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Relative Commutants in Compact Convex Sets and Banach Spaces

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1. Introduction

Many of the advances in the theory of compact convex sets have been obtained by extending familiar ideas from C*-algebras. This applies particularly to the centre, which for C*-algebras is an elementary algebraic concept but may be characterised in other ways using the order or norm structure. Alfsen and Andersen [2, 3] and Wils [16] introduced a space associated with an ordered Banach space of a certain type, which corresponds to the centre for C*-algebras. For the space $A(K)$ of continuous affine real-valued functions on a compact convex set K, this space could be described in various ways, either as functions in $A(K)$ multiplying with $A(K)$ on the extreme boundary $\partial_e K$ of K, or as continuous functions on $\partial_e K$ in the "facial" topology, or as order-bounded operators on $A(K)$. Alfsen and Effros [4] found a class of operators on an arbitrary Banach space, which reduce to the order-bounded operators in the case of $A(K)$, and which may be characterised by a multiplicative property or by a boundedness property, and may be identified with a space of continuous functions. This class of operators may be regarded as the centre of the Banach space, and is known as its *centraliser*.

Product structures in general and tensor products in particular are natural objects of study, and the question was raised as to whether the centre of a tensor product is the tensor product of the centres in some sense. This problem was considered by Haydon and Wassermann [11], Archbold [5] and Vincent-Smith [14] for C*-algebras, by Chu [9], Vincent-Smith and the author [7] for compact convex sets, and by Wickstead [15] for arbitrary Banach spaces.

It is also appropriate to consider relative commutants of subalgebras of C*-algebras. A recent result of Aizenman, Davies and Lieb [1] describes the class of multiplication operators associated with a relative commutant, and this paper is concerned with extensions of this to Banach spaces. Thus for a closed subspace X of a real Banach space Y, it is possible to define in a natural way a class of operators of X into Y, which will be called the *commutator* of X in Y, and which consists of the multiplication operators for C*-algebras. As is only to be

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expected, the results for commutators do not in general form such a neat theory as for centres, but a significant amount of the theory is recoverable. In particular in Theorem 3.2 a concrete description is given of the commutator of a Banach space Y in a Banach space tensor product $X \otimes Y$ where X contains a unit vector x_0 of a special type, and Y is identified with $x_0 \otimes Y$. This is a development of a problem considered in [11] and [6].

Behrends [8] has used the centraliser as an invariant to study isomorphisms between tensor products $X \otimes Y$ and $X' \otimes Y$ for commutative C*-algebras X and X'. Similarly the commutator could be used as an invariant for isomorphisms which themselves have an invariance property. However such complex machinery is unnecessary, since more elementary techniques lead to the same results, which are given in the final section of the paper.

2. Commutators of Banach Spaces

Let Y be a (real) Banach space with closed linear subspace X, and let $B(X;Y)$ (or $B(X)$ if X and Y coincide) denote the space of all bounded linear operators T of X into Y. The operator norm of T will be denoted by $\|T\|$. Such an operator T is *M-bounded* (resp. *strongly M-bounded*) if there is a positive real number λ such that for x and y in X (resp. for x in X and y in Y)

$$\|y - Tx\| \leqq \mathrm{Max}(\|y + \lambda x\|, \|y - \lambda x\|).$$

In this case, the smallest possible number λ will be denoted by $\|T\|_M$ (resp. $\|T\|'_M$). The class of all (strongly) M-bounded operators of X into Y will be called the *(strong) commutator of X in Y*, and denoted by $Z(X;Y)$ (resp. $Z'(X;Y)$). It is easily seen that $Z(X;Y)$ and $Z'(X;Y)$ are linear subspaces of $B(X;Y)$, and that $\| \ \|_M$ and $\| \ \|'_M$ are norms larger than $\| \ \|$, and it follows easily that they are complete. The commutator of X in itself is the *centraliser* $Z(X)$ of X, as defined by Alfsen and Effros [4, §4].

An ordered Banach space Y with closed positive cone Y^+ is an *F-space* if:

(i) $\|x\| \leqq \mathrm{Max}(\|y\|, \|z\|)$ whenever $y \leqq x \leqq z$

(ii) the open unit ball of Y is directed upwards in the relative ordering.

A vector u in an F-space Y is an *order-unit* if u is the largest vector in the closed unit ball of Y. It is shown in [4, Lemma 6.1] that an F-space Y with order-unit u is isometrically order-isomorphic to the space $A(K)$ of continuous affine real-valued functions on the weak*-compact convex set K of positive linear functionals of norm 1 on Y.

Now suppose that Y is any ordered Banach space. A positively-generated subspace X is an *order-ideal* if y belongs to X whenever $0 \leqq y \leqq x$ for some x in X. A linear operator T of any positively-generated subspace X into Y is *order-bounded* if there is a positive real number μ such that $-\mu x \leqq Tx \leqq \mu x$ for all x in X^+. The smallest such number μ will be denoted by $\|T\|_{\mathrm{ob}}$. The class of all order-bounded operators of X into Y is a linear space denoted by $W(X;Y)$, and $\| \ \|_{\mathrm{ob}}$ is a norm on $W(X;Y)$. In the special case when X coincides with Y, this space is known as the *ideal centre* of X, and has been studied by Wils [16].

If Y is an F-space, then $\| \|_{ob}$ is complete on $W(X;Y)$. If in addition, X and Y coincide, then $W(X;X)$ coincides with $Z(X)$ and the norms $\| \|$, $\| \|_M$ and $\| \|_{ob}$ all coincide on $Z(X)$ ([4, §6], [16, Lemma 2.3]).

Proposition 2.1. *Let Y be an F-space and X be a closed linear subspace which is an F-space in the relative ordering. Then any order-bounded operator T in $W(X;Y)$ is M-bounded and*

$$\|T\|_M \leqq \|T\|_{ob}.$$

Any strongly M-bounded operator S of X into Y is order-bounded with

$$\|S\|_{ob} \leqq \|S\|'_M.$$

If Y has an order-unit u which belongs to X, then $W(X;Y)$ and $Z(X;Y)$ coincide, and the norms $\| \|_{ob}$ and $\| \|_M$ also coincide.

Proof. We may suppose that $\|T\|_{ob}=1$. If $\|x \pm x'\|<1$ for some x and x' in X, then there exists y_1 in X with $x \pm x' \leqq y_1$ and $\|y_1\|<1$. Then

$$-Ty_1 - Tx' + Tx \leqq y_1 + x' - x,$$

$$Ty_1 - Tx' - Tx \leqq y_1 - x' - x.$$

Hence $x - Tx' \leqq y_1$. Similarly there exists y_2 in X with $\|y_2\|<1$ and $y_2 \leqq x - Tx'$, so $\|x - Tx'\|<1$.

Now suppose that S is strongly M-bounded with $\|S\|'_M=1$. For x in X^+ with $\|x\|<1$, we shall construct inductively a sequence y_n in Y with

$$\|y_n\|<1, \quad y_n \geqq x, \quad y_n \geqq nSx-(n-1)x.$$

We may take $y_0=x$. For y_n satisfying these conditions, $\|x-y_n \pm x\|<1$ since $-y_n \leqq 2x-y_n \leqq x$. Hence $\|x-y_n-Sx\|<1$, so there exists y_{n+1} in Y with $\|y_{n+1}\|<1$, $y_{n+1} \geqq x$, $y_{n+1} \geqq Sx+y_n-x \geqq (n+1)Sx-nx$.

Now $Sx \leqq n^{-1}(y_n+(n-1)x)$ for all n, so $Sx \leqq x$ since Y^+ is closed. Similarly $Sx \geqq -x$.

In the case when Y has an order-unit u lying in X, we may take $y_0=y_1=u$, and the above argument shows that $u \geqq Sx+u-x$, so $Sx \leqq x$.

Corollary 2.2. *Let A^s be the self-adjoint part of a C^*-algebra A with unit 1, and let B be a C^*-subalgebra of A containing 1. For z in A^s such that $zx=xz$ for all x in B, define*

$$T_z x = xz \quad (x \in B^s).$$

Then $Z(B^s;A^s)$ consists of the operators of the form T_z.

Proof. It is clear that T_z is order-bounded and $\|T_z\|_{ob}=\|z\|$. Conversely for T in $W(B^s;A^s)$, the argument of [1] shows that $T=T_z$ where $z=T1$.

Example 2.3. Let Y be any Banach space, x_0 be a non-zero vector in Y, and X be the one-dimensional space spanned by x_0. Then it is trivial to verify that $Z(X;Y)=B(X;Y)$.

In particular, taking Y to be an F-space, and x_0 to be an element of a proper order-ideal in Y^+, there may exist M-bounded operators of X into Y which are not order-bounded. Also taking $Y=A^s$ as in Corollary 2.2 and x_0 to be a non-trivial projection, Corollary 2.2 may fail if B is not assumed to contain the unit. However it is still true that $W(B^s; A^s)$ consists exactly of the operators of the form T_z whenever B is a C*-subalgebra of A.

Example 2.4. Let K be a compact convex set, Y be the F-space $A(K)$ and 1 be the constant function with value 1 in $A(K)$, so that 1 is an order-unit of Y. Let x_0 be a positive function of norm 1 in $A(K)$ with 0 and 1 in its range, and X be the two-dimensional space spanned by 1 and x_0. Since

$$\alpha x_0+\beta(1-x_0)\geqq 0 \Leftrightarrow \alpha\geqq 0, \quad \beta\geqq 0$$

it is easy to see that $W(X;Y)$ and hence $Z(X;Y)$ consist of the linear operators T of X into Y for which Tx_0 and $T(1-x_0)$ lie in the order-ideals generated by x_0 and by $1-x_0$ respectively.

It is now easy to construct examples to show that $Z(X;Y)$ may not be closed in $B(X;Y)$ in the strong operator topology, and that $\|\ \|$ and $\|\ \|_M$ may not coincide on $Z(X;Y)$. Also Corollary 2.2 may fail if B is not assumed to be a subalgebra of A. It would be interesting to find conditions on a closed linear subspace B of a C*-algebra A, which can be expressed in terms of the Banach space structure of A and B (and perhaps also the order structure), and which ensure that Corollary 2.2 holds. A necessary condition is that $\|T1\|=\|T\|_M$ for all T in $Z(B^s; A^s)$, but it is not clear whether this is sufficient.

For an F-space Y with unit u, and closed linear subspace X containing u, the *relative commutant* of X in Y is defined to be

$$RC(X;Y)=\{Tu\colon T\in Z(X;Y)\}.$$

Also u is said to *determine* $Z(X;Y)$ if $\|Tu\|=\|T\|_M$ for all T in $Z(X;Y)$. It is simple to check, using Proposition 2.1, that u determines $Z(X;Y)$ if and only if $Tx\geqq 0$ for all x in X^+ and T in $Z(X;Y)$ with $Tu\geqq 0$. In the special case when X is the whole of Y, $RC(X;X)$ is the *centre* of X, and u determines $Z(X)$ [2, 3].

Proposition 2.5. *Let Y be an F-space with order-unit u, and X be a closed linear subspace of Y containing u, and put $X'=RC(X;Y)$. Then X' is closed if and only if there is a constant c such that*

$$\inf\{\|T-S\|_M\colon S\in Z(X;Y),\ Su=0\}\leqq c\,\|Tu\|$$

for all T in $Z(X;Y)$. If u determines $Z(X;Y)$, then X' is closed, and X is contained in $RC(X';Y)$.

Proof. Let $Z_0=\{S\in Z(X;Y)\colon Su=0\}$. The mapping $T\to Tu$ induces a bounded linear isomorphism of $Z(X;Y)/Z_0$ onto X'. By the Open Mapping Theorem, X' is closed if and only if this mapping has bounded inverse.

If u determines $Z(X;Y)$, then $Z_0=(0)$ and c can be taken to be 1. Also u belongs to X' since the identity mapping belongs to $Z(X;Y)$. For x in X with $0\leqq x\leqq u$ and x' in X', there is a unique operator $T_{x'}$ in $Z(X;Y)$ with $T_{x'}u=x'$.

Put $Sx' = T_{x'}x$. Then S is linear, $Su = x$ and if $x' \geqq 0$ then $T_{x'} \geqq 0$, so $0 \leqq Sx' \leqq x'$. Thus $S \in Z(X'; Y)$ and $x = Su \in RC(X'; Y)$.

Again Example 2.4 may be used to construct examples, firstly in which $RC(X; Y)$ is not closed, and secondly in which $RC(X; Y) = Y$ but X is not contained in the centre of Y.

For a general Banach space X, Alfsen and Effros [4, Theorem 4.8] characterised the centraliser of X in terms of multiplicative properties on the extreme boundary P_X of the unit ball K_X of X^*. In the case when X is a closed subspace of Y, let

$$P_X(Y) = \{\phi \in K_Y : \phi|_X \in P_X\}.$$

We shall say that X is a *pure subspace* of Y if K_Y is the weak*-closed convex hull of $P_X(Y)$. By the bipolar theorem, this is equivalent to the following expression for the norm holding for all y in Y:

$$\|y\| = \sup\{|\phi(y)| : \phi \in P_X(Y)\}.$$

Note that X is a pure subspace of itself by the Krein-Milman theorem.

An operator T in $B(X; Y)$ is a *multiplier* if for each ϕ in $P_X(Y)$, $T^*\phi$ is a scalar multiple of $\phi|_X$. The space of all multipliers of X into Y will be denoted by $M(X; Y)$. It is easy to see that $M(X; Y)$ is closed in $B(X; Y)$ in the strong operator topology, so $M(X; Y)$ may differ from $Z(X; Y)$. Also [4, Theorem 4.8] shows that $M(X; X) = Z(X)$.

Theorem 2.6. *For any closed subspace X of a Banach space Y, $Z(X; Y)$ is contained in $M(X; Y)$. If X is a pure subspace of Y, then $Z'(X; Y) = Z(X; Y) = M(X; Y)$, and the norms $\| \ \|'_M$, $\| \ \|_M$ and $\| \ \|$ all coincide on $Z(X; Y)$.*

Proof. The proof of the first statement is similar to that in [4]. Thus for ϕ in $P_X(Y)$, x in X with $\phi(x) = 0$, and $\varepsilon > 0$, there is by [2, Proposition I.4.1] a function a in $A(K_X)$ with

$$a(\phi|_X) < \varepsilon$$

$$a(\psi) \geqq \mathrm{Max}(0, 2\psi(x)) \qquad (\psi \in K_X).$$

But then $a(\psi) = \psi(x') + \alpha$ for some x' in X and real number α. It is easy to see that $\|x'\| \leqq \alpha$ and $\|x' - 2x\| \leqq \alpha$. For any T in $Z(X; Y)$ with $\|T\|_M = 1$, $\|x' - x - Tx\| \leqq \alpha$, so

$$\phi(Tx) \leqq \alpha + \phi(x' - x) < \varepsilon.$$

Replacing ϕ by $-\phi$, and letting ε tend to 0, it follows that $\phi(Tx) = 0$ whenever $\phi(x) = 0$, so $T^*\phi$ is a scalar multiple of $\phi|_X$.

Now let T be a multiplier of norm 1, and take x in X and y in Y with $\|y \pm x\| \leqq 1$. For ϕ in $P_X(Y)$, $|\phi(y) \pm \phi(x)| \leqq 1$ and $T^*\phi = \lambda\phi|_X$ where λ is a real number with $|\lambda| = \|T^*\phi\| \leqq 1$. Hence $|\phi(y) - \lambda\phi(x)| \leqq 1$, so $|\phi(y - Tx)| \leqq 1$. Thus if X is a pure subspace, then $\|y - Tx\| \leqq 1$, and T is strongly M-bounded with $\|T\|'_M \leqq 1$.

It is easy to construct examples of multipliers which are not M-bounded. Indeed if the linear span of $P_X(Y)$ is not weak*-dense in Y^*, then the space Z of

vectors in Y annihilated by all functionals in $P_X(Y)$ is non-zero, and any operator in $B(X;Z)$ also belongs to $M(X;Y)$.

3. Commutators in Tensor Products

Let X and Y be Banach spaces, and $X \otimes Y$ be their injective Banach space tensor product, i.e. the completion of the algebraic tensor product $X \odot Y$ in the "least" cross-norm of Schatten [13] defined by:

$$\|\sum x_i \otimes y_i\| = \sup\{\|\sum \phi(x_i) y_i\| : \phi \in K_X\}.$$

For any ϕ in X^* and ψ in Y^*, there is a unique functional $\phi \otimes \psi$ in $(X \otimes Y)^*$ with

$$(\phi \otimes \psi)(x \otimes y) = \phi(x)\psi(y) \qquad (x \in X, y \in Y).$$

For a fixed unit vector x_0 in X, let

$$x_0 \otimes Y = \{x_0 \otimes y : y \in Y\}.$$

Then $x_0 \otimes Y$ is a closed linear subspace of $X \otimes Y$ isometrically isomorphic to Y, and there is a natural embedding of $X \odot B(Y)$ in $B(x_0 \otimes Y; X \otimes Y)$ given by

$$(x \otimes S)(x_0 \otimes y) = x \otimes Sy.$$

Proposition 3.1. *For the above embedding, $X \odot Z(Y)$ is contained in $Z(x_0 \otimes Y; X \otimes Y)$.*

Proof. Take x in X and S in $Z(Y)$ with $\|x\| = \|S\|_M = 1$. Suppose that $\|y_1 \pm 2y_2\| \leqq 1$ for some y_1 and y_2 in Y. Then $\|y_1 \pm 2Sy_2\| \leqq 1$, $\|y_1\| \leqq 1$ and $\|Sy_2\| \leqq \frac{1}{2}$. Take ϕ in K_X. If $|\phi(x_0)| \geqq \frac{1}{2}$, then

$$\left|\frac{\phi(x)}{\phi(x_0)}\right| \leqq 2$$

so

$$\|\phi(x_0) y_1 - \phi(x) S y_2\| \leqq \left\| y_1 - \frac{\phi(x)}{\phi(x_0)} S y_2 \right\| \leqq 1.$$

If on the other hand, $|\phi(x_0)| \leqq \frac{1}{2}$, then

$$\|\phi(x_0) y_1 - \phi(x) S y_2\| \leqq |\phi(x_0)| \, \|y_1\| + |\phi(x)| \, \|S y_2\| \leqq 1.$$

Thus $\|x_0 \otimes y_1 - x \otimes S y_2\| \leqq 1$.

It is easy to construct examples (even in C^*-algebras) in which $Z(x_0 \otimes Y; X \otimes Y)$ is much larger than $X \otimes Z(Y)$. This happens because of the behaviour of x_0 on P_X and is avoided when x_0 is an order-unit of an F-space.

Let K be a compact convex set, X be a Banach space and $C(K;X)$ (resp. $A(K;X)$) be the Banach space of all continuous (affine) functions of K into X. There is a natural embedding of $C(K) \odot X$ in $C(K;X)$ (where $C(K) = C(K;\mathbb{R})$) such that

$$(f \otimes x)(k) = f(k)\, x.$$

The norm induced on $C(K)\odot X$ is the "least" cross-norm, and $C(K)\odot X$ is dense in $C(K;X)$. Hence $C(K)\otimes X$ may be identified with $C(K;X)$ and $A(K)\otimes X$ with the closure of $A(K)\odot X$ in $A(K;X)$ [10, p. 90]. In Theorem 3.2 we shall obtain a description of the commutator of $1\otimes X$ in $A(K)\otimes X$ and in $A(K;X)$. The method used is a modification of one used by Vincent-Smith (unpublished) to give a simplified proof of Wickstead's result [15] that the centraliser of the injective tensor product of two Banach spaces is the strong closure of the tensor product of their centralisers.

There is a topology on P_X, known as the *structure topology*, and an isometric isomorphism Φ_X of $Z(X)$ onto the space $C_s^b(P_X)$ of bounded real-valued structurally continuous functions on P_X, for which

$$T^*\phi=(\Phi_X(T)(\phi))\,\phi \qquad (\phi\in P_X,\ T\in Z(X)).$$

For details of this topology, the reader is referred to [4].

Therorem 3.2. *Let K be a compact convex set and X be a Banach space. Then $1\otimes X$ is a pure subspace of $A(K)\otimes X$ and of $A(K;X)$. The commutator of $1\otimes X$ in $A(K)\otimes X$ or in $A(K;X)$ is the strong closure of $A(K)\odot Z(X)$ in $B(1\otimes X;A(K;X))$.*

Proof. For k in K and ϕ in P_X define ϕ_k in $P_{1\otimes X}(A(K;X))$ by

$$\phi_k(b)=\phi(b(k)) \qquad (b\in A(K;X)).$$

Then

$$\|b\|=\sup\{|\phi_k(b)|:\ \phi\in P_X,\ k\in K\}$$

so $1\otimes X$ is a pure subspace of $A(K)\otimes X$ and of $A(K;X)$.

Theorem 2.6 and Proposition 3.1 and the fact that $M(1\otimes X;A(K)\otimes X)$ is strongly closed show that it suffices to prove that any operator T in $Z(1\otimes X;A(K;X))$ with $\|T\|_M=1$ belongs to the strong closure of $A(K)\odot Z(X)$. For k in K, define T_k in $B(X)$ by

$$T_k x=T(1\otimes x)(k).$$

Routine verification shows that T_k is M-bounded, so $\Phi_X(T_k)$ is a structurally continuous function on P_X.

For fixed ϕ in P_X, choose x_ϕ in X with $\phi(x_\phi)\neq 0$. Then

$$\Phi_X(T_k)(\phi)=\frac{\phi(T(1\otimes x_\phi)(k))}{\phi(x_\phi)}.$$

Thus there exists a_ϕ in $A(K)$ with $\|a_\phi\|\leqq 1$ such that $\Phi_X(T_k)(\phi)=a_\phi(k)$. Now an argument similar to that of [6, Theorem 1] shows that the function

$$(k,\phi)\to\Phi_X(T_k)(\phi)=a_\phi(k)$$

is jointly continuous on $K\times P_X$ for the given topology on K and the structure topology on P_X.

Now take x_j in X with $\|x_j\| \leqq 1$ $(1 \leqq j \leqq n)$, and $0 < \varepsilon < 1$, and let

$$V = \{\phi \in P_X : |\phi(x_j)| \geqq \varepsilon/3 \text{ for some } j\}.$$

Then V is structurally compact [15, p. 569], so $\phi \to a_\phi$ is structurally continuous from V into $A(K)$ and separates points of V only if they are also separated by $C_s^b(P_X)$.

Hence if $\tilde{V}$ is the quotient of V obtained by identifying points of V not separated by $C_s^b(P_X)$, then there is a function $\tilde{a}$ in $C(\tilde{V}; A(K))$ such that $\tilde{a}(\pi(\phi)) = a_\phi$, where π is the quotient map of V onto $\tilde{V}$. Hence there exist functions f_i in $C(\tilde{V})$ and a_i in $A(K)$ $(1 \leqq i \leqq m)$ such that

$$\left\| \sum_{i=1}^{m} f_i(\pi(\phi))\, a_i - a_\phi \right\| < \varepsilon \qquad (\phi \in V).$$

But by the Stone-Weierstrass theorem, the functions induced on $\tilde{V}$ by functions in $C_s^b(P_X)$ are dense in $C(\tilde{V})$, so we may assume that there exist g_i in $C_s^b(P_X)$ such that $g_i(\phi) = f_i(\pi(\phi))$ for ϕ in V. Replacing g_i by g_i' where

$$g_i'(\phi) = \frac{2 g_i(\phi)}{\operatorname{Max}(\|\sum g_j(\phi)\, a_j\|, 2)}$$

we may assume that $\|\sum g_i(\phi)\, a_i\| \leqq 2$ for all ϕ in P_X. Now let $S_i = \Phi_X^{-1}(g_i)$ in $Z(X)$. Then for k in K, ϕ in P_X, and $1 \leqq j \leqq n$,

$$\phi(T(1 \otimes x_j)(k)) = a_\phi(k)\, \phi(x_j)$$

$$\phi((\sum_i a_i \otimes S_i)(1 \otimes x_j)(k)) = \sum_i g_i(\phi)\, a_i(k)\, \phi(x_j).$$

Thus, for ϕ in V,

$$|\phi((T - \sum_i a_i \otimes S_i)(1 \otimes x_j)(k))| \leqq \|\sum_i g_i(\phi)\, a_i - a_\phi\| < \varepsilon$$

while for ϕ in $P_X \setminus V$,

$$|\phi(T(1 \otimes x_j)(k))| \leqq \|a_\phi\|\, |\phi(x_j)| < \varepsilon/3$$

$$|\phi((\sum_i a_i \otimes S_i)(1 \otimes x_j)(k))| \leqq \|\sum_i g_i(\phi)\, a_i\|\, |\phi(x_j)| < 2\varepsilon/3.$$

Hence $\|(T - \sum a_i \otimes S_i)(1 \otimes x_j)\| < \varepsilon$. Thus T is strongly approximated by operators of the required form.

The above proof may be modified to show that for arbitrary Banach spaces X and Y and a unit vector x_0 in X, $M(x_0 \otimes Y; X \otimes Y)$ is the strong closure of $X \odot Z(Y)$ provided that the weak*-closed convex hull C_{x_0} of $\{\phi \in K_X : |\phi(x_0)| = 1\}$ is a neighbourhood of 0 in the norm-topology of X^*. But in this case, X may be ordered and given an equivalent norm so that it becomes an F-space with x_0 as order-unit. It is also easy to verify that $x_0 \otimes Y$ is a pure subspace of $X \otimes Y$ if and only if C_{x_0} is the whole unit ball of X^*, and this is equivalent to the existence of an ordering on X making it into an F-space with x_0 as order-unit in the original

norm. Furthermore any Banach space X can be identified with the space $A_0(K_X)$ of functions in $A(K_X)$ vanishing at 0. Thus the loss of generality in stating Theorem 3.2 in the given form is not serious.

If K and K' are compact convex sets, then $A(K; A(K'))$ and hence $A(K)\otimes A(K')$ have a natural pointwise ordering in which they become F-spaces with order-unit $1\otimes 1'$, where 1 and $1'$ are the order-units of $A(K)$ and $A(K')$.

Corollary 3.3. *The commutator of* $1\otimes A(K')$ *in either* $A(K)\otimes A(K')$ *or* $A(K; A(K'))$ *is determined by the order-unit* $1\otimes 1'$. *In each case the relative commutant of* $1\otimes A(K')$ *is* $A(K; Z(K'))$, *where* $Z(K')$ *is the centre of* $A(K')$.

Proof. For a' in $A(K')$ and z' in $Z(K')$, let $z'\cdot a'$ denote the unique function in $A(K')$ such that $(z'\cdot a')(k')=z'(k')a'(k')$ for k' in $\partial_e K'$. For S in $Z(A(K'))$, $S1'$ belongs to $Z(K')$ and $Sa'=(S1')\cdot a'$ [2, Theorem II.7.10]. Thus for a in $A(K)$ and k in K,

$$\begin{aligned}((a\otimes S)(1\otimes a'))(k)&=a(k)(Sa')=a(k)((S1')\cdot a')\\&=((a\otimes S)(1\otimes 1')(k))\cdot a'.\end{aligned}$$

By Theorem 3.2, both relative commutants contain $A(K)\odot Z(K')$, and are contained in $A(K)\otimes Z(K')$, and for any M-bounded operator T of $1\otimes A(K')$ into $A(K; A(K'))$ satisfies

$$T(1\otimes a')(k)=T(1\otimes 1')(k)\cdot a'$$

so that $\|T\|=\|T(1\otimes 1')\|$. But by Theorems 2.6 and 3.2, $\|\mathrm{T}\|=\|\mathrm{T}\|_M$, so $1\otimes 1'$ determines the commutators of $1\otimes A(K')$. By Proposition 2.5, the relative commutants are norm-closed and therefore coincide with $A(K)\otimes Z(K')$. But $A(K)\otimes Z(K')=A(K; Z(K'))$ [6, Theorem 1].

4. Isomorphisms of Tensor Products

The centraliser of a Banach space is clearly an invariant under isometric isomorphisms. This fact has been used by Behrends [8] to study isometric isomorphisms between spaces of the form $X\otimes Y$ and $X'\otimes Y$, where X and X' are (the self-adjoint parts of) commutative C*-algebras. He showed that if $Z(Y)$ is one-dimensional, then the existence of such an isomorphism implies that the spectra of X and X' are homeomorphic, so that X and X' are also isometrically isomorphic, and he was able to extend this to include some, but not all, cases when $Z(Y)$ is finite-dimensional. It also follows immediately from the result of [15] that if $Z(Y)$ is one-dimensional and X and X' are any Banach spaces such that $X\otimes Y$ and $X'\otimes Y$ are isometrically isomorphic, then so too are $Z(X)$ and $Z(X')$.

In a similar fashion, Theorem 3.2 may be used to deduce that if K and K' are compact convex sets, and Y is a Banach space with $Z(Y)$ one-dimensional, and there is an isometric isomorphism of $A(K)\otimes Y$ (or $A(K; Y)$) onto $A(K')\otimes Y$ (or $A(K'; Y)$) carrying $1\otimes Y$ onto $1'\otimes Y$, then K and K' are affinely homeomorphic. However a more general result can be obtained by more elementary means.

The following lemma may be compared with [12, Lemma 2.1], [6, Lemma 3] and the lemma in [15].

Lemma 4.1. *Let X and Y be Banach spaces, x_0 be a unit vector in X and ϕ_0 in $K_{X\otimes Y}$ and ψ_1 in P_Y be linear functionals satisfying*

$$\phi_0(x_0\otimes y)=\psi_1(y) \quad (y\in Y).$$

Then there is a functional ψ_0 in K_X such that $\phi_0=\psi_0\otimes\psi_1$ and $\psi_0(x_0)=1$.

Proof. Let $F=\{\phi\in K_{X\otimes Y}: \phi(x_0\otimes y)=\psi_1(y)\ (y\in Y)\}$. Then F is a non-empty weak*-closed face of $K_{X\otimes Y}$, and by the Krein-Milman theorem, F is the weak*-closed convex hull of $\partial_e F$, which equals $F\cap P_{X\otimes Y}$. But $P_{X\otimes Y}$ consists of functionals $\psi\otimes\psi'$ where ψ belongs to P_X and ψ' to P_Y, by the lemma in [15], so $F\cap P_{X\otimes Y}$ is contained in $\{\psi\otimes\psi_1: \psi\in K_X, \psi(x_0)=1\}$. But this set is weak*-closed and convex, so the lemma follows.

Theorem 4.2. *Let K and K' be compact convex sets, X be a Banach space, and suppose that there is an isometric isomorphism Ψ of $A(K)\otimes X$ (or $A(K;X)$) onto $A(K')\otimes X$ (or $A(K';X)$) carrying $1\otimes X$ onto $1'\otimes X$. Then K and K' are affinely homeomorphic.*

Proof. There is an isometric isomorphism Ψ_0 of X such that

$$\Psi(1\otimes\Psi_0(x))=1'\otimes x.$$

There is also an isometric isomorphism Ψ_1 of $A(K';X)$, leaving $A(K')\otimes X$ invariant, defined by

$$(\Psi_1(b))(k')=\Psi_0(b(k')).$$

Composing Ψ with Ψ_1, we may assume that $\Psi(1\otimes x)=1'\otimes x$.

Fix ψ_1 in P_X. Let $B(=A(K)\otimes X$ or $A(K;X))$ be the domain of Ψ and B' be its range, and put

$$F=\{\phi\in K_B: \phi(1\otimes x)=\psi_1(x)\}$$
$$F'=\{\phi'\in K_{B'}: \phi'(1'\otimes x)=\psi_1(x)\}.$$

Then $\Psi^*F'=F$. But identifying $C(K)\otimes X$ with $C(K;X)$, $1\otimes X\subset B\subset C(K)\otimes X$. It follows from Lemma 4.1 and the Hahn-Banach theorem that F consists of the functionals ϕ of the form

$$\phi(b)=\int\psi_1(b(k))\,d\mu(k) \quad (b\in B)$$

for some Radon probability measure μ on K. But then $\phi(b)=\psi_1(b(k))$ where k is the barycentre of μ. Similarly F' consists of functionals of the form $b'\to\psi_1(b'(k'))$ for k' in K'. Hence Ψ^* induces a bijection θ of K' onto K uniquely determined by the identity

$$\psi_1(\Psi(b)(k'))=\psi_1(b(\theta(k'))) \quad (b\in B,\ k'\in K').$$

It is clear that θ is an affine homeomorphism.

Corollary 4.3. *Let X, X' and Y be Banach spaces, and suppose that there is an isometric isomorphism Ψ of $X \otimes Y$ onto $X' \otimes Y$ such that for any w in $X \otimes Y$ and y in Y,*

$$\sup\{|(\phi \otimes \psi)(w) + \psi(y)| : \phi \in K_X, \psi \in K_Y\}$$
$$= \sup\{|(\phi' \otimes \psi)(\Psi(w)) + \psi(y)| : \phi' \in K_{X'}, \psi \in K_Y\}.$$

Then X and X' are isometrically isomorphic.

Proof. If X is identified with $A_0(K_X)$, then $X \otimes Y$ may be identified with a subspace of $A(K_X) \otimes Y$. Thus we may define a linear isomorphism $\tilde{\Psi}$ of $A(K_X) \otimes Y$ onto $A(K_{X'}) \otimes Y$ by

$$\tilde{\Psi}(w + 1 \otimes y) = \Psi(w) + 1' \otimes y \qquad (w \in X \otimes Y, y \in Y).$$

The given condition ensures that $\tilde{\Psi}$ is isometric. By Theorem 4.2 and its proof, for any ψ_1 in P_Y, there is an affine homeomorphism θ of $K_{X'}$ onto K_X such that

$$(\theta(0) \otimes \psi_1)(x \otimes y) = 0 \qquad (x \in X, y \in Y).$$

Hence $\theta(0) = 0$, so θ induces an isometric isomorphism of $A_0(K_X)$ onto $A_0(K_{X'})$.

If Ψ_1 is an isometric isomorphism of X onto X', then there is a unique isometric isomorphism Ψ of $X \otimes Y$ onto $X' \otimes Y$ satisfying

$$\Psi(x \otimes y) = \Psi_1(x) \otimes y.$$

Then Ψ satisfies the conditions of Corollary 4.3. However the existence of an isometric isomorphism between $X \otimes Y$ and $X' \otimes Y$ without further restriction does not imply that X and X' are isometrically isomorphic, even when X, X' and Y are (the self-adjoint parts of) commutative C*-algebras and Y is two-dimensional [8].

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