \to Y^{(1)}$ by $a(z) = a'(z^m)$. Then a is in the homotopy class r, and we use a to attach the 2-cell corresponding to r.

Now let $\tilde{Y}$ be the universal covering of Y. Call two 2-cells of $\tilde{Y}$ *equivalent* if their attaching maps are homotopic in $\tilde{Y}^{(1)}$. It follows from the Identity Property that, for every 2-cell e of $\tilde{Y}$, there is a finite cyclic subgroup G_e of G which regularly permutes the equivalence class containing e. Choose a representative set E of the G-orbits of 2-cells of $\tilde{Y}$, and for each e in E a left transversal T_e of G_e in G. Now let X be the subcomplex of $\tilde{Y}$ consisting of $\tilde{Y}^{(1)}$, together with the 2-cells $\{t(e);\ e \in E,\ t \in T_e\}$.

Clearly G maps equivalent 2-cells to equivalent 2-cells. Hence X contains precisely one 2-cell from each equivalence class, and so is simply connected. From the Identity Property for $\langle U|R\rangle$, it follows that X is acyclic. Hence X is contractible.

The action of the finite cyclic group G_e on the unit disc by rotations defines an action of G_e on the interior of e. By the particular choice of attaching maps in Y, this extends to an action on $\tilde{Y}^{(1)} \cup e$ which restricts to the natural G_e-action on $\tilde{Y}^{(1)}$. For $t \in T_e$, the automorphism of $\tilde{Y}$ determined by t restricts to a homeomorphism $\tilde{Y}^{(1)} \cup e \to \tilde{Y}^{(1)} \cup t(e)$. Piecing these homeomorphisms together, we extend the natural free G-action on $\tilde{Y}^{(1)}$ to a 1-free action on X.

Conversely, suppose G acts 1-freely on the contractible 2-complex X. Since G acts freely on the graph $X^{(1)}$, we may form the quotient graph $Y = G \setminus X^{(1)}$. Let p: $X^{(1)} \to Y$ denote the quotient map, fix a 0-cell v of $X^{(1)}$ as a base-point, and define $N = \pi_1(X^{(1)}, v)$, $F = \pi_1(Y, p(v))$. Then the homotopy exact sequence of p has the form

$$1 \to N \xrightarrow{p_*} F \xrightarrow{d} G \to 1.$$

An explicit set of defining relators in F for a presentation of G is given as follows. Choose a representative 2-cell e of X from each G-orbit $[e]$. Choose a closed path a_e, based at v, representing the free homotopy class of the attaching map of e. Now define $r_{[e]}$ to be the element of F represented by the closed path $p(a_e)$ in Y. The element $r_{[e]}$ is well-defined up to conjugacy in F.

Now let R denote the set of all $r_{[e]}$ as $[e]$ runs through the set of G-orbits of 2-cells of X. Since X is contractible, it follows that R generates $p_*(N)$ as a normal subgroup of F, so that R is a set of defining relators for a presentation of G. The Identity Property for this presentation is equivalent to the property that the stabilizer in G of any representative 2-cell e is the image under d of the centralizer in F of $r_{[e]}$.

If $g(e)=e$, then $g(a_e)$ and a_e represent the same free homotopy class of maps $S^1 \to X^{(1)}$. Hence, for a suitably chosen path x in $X^{(1)}$ from v to $g(v)$, the paths $x \cdot g(a_e) \cdot x^{-1}$ and a_e represent the same element of N. If f is the element of F represented by the closed path $p(x)$ in Y, then f commutes with $r_{[e]}$, and $d(f)=g$. Conversely, if $f \in F$ commutes with $r_{[e]}$, then $d(f)(a_e)$ and a_e represent the same free homotopy class and, since X is contractible, it follows that $d(f)(e)=e$.

3. Examples

Suppose (M, T) is a triangulated n-manifold without boundary. If a group G acts on M in such a way that T is G-invariant and for all $g \neq 1$ in G, the only fixed points of the associated map $\tilde{g}: M \to M$ are vertices of T, then the G-action induced on the dual complex T^* is $(n-1)$-free. In particular, if M is contractible, then $\operatorname{qpd} G \leqq n+1$.

This criterion covers all the examples below.

3.1. Suppose Λ is an order in some n-dimensional $\mathbb{Q}$-algebra A. Let F be a finite subgroup of the group of units of Λ. The group F acts on the abelian group Λ by left multiplication, and for $f \neq 1$ in F, the action of f has no fixed point in Λ except 0, provided $(1-f)$ is not a zero-divisor in Λ (in partiular, if A is a division-algebra). If we regard A as $\mathbb{Q}^n \subset \mathbb{R}^n$, then the F-action on Λ extends to a linear representation of F, $\rho: F \to GL(n, \mathbb{R})$. Now the semidirect product $\Lambda \rtimes F$ acts on $\mathbb{R}^n$ by affine transformations: $(\underline{\lambda}, f)\underline{x} = \underline{\lambda} + \rho(f)(\underline{x})$ $(\underline{\lambda} \in \Lambda, f \in F, \underline{x} \in \mathbb{R}^n)$. It is possible to construct a cellular subdivision C of $\mathbb{R}^n$ with convex cells, such that C is preserved by the action of $\Lambda \rtimes F$, and any fixed point of any $(\underline{\lambda}, f) \neq (0, 1)$ in $\Lambda \rtimes F$ is a vertex of C. Hence the action of $\Lambda \rtimes F$ on the dual C^* is $(n-1)$-free, and $\operatorname{qpd}(\Lambda \rtimes F) \leqq n+1$.

Particular examples for F and Λ are the following:

(a) $F = \mathbb{Z}/2\mathbb{Z}$ as the group of units of $\Lambda = \mathbb{Z} \subset \mathbb{Q}$.

(b) For $m > 2$, $F = \mathbb{Z}/m\mathbb{Z}$ as a group of units of $\Lambda = \mathbb{Z}[\exp 2\pi i/m]$, in the $\varphi(m)$-dimensional $\mathbb{Q}$-algebra $A = \mathbb{Q}[\exp 2\pi i/m]$, where φ is the Euler function.

(c) The quaternion group F as the group of units of $\Lambda = \mathbb{Z}[i, j] \subset \mathbb{Q}[i, j]$ in the quaternions $\mathbb{H}$.

(d) The binary tetrahedral group F as the group of units of

$\Lambda = \mathbb{Z}[i, \frac{1}{2}(1+i+j+k)] \subset \mathbb{Q}[i,j] \subset \mathbb{H}$.

Note that if a subgroup F of the units of Λ has the property that for $f \neq 1$ in F, $(1-f)$ is not a zero-divisor in Λ, then for any positive integer r, the diagonal embedding of F in the units of Λ^r has the same property. Hence $\mathrm{qpd}(\Lambda^r] F) \leqq rn + 1$.

3.2. A Fuchsian group G is a discrete group of orientation-preserving isometries of the hyperbolic plane, which preserves some hyperbolic tessellation T with the property that the only fixed points of this action are vertices of T. It follows that G has the Identity Property. Fuchsian groups have been investigated by many authors, and their properties are well-known [4, 6].

A case of particular interest is the triangle group

$$G = T(l, m, n) = \langle a, b \mid a^l = b^m = (ab)^n = 1 \rangle \qquad (1/l + 1/m + 1/n < 1),$$

because G is an infinite group generated by two torsion elements, but is not a free product of cyclic groups. Thus, for example, G cannot be embedded in a 1-relator group.

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