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Autor: Switzer, Robert W.

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Kontakt/Contact

Digizeitschriften e.V.
SUB Göttingen
Platz der Göttinger Sieben 1
37073 Göttingen

info@digizeitschriften.de

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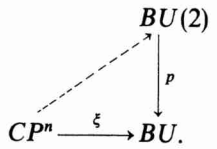
Postnikov Towers Associated with Complex 2-Plane and Symplectic Line Bundles

Robert M. Switzer

Mathematisches Institut der Universität, Bunsenstraße 3–5, D-3400 Göttingen, Federal Republic of Germany

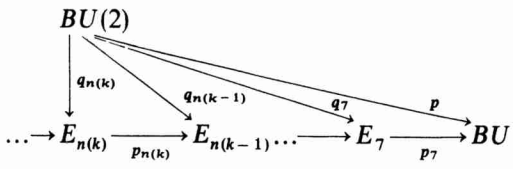
1. Introduction and Statement of Results

In recent years there has been some interest in complex bundles of rank 2 over complex projective spaces (cf. [1, 2, 4, 5]). We wish to attack the problem of investigating such bundles as follows: first one determines for which Chern classes c_1, c_2 there is a stable bundle $\xi \downarrow CP^n$ with $c_1\xi = c_1$, $c_2\xi = c_2$, $c_k\xi = 0$, $k \geqq 3$. Then for a fixed such stable class ξ one asks whether there are rank 2 representatives in it and if so how many (cf. [6, 7]). In homotopy theoretic terms this latter question means investigating liftings of the following sort.



The standard method of dealing with such a lifting problem is to construct the Postnikov tower of p as far as needed or as far as one can and then see what obstructions occur. That is what we propose to do in this paper.

The Postnikov tower of p is a factorization of p



such that $p_{n(k)}\colon E_{n(k)} \to E_{n(k-1)}$ is a principal fibration and each $q_{n(k)}$ is an $(n(k)-1)$-equivalence.

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We shall construct the tower for $BU(2)\to BU$ stage by stage. The result is shown on p. 89. If one calculates the indeterminacy of the various obstructions using for example the method in [8], then one gets the following result.

Theorem 1. *Let $\xi\colon X\to BU$ be a map; for $\dim X\leqq 14$ the obstructions to finding a lifting $\eta\colon X\to BU(2)$ of ξ are as follows.*

$$(c_3(\xi), c_4(\xi))\in H^6(X;\mathbb{Z})\oplus H^8(X;\mathbb{Z}),$$

$$(\omega_7,\omega_9,\omega_{11})\in \frac{\ker[\Delta_1\colon H^7(X;\mathbb{Z}_2)\oplus H^9(X;\mathbb{Z}_2)\oplus H^{11}(X;\mathbb{Z}_3)\to}{\operatorname{im}[\Delta_2\colon H^5(X;\mathbb{Z})\oplus H^7(X;\mathbb{Z})\to} \atop \frac{\to H^9(X;\mathbb{Z}_2)\oplus H^{11}(X;\mathbb{Z}_2)\oplus H^{14}(X;\mathbb{Z}_2)]}{\to H^7(X;\mathbb{Z}_2)\oplus H^9(X;\mathbb{Z}_2)\oplus H^{11}(X;\mathbb{Z}_3)]},$$

$$(\omega_8,\omega_{10},\omega_{13})\in \frac{\ker[\Delta_3\colon H^8(X;\mathbb{Z}_2)\oplus H^{10}(X;\mathbb{Z}_2)\oplus H^{13}(X;\mathbb{Z}_2)\to}{\operatorname{im}[\Delta_1\colon H^6(X;\mathbb{Z}_2)\oplus H^8(X;\mathbb{Z}_2)\oplus H^{10}(X;\mathbb{Z}_3)\to} \atop \frac{\to H^{11}(X;\mathbb{Z})\oplus H^{13}(X;\mathbb{Z})\oplus H^{15}(X;\mathbb{Z}_2)]}{\to H^8(X;\mathbb{Z}_2)\oplus H^{10}(X;\mathbb{Z}_2)\oplus H^{13}(X;\mathbb{Z}_2)]},$$

$$(c_5/24, c_6/8, \omega_{14})\in \frac{H^{10}(X;\mathbb{Z})\oplus H^{12}(X;\mathbb{Z})\oplus}{\operatorname{im}[\Delta_3\colon H^7(X;\mathbb{Z}_2)\oplus H^9(X;\mathbb{Z}_2)\oplus H^{12}(X;\mathbb{Z}_2)\to} \atop \frac{\oplus H^{14}(X;\mathbb{Z}_2)}{\to H^{10}(X;\mathbb{Z})\oplus H^{12}(X;\mathbb{Z})\oplus H^{14}(X;\mathbb{Z}_2)]},$$

$$c_7/360\in \frac{H^{14}(X;\mathbb{Z})}{(\beta'_3P_3^1+\beta'_2Sq^4+c_1\beta'_2Sq^2)\cdot H^9(X;\mathbb{Z})}$$

where the homomorphisms Δ_i are given by the following matrices:

$$\Delta_1=\begin{pmatrix} Sq^2+c_1 & 0 & 0\\ c_2 & Sq^2 & 0\\ (Sq^4+c_1^2)Sq^2Sq^1 & 0 & 0\end{pmatrix},$$

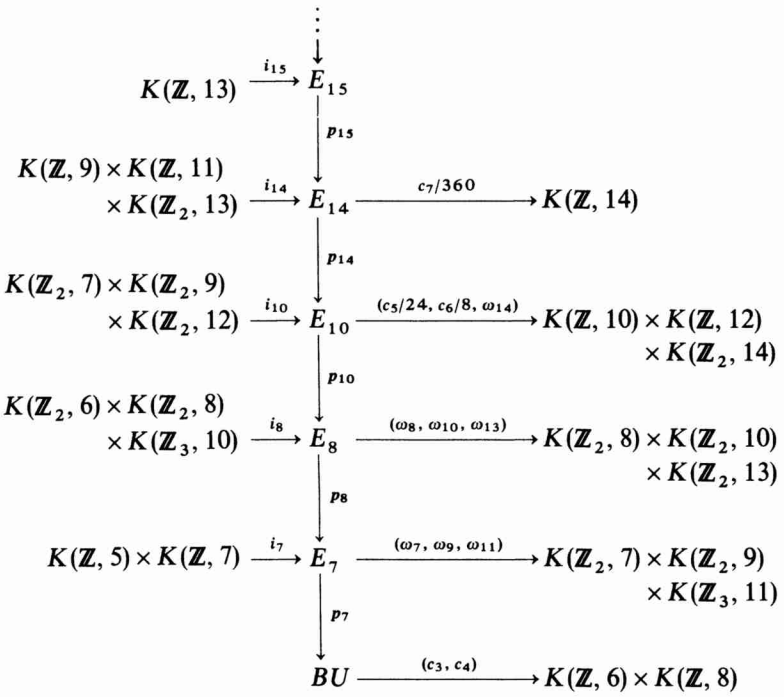
$$\Delta_2=\begin{pmatrix} Sq^2+c_1 & 0\\ Sq^4+c_2 & Sq^2\\ c_1(c_1^2+c_2-P_3^1) & P_3^1+c_1^2-c_2\end{pmatrix},$$

$$\Delta_3=\begin{pmatrix} \beta'_2(Sq^2+c_1) & 0 & 0\\ c_2\beta'_2 & \beta'_2Sq^2 & 0\\ Sq^7+Sq^4Sq^2Sq^1+c_1^2Sq^2Sq^1 & 0 & Sq^2+c_1\end{pmatrix}.$$

Here c_1, c_2 etc. means the operation of taking $\cup$-product with $c_1(\xi)$ resp. $c_2(\xi)$ etc., β'_2 means the Bockstein operator $\beta'_2\colon H^q(X;\mathbb{Z}_2)\to H^{q+1}(X;\mathbb{Z})$ associated to the coefficient sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow 0$$

and P_3^1 the first mod 3 p^{th} power operation.



Using the Postnikov tower we can also say something about the number of homotopy classes of liftings of a given ξ. If $\eta_1, \eta_2 : X \to BU(2)$ are two liftings of a fixed $\xi : X \to BU$, then we describe the obstructions $\delta_k(\eta_1, \eta_2)$ to a homotopy $\eta_1 \simeq \eta_2$ as subsets of certain quotient groups. The result is as follows.

Theorem 2. *Let $\eta_1, \eta_2 : X \to BU(2)$ be two liftings of a given map $\xi : X \to BU$. For $\dim X \leqq 12$ the obstructions to finding a homotopy $\eta_1 \simeq \eta_2$ are as follows.*

$$(\delta_5, \delta_7) \subset \ker [\Delta_2 : H^5(X; \mathbb{Z}) \oplus H^7(X; \mathbb{Z}) \to H^7(X; \mathbb{Z}_2) \oplus H^9(X; \mathbb{Z}_2) \oplus H^{11}(X; \mathbb{Z}_3)]$$

$$(\delta_6, \delta_8, \delta_{10}) \subset \frac{\ker [\Delta_1 : H^6(X; \mathbb{Z}_2) \oplus H^8(X; \mathbb{Z}_2) \oplus H^{10}(X; \mathbb{Z}_3) \to}{\operatorname{im} [\Delta_2 : H^4(X; \mathbb{Z}) \oplus H^6(X; \mathbb{Z}) \to \\ \to H^8(X; \mathbb{Z}_2) \oplus H^{10}(X; \mathbb{Z}_2) \oplus H^{13}(X; \mathbb{Z}_2)]} \\ \to H^6(X; \mathbb{Z}_2) \oplus H^8(X; \mathbb{Z}_2) \oplus H^{10}(X; \mathbb{Z}_3)]}$$

$$(\delta_7', \delta_9, \delta_{12}) \subset \frac{\ker [\Delta_3 : H^7(X; \mathbb{Z}_2) \oplus H^9(X; \mathbb{Z}_2) \oplus H^{12}(X; \mathbb{Z}_2) \to}{\operatorname{im} [\Delta_1 : H^5(X; \mathbb{Z}_2) \oplus H^7(X; \mathbb{Z}_2) \oplus H^9(X; \mathbb{Z}_3) \to \\ \to H^{10}(X; \mathbb{Z}) \oplus H^{12}(X; \mathbb{Z}) \oplus H^{14}(X; \mathbb{Z}_2)]} \\ \to H^7(X; \mathbb{Z}_2) \oplus H^9(X; \mathbb{Z}_2) \oplus H^{12}(X; \mathbb{Z}_2)]}$$

$$(\delta_9', \delta_{11}, \delta_{13}) \subset \frac{H^9(X;\mathbb{Z})\oplus H^{11}(X;\mathbb{Z})\oplus}{\operatorname{im}[\Delta_3: H^6(X;\mathbb{Z}_2)\oplus H^8(X;\mathbb{Z}_2)\oplus H^{11}(X;\mathbb{Z}_2)\to} \frac{\oplus H^{13}(X;\mathbb{Z}_2)}{\to H^9(X;\mathbb{Z})\oplus H^{11}(X;\mathbb{Z})\oplus H^{13}(X;\mathbb{Z}_2)]},$$

where Δ_1, Δ_2, Δ_3 *are as in Theorem* 1.

The fibre of $p: BU(2)\to BU$ is the Stiefel manifold $V=U/U(2)$. As a by-product of our calculations we get information on the homotopy of V.

Theorem 3.

$$\pi_q(U/U(2))\cong\begin{cases}0 & q<5\\ \mathbb{Z} & q=5, 11\\ \mathbb{Z}_2 & q=6, 8, 12\\ \mathbb{Z}\oplus\mathbb{Z}_2 & q=7, 9, 13\\ \mathbb{Z}_3 & q=10.\end{cases}$$

Remark. From the exact homotopy sequence

$$0=\pi_{2q}(U)\to\pi_{2q}(V)\to\pi_{2q-1}(U(2))\to\pi_{2q-1}(U)\cong\mathbb{Z}$$

and the fact that $\pi_{2q-1}(U(2))\cong\pi_{2q-1}(S^3)$ is finite for $q\geqq 3$ we see that $\pi_{2q}(V)\cong\pi_{2q-1}(S^3)$ for $q\geqq 3$. Thus Theorem 3 gives new information at most in odd dimensions.

Now entirely analogous but easier calculations give corresponding results on the Postnikov tower of $BSp(1)\to BSp$. We state the results here without proof. The Postnikov tower looks as shown on p. 91.

Theorem 4. *Let* $\xi: X\to BSp$ *be a map and* $\dim X\leqq 13$. *Then the obstructions to a lifting* $\xi: X\to BSp(1)$ *of* ξ *are as follows.*

$$p_2(\xi)\in H^8(X;\mathbb{Z}),$$

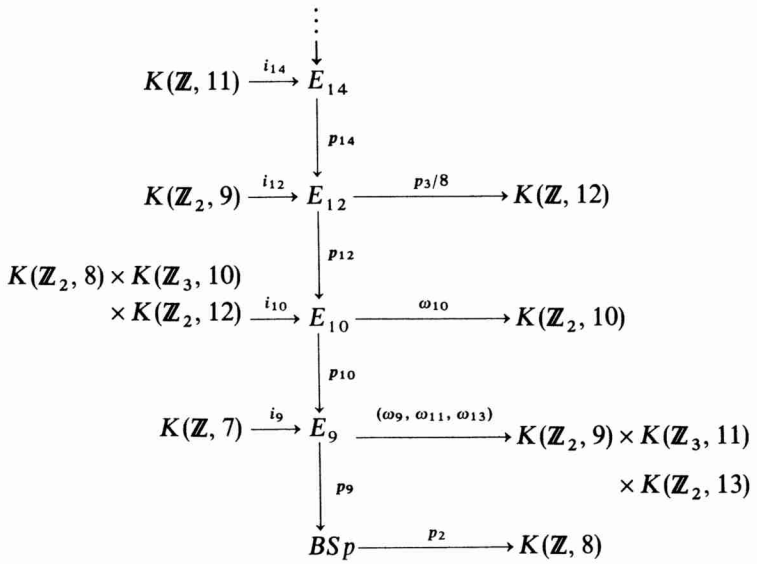
$$\omega_9\in\frac{\ker[Sq^2: H^9(X;\mathbb{Z}_2)\to H^{11}(X;\mathbb{Z}_2)]}{\operatorname{im}[Sq^2: H^7(X;\mathbb{Z})\to H^9(X;\mathbb{Z}_2)]},$$

$$\omega_{10}\in\frac{\ker[Sq^3: H^{10}(X;\mathbb{Z}_2)\to H^{13}(X;\mathbb{Z}_2)]}{\operatorname{im}[Sq^2: H^8(X;\mathbb{Z}_2)\to H^{10}(X;\mathbb{Z}_2)]},$$

$$\omega_{11}\in H^{11}(X;\mathbb{Z}_3)/(P_3^1+p_1)\,H^7(X;\mathbb{Z}),$$

$$p_3/8\in H^{12}(X;\mathbb{Z}),$$

$$\omega_{13}\in H^{13}(X;\mathbb{Z}_2)/Sq^6\,H^7(X;\mathbb{Z}).$$



Theorem 5. *Let* $\zeta_1, \zeta_2: X \to BSp(1)$ *be two liftings of a given map* $\xi: X \to BSp$. *For* $\dim X \leqq 12$ *the obstructions to finding a homotopy* $\zeta_1 \simeq \zeta_2$ *are as follows.*

$$\delta_7 \in H^7(X; \mathbb{Z})/\{f^* \lambda_7 + p_1 \xi \cdot f^* \lambda_3 \mid f \in [X; Sp]\},$$

$$\delta_8 \subset \frac{\ker [Sq^2: H^8(X; \mathbb{Z}_2) \to H^{10}(X; \mathbb{Z}_2)]}{\operatorname{im} [Sq^2: H^6(X; \mathbb{Z}) \to H^8(X; \mathbb{Z}_2)]},$$

$$\delta_9 \subset H^9(X; \mathbb{Z}_2)/Sq^2 H^7(X; \mathbb{Z}_2),$$

$$\delta_{10} \subset H^{10}(X; \mathbb{Z}_3)/(P_3^1 + p_1) H^6(X; \mathbb{Z}),$$

$$\delta_{11} \subset H^{11}(X; \mathbb{Z})/\beta_2' Sq^2 Sq^1 H^8(X; \mathbb{Z}_2),$$

$$\delta_{12} \subset H^{12}(X; \mathbb{Z}_2)/Sq^6 H^6(X; \mathbb{Z}).$$

λ_3, λ_7 are generators of the exterior algebra

$$H^*(Sp; \mathbb{Z}) \cong E(\lambda_3, \lambda_7, \ldots).$$

Theorem 6.

$$\pi_q(Sp/Sp(1)) \simeq \begin{cases} 0 & q < 7 \\ \mathbb{Z} & q = 7, 11 \\ \mathbb{Z}_2 & q = 8, 9, 12 \\ \mathbb{Z}_3 & q = 10. \end{cases}$$

2. Proof of Theorem 1

Once we have constructed E_n the kernel of $q_n^*: H^*(E_n) \to H^*(BU(2))$ will lie in dimensions greater than or equal to n. We then construct E_{n+1} so as to kill the part of the kernel in dimension n. For reasons which will be clear later we choose the first stage of the Postnikov tower as follows.

$$\begin{array}{ccc} K(\mathbb{Z}, 5) \times K(\mathbb{Z}, 7) & \xrightarrow{i_7} & E_7 \\ & & \downarrow p_7 \\ & & BU \xrightarrow{(c_3, c_4)} K(\mathbb{Z}, 6) \times K(\mathbb{Z}, 8). \end{array}$$

We shall use the Serre spectral sequence of the fibration $p_n: E_n \to E_{n-1}$ to compute as much of the cohomology of E_n as we need. It proves to be very important that one carry out cohomology calculations simultaneously for coefficients $\mathbb{Z}$ and $\mathbb{Z}_p$ for all relevant primes p (in our case $p=2, 3, 5$). Furthermore one must do careful bookkeeping on the action of the Steenrod algebra on the various cohomology generators.

The starting information is of course the cohomology

$$H^*(BU; \mathbb{Z}) = \mathbb{Z}[c_1, c_2, \ldots]$$

of BU together with the action of the Steenrod algebra on it. In particular we shall need

$$\begin{aligned} &Sq^4 c_3 = Sq^2 c_4 = c_5, \quad Sq^4 c_4 = c_6, \quad Sq^2 c_6 = c_7 && (p=2), \\ &P^1 c_3 = -c_5, \quad P^2 c_3 = c_7, \quad P^1 c_5 = c_7, \quad P^1 c_6 = -c_8 && (p=3), \\ &P^1 c_3 = 2c_7, \quad P^1 c_4 = -2c_8 && (p=5) \end{aligned}$$

modulo decomposables.

The mod p cohomology at each stage can be dealt with fairly simply using the following result.

Proposition 1. *Let $F \to E \to B$ be a fibration such that $H^*(F; \mathbb{Z}_p) \cong \mathbb{Z}_p[x_1, x_2, \ldots] \otimes S$, where S has a simple system of generators $\{y_1, y_2, \ldots\}$. Suppose that*

i) *only finitely many $x_1, x_2, \ldots, y_1, y_2, \ldots$ lie in any dimension;*
ii) *all $x_1, x_2, \ldots, y_1, y_2, \ldots$ are transgressive;*
iii) $\tau x_i = 0$, $i \geqq 1$;
iv) $H^*(B; \mathbb{Z}_p) \cong \mathbb{Z}_p[\tau y_1, \tau y_2, \ldots, u_1, u_2, \ldots]$.

Then $H^*(E; \mathbb{Z}_p) \cong \mathbb{Z}_p[u_1, u_2, \ldots, x_1, x_2, \ldots]$. If the hypotheses hold only in dimensions $\leqq n$, then the conclusion holds in dimensions $< n$.

The proof is a straightforward exercise with spectral sequences and is omitted.

We apply the proposition to the fibration

$$K(\mathbb{Z}, 5) \times K(\mathbb{Z}, 7) \to E_7 \to BU.$$

$H^*(K(\mathbb{Z}, 5); \mathbb{Z}_2)$ is a polynomial algebra generated by the $Sq^I \iota_5$, $\iota_5 \in H^5(K(\mathbb{Z}, 5); \mathbb{Z})$ the fundamental class, $I=(i_1, i_2, \ldots, i_s)$ admissible of excess $e(I)<5$ and $i_s>1$. The description of $H^*(K(\mathbb{Z}, 7); \mathbb{Z}_2)$ is analogous. If we take

$$y_1=\iota_5, \quad y_2=\iota_7, \quad y_3=Sq^4\iota_5, \quad y_4=Sq^4\iota_7, \quad y_5=Sq^6\iota_7$$

and adjoin

$$\iota_5^2=Sq^5\iota_5, \quad \iota_7^2=Sq^7\iota_7, \quad (Sq^4\iota_5)^2=Sq^9Sq^4\iota_5,$$

etc. to the polynomial generators, then since

$$\begin{aligned}\tau(\iota_5)&=c_3,\\ \tau(\iota_7)&=c_4,\\ \tau(Sq^4\iota_5)&=c_5,\\ \tau(Sq^4\iota_7)&=c_6,\\ \tau(Sq^6\iota_7)&=c_7\end{aligned}$$

we see that

$$H^*(E_7; \mathbb{Z}_2) \cong H^*(BU; \mathbb{Z}_2)/(c_3, c_4, c_5, c_6, c_7) \otimes A_2^* \cdot (\omega_7, \omega_9)$$

in dimensions $\leqq 16$, where

$$\omega_7 \in H^7(E_7; \mathbb{Z}_2), \quad \omega_9 \in H^9(E_7; \mathbb{Z}_2)$$

are characterized by

$$i_7^*\omega_7=Sq^2\iota_5, \quad i_7^*\omega_9=Sq^2\iota_7+Sq^4\iota_5.$$

Let $\mu: K(\mathbb{Z}, 5) \times K(\mathbb{Z}, 7) \times E_7 \to E_7$ denote the homotopy action of $K(\mathbb{Z}, 5) \times K(\mathbb{Z}, 7)$ on E_7. By the method of [8] one readily sees that

$$\begin{aligned}\mu^*\omega_7&=1\times\omega_7+\iota_5\times c_1+Sq^2\iota_5\times 1,\\ \mu^*\omega_9&=1\times\omega_9+\iota_5\times c_2+(Sq^2\iota_7+Sq^4\iota_5)\times 1.\end{aligned}$$

From these equations we can deduce the relations

$$\begin{aligned}&Sq^2\omega_7=c_1\omega_7, \quad Sq^2\omega_9=c_2\omega_7,\\ &Sq^4Sq^2Sq^1\omega_7=Sq^7\omega_7+c_1^2Sq^2Sq^1\omega_7,\\ &Sq^4Sq^2Sq^1\omega_9=Sq^7\omega_9+c_1c_2Sq^2Sq^1\omega_7+c_2^2Sq^1\omega_7(+c_8?).\end{aligned}$$

One uses the fact that the map $\nu=\mu\circ(\mathrm{id}\times q_7)$

$$K(\mathbb{Z}, 5)\times K(\mathbb{Z}, 7)\times BU(2) \xrightarrow{\mu\circ(\mathrm{id}\times q_7)} E_7$$

induces a monomorphism in cohomology (resp. $\ker \nu^* \cong \mathbb{Z}_2$ generated by c_8 in dimension 16).

The spectral sequences for coefficients $\mathbb{Z}_3, \mathbb{Z}_5$ are quite straightforward. The results in dimensions $\leqq 16$ are entered in Table 1. One computes

Table 1

q	7	8	9	10	11	12	13	14	15	16	17
$H^q(E_7; \mathbf{Z}_2)/(c_1, c_2)$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_2^2$	$\mathbf{Z}_2$	$\mathbf{Z}_2^2$	$\mathbf{Z}_2^2$	$\mathbf{Z}_2^4$	$\mathbf{Z}_2^2$	$\mathbf{Z}_2^4$	$\mathbf{Z}_2^2$
Generators	ω_7	$Sq^1\omega_7$	ω_9	$Sq^2Sq^1\omega_7$ $Sq^1\omega_9$	$Sq^4\omega_7$	$Sq^5\omega_7$ $Sq^2Sq^1\omega_9$	$Sq^6\omega_7$ $Sq^4\omega_9$	$Sq^7\omega_7$ $Sq^6Sq^1\omega_7$ $Sq^5\omega_9$ $c_7/2$	$Sq^7Sq^1\omega_7$ $Sq^6\omega_9$	$Sq^8Sq^1\omega_7$ $Sq^7\omega_9$ $Sq^6Sq^1\omega_9$ c_8	$Sq^7Sq^1\omega_9$ $Sq^8\omega_9$
$H^q(E_7; \mathbf{Z}_3)/(c_1, c_2)$	0	0	0	$\mathbf{Z}_3$	$\mathbf{Z}_3$	$\mathbf{Z}_3^2$	0	$\mathbf{Z}_3$	$\mathbf{Z}_3$	$\mathbf{Z}_3^2$	$\mathbf{Z}_3$
Generators				$c_5/3$	ω_{11}	$\beta\omega_{11}$ c_6		$c_7/3$	$P^1\omega_{11}$	$\beta P^1\omega_{11}$ c_8	
$H^q(E_7; \mathbf{Z}_5)/(c_1, c_2)$	0	0	0	$\mathbf{Z}_5$	0	$\mathbf{Z}_5$	0	$\mathbf{Z}_5$	0	$\mathbf{Z}_5$	0
Generators				c_5		c_6		$c_7/5$		$c_8/5$	
$H^q(E_7; \mathbf{Z})/(c_1, c_2)$	0	$\mathbf{Z}_2$	0	$\mathbf{Z}\oplus\mathbf{Z}_2$	0	$\mathbf{Z}\oplus\mathbf{Z}_2\oplus\mathbf{Z}_3$	0	$\mathbf{Z}\oplus\mathbf{Z}_2^2$	$\mathbf{Z}_2$	$\mathbf{Z}\oplus\mathbf{Z}_2^2\oplus\mathbf{Z}_3$	$\mathbf{Z}_2$
Generators		$\beta'_2\omega_7$		$c_5/6$ $\beta'_2\omega_9$		$c_6/2$ $\beta'_2Sq^4\omega_7$ $\beta'_3\omega_{11}$		$c_7/30$ $\beta'_2Sq^6\omega_7$ $\beta'_2Sq^4\omega_9$	$\beta'_2Sq^6Sq^1\omega_7$	$c_8/5$ $\beta'_2Sq^6\omega_9$ $\beta'_3P^1_3\omega_{11}$ $(\beta'_2\omega_7)^2$	$\beta'_2Sq^6Sq^1\omega_9$

$$\mu^* \omega_{11} = 1 \times \omega_{11} + \iota_5 \times c_1(c_1^2 + c_2) + \iota_7 \times (c_1^2 - c_2) - P_3^1 \iota_5 \times c_1 + P_3^1 \iota_7 \times 1.$$

We come now to the integral cohomology of E_7. Of course we shall need the cohomology $H^q(K(\mathbb{Z}, 5) \times K(\mathbb{Z}, 7); \mathbb{Z})$ of the fibre for $q \leqq 16$. We have the Bockstein sequences

$$\begin{array}{ccccccccc} \ldots \to & H^q(X;\mathbb{Z}) & \xrightarrow{\times p} & H^q(X;\mathbb{Z}) & \xrightarrow{\rho_p} & H^q(X;\mathbb{Z}_p) & \xrightarrow{\beta'_p} & H^{q+1}(X;\mathbb{Z}) & \to \ldots \\ & \downarrow{\scriptstyle \rho_p} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \rho_p} & \\ \ldots \to & H^q(X;\mathbb{Z}_p) & \longrightarrow & H^q(X;\mathbb{Z}_{p^2}) & \longrightarrow & H^q(X;\mathbb{Z}_p) & \xrightarrow{\beta_p} & H^{q+1}(X;\mathbb{Z}_p) & \to \ldots \end{array}$$

and the knowledge of $H^*(K(\mathbb{Z}, 5) \times K(\mathbb{Z}, 7); \mathbb{Z}_p)$, which can be compared with $H^*(K(\mathbb{Z}, 5) \times K(\mathbb{Z}, 7); \mathbb{Z})$ by means of the universal coefficient theorem. The elements of order p in $H^q(X; \mathbb{Z})$ are the elements in $\operatorname{im} \beta'_p$ and are mapped under reduction mod p (ρ_p) onto

$$\operatorname{im}[\beta_p : H^{q-1}(X;\mathbb{Z}_p) \to H^q(X;\mathbb{Z}_p)].$$

Elements of order p^2 are possible where

$$H^{q-1}(X;\mathbb{Z}_p) \xrightarrow{\beta_p} H^q(X;\mathbb{Z}_p) \xrightarrow{\beta_p} H^{q+1}(X;\mathbb{Z}_p)$$

is not exact. However in the range in which we are interested and for $X = K(\mathbb{Z}, 5) \times K(\mathbb{Z}, 7)$ β_p is exact.

The differentials in the integral spectral sequence are all straightforward with the possible exception of

$$d_{10}(\beta'_2(Sq^2 \iota_5 \cdot Sq^4 \iota_5)) = \beta'_2 Sq^2 \iota_5 \cdot c_5.$$

The extension problems are solved by comparison with the already known results for mod p cohomology. Thus $H^{10}(E_7; \mathbb{Z})/(c_1, c_2)$ is $\mathbb{Z} \oplus \mathbb{Z}_2$ and not $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, $\mathbb{Z} \oplus \mathbb{Z}_6$ or $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_6$, for otherwise we should have one summand too many in $H^9(E_7; \mathbb{Z}_p)/(c_1, c_2)$ for $p = 2$ or 3. The resulting integral cohomology groups modulo the ideal (c_1, c_2) are entered in Table 2.

Note that $P_3^1(c_5/6) = c_7/30$ in $H^{14}(E_7; \mathbb{Z}_3)$.

The next stage of our tower should now look as follows.

$$\begin{array}{ccc} K(\mathbb{Z}_2, 6) \times K(\mathbb{Z}_2, 8) & & \\ \quad \times K(\mathbb{Z}_3, 10) & \xrightarrow{i_8} & E_8 \\ & & \downarrow{\scriptstyle p_8} \\ & & E_7 \xrightarrow{(\omega_7, \omega_9, \omega_{11})} K(\mathbb{Z}_2, 7) \times K(\mathbb{Z}_2, 9) \\ & & \qquad \times K(\mathbb{Z}_3, 11). \end{array}$$

The $\mathbb{Z}_p$-cohomology causes us no difficulties whatever. The method described above for computing $H^*(K(\mathbb{Z}, 5) \times K(\mathbb{Z}, 7); \mathbb{Z})$ works equally well for $H^*(K(\mathbb{Z}_2, 6) \times K(\mathbb{Z}_2, 8) \times K(\mathbb{Z}_3, 10); \mathbb{Z})$. In this case we have $Sq^1 Sq^6 \iota_6 = Sq^7 \iota_6$

Table 2

q	8	9	10	11	12	13	14	15	16
$H^q(E_8; \mathbf{Z}_2)/(c_1, c_2)$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_2^2$	$\mathbf{Z}_2^2$	$\mathbf{Z}_2^3$	$\mathbf{Z}_2^3$	$\mathbf{Z}_2^5$	$\mathbf{Z}_2^5$	$\mathbf{Z}_2^6$
Generators	ω_8	$Sq^1\omega_8$	$Sq^2\omega_8$ ω_{10}	$Sq^2Sq^1\omega_8$ $Sq^1\omega_{10}$	$Sq^3Sq^1\omega_8$ $Sq^4\omega_8$ $Sq^2\omega_{10}$	$Sq^5\omega_8$ $Sq^2Sq^1\omega_{10}$ ω_{13}	$Sq^6\omega_8$ $Sq^4Sq^2\omega_8$ $Sq^4\omega_{10}$ $Sq^3Sq^1\omega_{10}$ $c_7/30$	$Sq^7\omega_8$ $Sq^6Sq^1\omega_8$ $Sq^4Sq^2Sq^1\omega_8$ $Sq^5\omega_{10}$ ω_{15}	$Sq^7Sq^1\omega_8$ $Sq^6Sq^2\omega_8$ $Sq^5Sq^2Sq^1\omega_8$ $Sq^6\omega_{10}$ $Sq^4Sq^2\omega_{10}$ $c_8/5$
$H^q(E_8; \mathbf{Z}_3)/(c_1, c_2)$	0	0	$\mathbf{Z}_3$	0	$\mathbf{Z}_3$	0	$\mathbf{Z}_3$	0	$\mathbf{Z}_3$
Generators			$c_5/12$		$c_6/4$		$c_7/30$		$c_8/5$
$H^q(E_8; \mathbf{Z})/(c_1, c_2)$	0	$\mathbf{Z}_2$	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}\oplus\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}\oplus\mathbf{Z}_2^2$	$\mathbf{Z}_2^2$	$\mathbf{Z}\oplus\mathbf{Z}_2^3$
Generators		$\beta'_2\omega_8$	$c_5/12$	$\beta'_2\omega_{10}$	$c_6/4$ $\beta'_2Sq^2Sq^1\omega_8$	$\beta'_2Sq^4\omega_8$	$c_7/30$ $\beta'_2Sq^2Sq^1\omega_{10}$ $\beta'_2\omega_{13}$	$\beta'_2Sq^6\omega_8$ $\beta'_2Sq^4\omega_{10}$	$c_8/5$ $\beta'_2Sq^6Sq^1\omega_8$ $\beta'_2Sq^4Sq^2Sq^1\omega_8$ $\beta'_2\omega_{15}$

$=0$ but $Sq^6 \iota_6 \notin \operatorname{im} Sq^1$. Suppose $Sq^6 \iota_6$ were integral – say $\rho_2 y = Sq^6 \iota_6$ for some $y \in H^{12}(K(\mathbb{Z}_2, 6) \times K(\mathbb{Z}_2, 8) \times K(\mathbb{Z}_3, 10); \mathbb{Z})$. Since $H^q(K(\mathbb{Z}_2, 6) \times K(\mathbb{Z}_2, 8) \times K(\mathbb{Z}_3, 10); \mathbb{Q}) = 0$ for $q > 0$, y must be of order 2^s for some s. But then the universal coefficient theorem would give a $\mathbb{Z}_2$-summand too many in $H^{11}(K(\mathbb{Z}_2, 6) \times K(\mathbb{Z}_2, 8) \times K(\mathbb{Z}_3, 10); \mathbb{Z}_2)$. Hence the only possibility is an element $x_{13} \in H^{13}(K(\mathbb{Z}_2, 6) \times K(\mathbb{Z}_2, 8) \times K(\mathbb{Z}_3, 10); \mathbb{Z})$ of order 4 with

$$\beta'_2 Sq^6 \iota_6 = 2x_{13}, \quad \rho_2 x_{13} = Sq^6 Sq^1 \iota_6 + \iota_6 Sq^1 \iota_6.$$

Thus in the integral spectral sequence we have

$$d_7 x_{13} = \beta'_2 \iota_6 \cdot \omega_7$$
$$d_{14} 2x_{13} = d_{14} \beta'_2 Sq^6 \iota_6 = \beta'_2 Sq^6 \omega_7.$$

The other differentials are unproblematical. Again the extension problems are solved by comparison with the mod p cohomology. $H^q(E_8; \mathbb{Z})/(c_1, c_2)$ is entered in Table 2 for $q \leqq 16$.

The method of [8] gives us

$$\mu^* \omega_8 = 1 \times \omega_8 + \iota_6 \times c_1 + Sq^2 \iota_6 \times 1,$$
$$\mu^* \omega_{10} = 1 \times \omega_{10} + \iota_6 \times c_2 + Sq^2 \iota_8 \times 1,$$
$$\mu^* \omega_{13} = 1 \times \omega_{13} + Sq^2 Sq^1 \iota_6 \times c_1^2 + Sq^4 Sq^2 Sq^1 \iota_6 \times 1$$

and hence

$$\beta'_2 Sq^2 \omega_8 = c_1 \beta'_2 \omega_8, \quad \beta'_2 Sq^2 \omega_{10} = c_2 \beta'_2 \omega_8,$$
$$Sq^4 Sq^2 \omega_8 = Sq^1 \omega_{13} + c_1 Sq^4 \omega_8 + c_1^3 \omega_8 (+ c_7/30?),$$
$$Sq^4 Sq^2 Sq^1 \omega_8 = Sq^2 \omega_{13} + c_1 \omega_{13} + Sq^7 \omega_8 + c_1^2 Sq^2 Sq^1 \omega_8.$$

With these methods we cannot answer the question whether $Sq^4 Sq^2 \omega_8$ contains a summand $c_7/30$. However, if one assumes it does not, then one is eventually led to the conclusion $\pi_{11} S^3 \cong \pi_{12} V \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, which is false. Thus we must assume there is indeed such a summand $c_7/30$.

We extend our tower one stage further as follows.

$$\begin{array}{l} K(\mathbb{Z}_2, 7) \times K(\mathbb{Z}_2, 9) \\ \qquad \times K(\mathbb{Z}_2, 12) \xrightarrow{i_{10}} E_{10} \\ \qquad\qquad\qquad\qquad \Big\downarrow p_{10} \\ \qquad\qquad\qquad\qquad E_8 \xrightarrow{(\omega_8, \omega_{10}, \omega_{13})} K(\mathbb{Z}_2, 8) \times K(\mathbb{Z}_2, 10) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times K(\mathbb{Z}_2, 13). \end{array}$$

The spectral sequences cause no difficulties at all. The cohomology of E_{10} appears in Table 3. The action μ works as follows.

$$\mu^* c_5/24 = 1 \times c_5/24 + \beta'_2 \iota_7 \times c_1 + \beta'_2 Sq^2 \iota_7 \times 1,$$
$$\mu^* c_6/8 = 1 \times c_6/8 + \beta'_2 \iota_7 \times c_2 + \beta'_2 Sq^2 \iota_9 \times 1,$$

$$\mu^* c_7/60 = 1 \times c_7/60 + \beta'_2 \iota_7 \times c_1^3 + \beta'_2 Sq^4 \iota_7 \times c_1 + \beta'_2 Sq^4 Sq^2 \iota_7 \times 1,$$

$$\mu^* \omega_{14} = 1 \times \omega_{14} + Sq^2 Sq^1 \iota_7 \times c_1^2 + \iota_{12} \times c_1 + (Sq^2 \iota_{12} + Sq^7 \iota_7 + Sq^4 Sq^2 Sq^1 \iota_7) \times 1,$$

$$Sq^4 c_5/24 = c_7/60 + c_1 Sq^2 c_5/24.$$

The last stage we shall work out is the following.

$$\begin{array}{l} K(\mathbb{Z}, 9) \times K(\mathbb{Z}, 11) \\ \quad \times K(\mathbb{Z}_2, 13) \xrightarrow{i_{14}} E_{14} \\ \qquad\qquad\qquad\qquad \downarrow p_{14} \\ \qquad\qquad\qquad\qquad E_{10} \xrightarrow{(c_5/24,\, c_6/8,\, \omega_{14})} K(\mathbb{Z}, 10) \times K(\mathbb{Z}, 12) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times K(\mathbb{Z}_2, 14). \end{array}$$

Table 3

q	10	11	12	13	14	15
$H^q(E_{10}; \mathbb{Z}_2)/(c_1, c_2)$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$
Generators	$c_5/24$		$Sq^2(c_5/24)$ $c_6/8$	$Sq^3(c_5/24)$	$Sq^2(c_6/8)$ $c_7/60$ ω_{14}	$Sq^3(c_6/8)$ $Sq^1\omega_{14}$ ω_{15}
$H^q(E_{10}; \mathbb{Z})/(c_1, c_2)$	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2^2$
Generators	$c_5/24$		$c_6/8$	$\beta'_2 Sq^2(c_5/24)$	$c_7/60$	$\beta'_2 Sq^2(c_6/8)$ $\beta'_2 \omega_{14}$

The Generators

$i_7^* \omega_7 = Sq^2 \iota_5$
$i_7^* \omega_9 = Sq^2 \iota_7$
$i_7^* c_5/6 = \beta'_2 Sq^4 \iota_5 + \beta'_3 P_3^1 \iota_5 \qquad \rho_2 c_5/6 = Sq^2 Sq^1 \omega_7$
$i_7^* \omega_{11} = P_3^1 \iota_7$
$i_7^* c_6/2 = \beta'_2 Sq^4 \iota_7 \qquad \rho_2 c_6/2 = Sq^2 Sq^1 \omega_9$
$i_7^* c_7/30 = \beta'_2 Sq^6 \iota_7 + \beta'_3 P_3^2 \iota_5 + \beta'_5 P_5^1 \iota_5$
$i_7^* c_8/5 = \beta'_5 P_5^1 \iota_7$

$i_8^* \omega_8 = Sq^2 \iota_6$
$i_8^* c_5/12 = \beta'_2 Sq^2 Sq^1 \iota_6 \qquad \rho_2 c_5/12 = Sq^2 \omega_8$
$i_8^* \omega_{10} = Sq^2 \iota_8$
$i_8^* c_6/4 = \beta'_2 Sq^2 Sq^1 \iota_8 \qquad \rho_2 c_6/4 = Sq^2 \omega_{10}$
$i_8^* \omega_{13} = Sq^4 Sq^2 Sq^1 \iota_6 \qquad i_8^* \omega_{15} = Sq^4 Sq^2 Sq^1 \iota_8$

$i_{10}^* c_5/24 = \beta'_2 Sq^2 \iota_7$
$i_{10}^* c_6/8 = \beta'_2 Sq^2 \iota_9$
$i_{10}^* c_7/60 = \beta'_2 Sq^4 Sq^2 \iota_7$
$i_{10}^* \omega_{14} = Sq^2 \iota_{12} + Sq^7 \iota_7 + Sq^4 Sq^2 Sq^1 \iota_7$

One calculates without difficulty that E_{14} is 13-connected and $H^{14}(E_{14};\mathbb{Z})\cong\mathbb{Z}$ generated by $c_7/360$, where

$$i_{14}^* c_7/360 = \beta'_2 Sq^4 \iota_9 + \beta'_3 P_3^1 \iota_9.$$

The action μ is given by

$$\mu^* c_7/360 = 1\times c_7/360 + \beta'_2 Sq^2 \iota_9 \times c_1 + (\beta'_2 Sq^4 \iota_9 + \beta'_3 P_3^1 \iota_9)\times 1.$$

Remark. The reader may have wondered why the obstructions were killed in what seems at first glance to be an arbitrary order. The principle is simple: kill each obstruction as soon as it arises. If one postpones the killing of an obstruction, one loses information about its indeterminacy.

One should resist the urge to kill c_5, c_6, c_7 already in the first stage, however, although this procedure would seem quite natural when one considers stable bundles ξ with $c_k\xi=0$ for $k\geqq 3$. The reason is that the obstructions which appear here as $c_5/24$, $c_6/8$, $c_7/360$ would then give rise to obstructions in $H^9(X;\mathbb{Z}_{24})$, $H^{11}(X;\mathbb{Z}_8)$, $H^{13}(X;\mathbb{Z}_{360})$ which would not be readily identifiable.

3. Proof of Theorem 2

To prove Theorem 2 we need an easy generalization of some work of James and Thomas [3]. They consider the enumeration of homotopy classes $[\xi']$ of liftings $\xi'\colon X\to E$ of a given map $\xi\colon X\to B$ if $p\colon E\to B$ is the fibration induced by a map $f\colon B\to C$ between two H-spaces B, C, where f is not assumed to be an H-map. That is often the situation in the first stage of a Postnikov tower; but in the higher stages the space E_n, which

$$\begin{array}{ccccc} & & \vdots & & \\ & & \downarrow & & \\ \Omega C_n & \xrightarrow{i_{n+1}} & E_{n+1} & & \\ & & \downarrow{\scriptstyle P_{n+1}} & & \\ \Omega C_{n-1} & \xrightarrow{i_n} & E_n & \xrightarrow{\omega_n} & C_n \\ & & \downarrow & & \\ & & \vdots & & \end{array}$$

now plays the rôle of B, is very unlikely to be an H-space. What one does have, however, is an H-action of the H-space ΩC_{n-1} on E_n. This we can exploit to get some of James' and Thomas' results in this situation as well.

The situation we want to consider then is the following: A, C are H-groups, B a space on which A H-acts and $f\colon B\to C$ an arbitrary map. We consider the space $\Omega^* B = B^{S^1}$ of free loops in B and the fibration

$$\Omega B \xrightarrow{i} \Omega^* B \xrightarrow{r} B,$$

where $r(\omega)=\omega(1)$, $1\in S^1$, $\omega: S^1\to B$ any loop. The following diagram commutes.

$$\begin{array}{ccccc} & & [X,\Omega^* B] & \xrightarrow{r_*} & [X,B] \\ & & \downarrow{\scriptstyle (\Omega^* f)_*} & & \downarrow{\scriptstyle f_*} \\ [X,\Omega C] & \xrightarrow{i_*} & [X,\Omega^* C] & \xrightarrow{r_*} & [X,C]. \end{array}$$

Definition. Let $\xi: X\to B$, $k: X\to \Omega C$ be maps. Then k is *f-correlated* to ξ if and only if there is a map $h: X\to \Omega^* B$ with $r_* h=\xi$, $(\Omega^* f)_* h=i_* k$ (we do not distinguish between a map and its homotopy class).

Theorem (James, Thomas). *Let $p: E\to B$ be the fibration induced from the path fibration over C by f. Given a lifting $\xi': X\to E$ of a map $\xi: X\to B$ and a map $k: X\to \Omega C$ we have $k\cdot\xi'=\xi'$ if and only if k is f-correlated to ξ.*

Here $k\cdot\xi'$ refers to the action of the group $[X,\Omega C]$ on the set $[X,E]$.

Now let $m: A\times B\to B$ be the H-action of A on B and $n: C\times C\to C$ the H-multiplication on C. Then we have induced maps

$$m': \Omega A\times B\to \Omega^* B$$
$$n': \Omega C\times C\to \Omega^* C$$

(for example $m'(\omega, b)(t)=m(\omega(t), b)$). James and Thomas remark that n' is a weak homotopy equivalence. We regard the following composite map

$$\begin{array}{ccc} [X,\Omega A]\times[X,B] & \xrightarrow{m'_*} & [X,\Omega^* B] \\ & & \downarrow{\scriptstyle (\Omega^* f)_*} \\ [X,\Omega C]\times[X,C] & \xrightarrow[\cong]{n'_*} & [X,\Omega^* C] \end{array}$$

where X is a CW-complex. We can describe this map by introducing

$$\Delta(f,\xi): [X,\Omega A]\to[X,\Omega C]$$

for $\xi\in[X,B]$, where

$$\Delta(f,\xi)(g,x,t)=n(fm(g(x,t),\xi(x)), vf\xi(x)),$$

$v: C\to C$ the H-inverse of the H-group C, $g: X\to\Omega A$, $x\in X$, $t\in[0,1]$ arbitrary. Simply by writing down the definitions we obtain the following.

Lemma. $(n'_*)^{-1}\circ(\Omega^* f)_*\circ m'_*(g,\xi)=(\Delta(f,\xi)\, g, f\circ\xi)$ *for all* $g: X\to\Omega A$.

Corollary 1. *Let $\xi': X\to E$ be a lifting of ξ and $k: X\to\Omega C$ of the form $k=\Delta(f,\xi)\, g$ for some $g: X\to\Omega A$. Then $k\cdot\xi'=\xi'$.*

Proof. Set $h=m'_*(g,\xi)\in[X,\Omega^* B]$. Then $r_* h=\xi$ and $(\Omega^* f)_* h=n'_*(\Delta(f,\xi)\, g, f\circ\xi)$ $=n'_*(\Delta(f,\xi)\, g,*)=i_*\Delta(f,\xi)\, g=i_* k$. Hence k is f-correlated to ξ.

Corollary 2. *If* $\Delta(f,\xi)[X,\Omega A]$ *is a subgroup of* $[X,\Omega C]$ *then* $[X,\Omega C]/\Delta(f,\xi)[X,\Omega A]$ *acts on* $p_*^{-1}\xi\in[X,E]$ *transitively.*

Remark. This action is even effective in the case that $r_*^{-1}\xi = m'_*([X,\Omega A]\times\xi)\subset[X,\Omega^* B]$. But we have little hope of being able to prove such a relation directly in the cases which interest us.

The next problem then is how to calculate $\Delta(f,\xi)$; here too a theorem of James and Thomas lets itself be adapted almost without change.

Theorem. *Let* $\alpha\in\tilde{H}^*(C)$ *be primitive with respect to*

$$n^*: H^*(C)\to H^*(C\times C),$$

i.e. $n^*\alpha = 1\times\alpha+\alpha\times 1$. *Suppose further that* $m^*f^*\alpha\in H^*(A\times B)$ *is of the form*

$$m^*f^*\alpha = 1\times f^*\alpha + i^*f^*\alpha\times 1+\sum_j u_j\times v_j$$

for appropriate $u_j\in\tilde{H}^*(A)$, $v_j\in\tilde{H}^*(B)$ ($i: A\to B$ *the map given by* $i(a)=m(a,*)$). *Then we have for all* $g\in[X,\Omega A]$

$$(\Delta(f,\xi)\,g)^*\,\sigma'\alpha = g^*\,\sigma'\,i^*f^*\alpha+\sum_j (g^*\,\sigma'\,u_j)\cdot(\xi^*\,v_j),$$

where $\sigma'\colon \tilde{H}^q(Y)\to\tilde{H}^{q-1}(\Omega Y)$ *denotes the usual "suspension".*

In particular if A, C are generalized Eilenberg-Mac Lane complexes

$$A=\prod_{i=1}^{s} K(\pi_i,m_i),$$

$$C=\prod_{i=1}^{t} K(G_i,n_i)$$

we can take the α in the theorem to be the fundamental class

$$\iota_{n_i} = 1\times\ldots\times 1\times\iota_{n_i}\times\ldots\times 1\in\tilde{H}^*(C;G_i)$$

of $K(G_i,n_i)$. If for example

$$m^*f^*(\iota_{n_i}) = 1\times f^*\iota_{n_i}+i^*f^*\iota_{n_i}\times 1+\sum_j u_j\times v_j$$

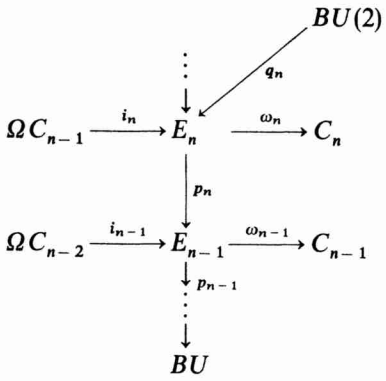
for $u_j\in\tilde{H}^*(A;G_i)$, $v_j\in\tilde{H}^*(B;\mathbb{Z})$, then for $g\in[X,\Omega A]=\prod_{i=1}^{s} H^{m_i-1}(X;\pi_i)$ we have

$$(\Delta(f,\xi)\,g)^*\,\iota_{n_i-1} = g^*\,\sigma'\,i^*f^*\,\iota_{n_i}+\sum_j (g^*\,\sigma'\,u_j)\cdot(\xi^*\,v_j),$$

this being the ith component of

$$\Delta(f,\xi)\,g\in[X,\Omega C]=\prod_{k=1}^{t} H^{n_k-1}(X;G_k).$$

Suppose now we have a section of Postnikov tower as follows.



Denote by $m_n: \Omega C_{n-1} \times E_n \to E_n$ the H-action for each n and suppose

$$m_{n-1}^* \omega_{n-1} = 1 \times \omega_{n-1} + i_{n-1}^* \omega_{n-1} \times 1 + \sum_j u_j \times v_j,$$

$$m_n^* \omega_n = 1 \times \omega_n + i_n^* \omega_n \times 1 + \sum_k x_k \times y_k.$$

If we have two liftings $\xi_1, \xi_2: X \to BU(2)$ of a given $\xi: X \to BU$, we can consider the liftings

$$\xi_i' = q_n \circ \xi_i: X \to E_n \qquad i = 1, 2$$

to E_n. By Cor. 2 we can find an element $\delta \in [X, \Omega C_{n-1}]$ variable within the coset $\Delta(\omega_{n-1}, \xi_1)\,[X, \Omega C_{n-2}]$ such that

$$\xi_2 \simeq m_n \circ (\delta, \xi_1): X \to E_n.$$

Since both ξ_1 and ξ_2 lift to E_{n+1} we have

$$0 = \xi_2^* \omega_n - \xi_1^* \omega_n = \delta^* i_n^* \omega_n + \sum_k (\delta^* x_k)(\xi^* y_k).$$

Thus we can regard δ as an element of the quotient

$$Q_n = \frac{\ker \{[\vartheta_n + \sum_k (\xi^* y_k) \cdot x_k]: [X, \Omega C_{n-1}] \to [X, C_n]\}}{\operatorname{im} \{[\sigma' \vartheta_{n-1} + \sum_j (\xi^* v_i) \cdot \sigma' u_j]: [X, \Omega^2 C_{n-2}] \to [X, \Omega C_{n-1}]\}},$$

where x_k, u_j, $\vartheta_n = \omega_n \circ i_n$, $\vartheta_{n-1} = \omega_{n-1} \circ i_{n-1}$ are regarded as cohomology operations. What we have actually shown is that there is a transitive action of this quotient Q_n on the set

$$(p_7 \circ p_8 \ldots \circ p_n)_*^{-1}(\xi) \cap \ker \omega_n^*$$

of homotopy classes of liftings $\xi': X \to E_n$ of ξ which lift further to E_{n+1}. Given two such liftings $\xi_1, \xi_2: X \to E_n$ which lift further, we shall regard the difference

obstruction $\delta_n(\xi_1, \xi_2)$ to a homotopy between them as the set of $x \in Q_n$ with $x\xi_1 = \xi_2$; thus $\delta_n(\xi_1, \xi_2)$ is a subset

$$\delta_n(\xi_1, \xi_2) \subset Q_n.$$

Theorem 2 now follows simply by looking up the effect of the H-action μ on each of the obstructions ω_n.

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