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Normal Characteristic Forms: Conformal Invariance and Duality

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This article is concerned with the application of characteristic forms to extrinsic differential geometry, and in particular, to the conformal geometry of submanifolds, and to isometric and conformal immersions. The first two sections give applications to elementary geometry of curves and hypersurfaces in Euclidean space, whereas the third section is more formal and treats the duality of characteristic forms in a universal context.

Section 1 proves the invariance of the normal connection forms and consequently all normal characteristic forms under conformal mappings of the ambient Euclidean space. Section 2 discusses the normal secondary Euler form for n-manifolds immersed in Euclidean $(n+2)$-space. For $n=1$, it uses the conformal invariance of the normal Euler form to give a generalization of the Scherrer theorem [14], that if all closed curves on a surface in three-space have their total torsion equal to zero, the surface is part of a sphere or a plane. For $n>1$, it extends the Ferus generalization [7] of Scherrer's theorem by putting it in a conformal geometric framework. The main tool of this section is a variational formula for the secondary Euler form (Theorem 2.3).

Section 3 is devoted to the duality of Chern and Pontryagin forms and their transgressed forms. It builds upon the ideas in the article of Chern and White [6], and extends some of the main results of that article. The philosophy is to work with differential forms in the "universal diagram" over the Grassmannian, and to pull back to obtain integral formulae for submanifolds. The main theorem of this section is closely related to the duality theorem for Simons characters, and has applications to the theory of conformal and isometric immersions.

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1. Conformal Invariance

Let $x: M^n \to E^{n+k}$ be a C^2 immersion of an n-dimensional differentiable manifold $M^n = M$ into Euclidean $(n+k)$-space. Let $e_1, \ldots, e_{n+k}$ be a smooth orthonormal

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frame field on M such that $e_1, \ldots, e_n$ are tangent to $x(M)$ and $e_{n+1}, \ldots, e_{n+k}$ are normal to $x(M)$. If we write $dx=\sum_A \varphi_A e_A$, $de_A=\sum_B \varphi_{AB} e_B$ where $\varphi_{AB}+\varphi_{BA}=0$, then the Cartan structure equations yield

$$d\varphi_A=\sum_B \varphi_B \wedge \varphi_{BA}.$$

Hereafter, for range of indices, we use $1 \leqq A, B, C \leqq n+k$, $1 \leqq i, j \leqq n$, $n+1 \leqq \alpha, \beta \leqq n+k$. This section is devoted to the proof of

Theorem 1.1. *The normal connection forms $\varphi_{\alpha\beta}$ are invariant under conformal transformations of the ambient Euclidean space E^{n+k}.*

Proof. We can assume that the frame field is extended to an open subset of E^{n+k}. Let $\bar{\varphi}_A=\lambda\varphi_A$, $\lambda \neq 0$, denote a representative conformal transformation. By the structure equations

$$d\bar{\varphi}_A=\sum_B \bar{\varphi}_B \wedge \bar{\varphi}_{BA}.$$

Hence,

$$d(\lambda\varphi_A)=\sum_B \lambda\varphi_B \wedge \bar{\varphi}_{BA}$$

giving

$$d\lambda \wedge \varphi_A+\sum_B \lambda\varphi_B \wedge \varphi_{BA}=\sum_B \lambda\varphi_B \wedge \bar{\varphi}_{BA}.$$

Writing $d\lambda=\sum_B \lambda_B \varphi_B$, we obtain

$$\sum_B \lambda_B \varphi_B \wedge \varphi_A+\sum_B \lambda\varphi_B \wedge \varphi_{BA}=\sum_B \lambda\varphi_B \wedge \bar{\varphi}_{BA}$$

or

$$\sum_B \varphi_B \wedge \{\lambda(\bar{\varphi}_{BA}-\varphi_{BA})-\lambda_B\varphi_A\}=0.$$

By Cartan's lemma, we have

$$\lambda(\bar{\varphi}_{BA}-\varphi_{BA})-\lambda_B\varphi_A=\sum_C h^C_{BA}\varphi_C \tag{1.1}$$

with

$$h^C_{BA}=h^B_{CA}. \tag{1.2}$$

Interchanging indices gives

$$\lambda(\bar{\varphi}_{AB}-\varphi_{AB})-\lambda_A\varphi_B=\sum_C h^C_{AB}\varphi_C \tag{1.3}$$

Adding (1.1) and (1.3) yields

$$-\lambda_B \varphi_A - \lambda_A \varphi_B = \sum_C (h^C_{BA} + h^C_{AB}) \varphi_C$$

Since on E^{n+k} the φ_A's are linearly independent

(1.4) $h^C_{BA} + h^C_{AB} = 0$ for $C \neq A, B$.

Finally, to complete the proof of the theorem, we note that when restricted to M^n, $\varphi_\alpha = 0$, so that Eq. (1.3) becomes for normal indices

$$\lambda(\bar{\varphi}_{\alpha\beta} - \varphi_{\alpha\beta}) = \sum_i h^i_{\alpha\beta} \varphi_i .$$

Using (1.2) and (1.4) we obtain

$$h^i_{\alpha\beta} = h^\alpha_{i\beta} = -h^\alpha_{\beta i} = -h^\beta_{\alpha i} = h^\beta_{i\alpha} = h^i_{\beta\alpha} = -h^i_{\alpha\beta} .$$

Therefore, $h^i_{\alpha\beta} = 0$ for all i. This gives $\bar{\varphi}_{\alpha\beta} - \varphi_{\alpha\beta} = 0$.

Remark. This theorem is known by several geometers but, since we have not found any proof in the literature, we present this proof for completeness.

2. The Euler Form

In this section we present a discussion of the normal secondary Euler form for immersions $x\colon M^n \to E^{n+2}$ of oriented differentiable n-manifolds into Euclidean $(n+2)$-space. If, as in section 1, $e_1, \ldots, e_{n+2}$ is an orthonormal frame field on M such that $e_1, \ldots, e_n$ are tangent to $x(M)$ and e_{n+1}, e_{n+2} are normal, then this form is simply the form $\varphi_{n+1\,n+2}$. We divide this section into two parts, first the case $n=1$, and second the case $n>1$.

The Case $n=1$

Let $x\colon C \to E^3$ be an immersed closed curve in Euclidean three-space. Let e_1 denote the unit tangent vector field, let e_2 be a unit normal vector field, and let $e_3 = e_1 \times e_2$ be the complementary unit normal vector field. The quantity $\frac{1}{2\pi}\int_C \varphi_{23}$ is called the *total twist*, $Tw(x)$, of e_2 along C. It is easy to show that, although $Tw(x)$ depends on the choice of unit normal vector field e_2, its reduction mod Z, $Tw(x)\tilde{}$, does not. Cf. [1]. Therefore, it immediately follows from the conformal invariance theorem of Section 1 that

Theorem 2.1. *Up to sign, $Tw(x)\tilde{}$ is invariant under conformal transformations of E^3.*

Remark. This theorem is valid *up to sign* because of the fact that a conformal transformation such as an inversion may reverse the orientation of the ambient Euclidean space. Cf. [1].

We next prove a generalization of the classical Scherrer theorem which states that if S is a surface in E^3 such that every closed curve on S has total torsion equal to zero, then S is part of a sphere or plane.

Theorem 2.2. *Let S be a surface in Euclidean three-space. Then $Tw(x)\tilde{}=0$ for all immersed closed curves x on S if and only if S is part of a sphere or plane.*

Remark. Because of Theorem 2.1, and since a sphere and plane are conformally equivalent, this theorem, quite properly, becomes a theorem of conformal differential geometry.

Proof of Theorem. Suppose S is part of a plane. Then clearly $Tw(x)\tilde{}=0$ since one may choose e_2 to be the constant vector field perpendicular to the plane. If S is part of a sphere, then one may map S by a conformal transformation onto a plane and then by Theorem 2.1 $Tw(x)\tilde{}=0$.

On the other hand suppose $Tw(x)\tilde{}=0$ for all immersed curves x on S. We consider variations of the total twist $\frac{1}{2\pi}\int_C \varphi_{23}$. Later in this section we show that if A is a variation vector for a variation of x and t the variation parameter,

$$(2.1)\quad \frac{\partial}{\partial t}\bigg|_A \int_C \varphi_{23} = -\int_C A \cdot \frac{d}{ds}\left(e_1 \times \frac{de_1}{ds}\right) ds$$

where s is the arc-length of the curve C. Note again that the right-hand is independent of the choice of vector field e_2. Now since $Tw(x)\tilde{}=0$ for all x on S, Eq. (2.1) implies that the vector $\frac{d}{ds}\left(e_1 \times \frac{de_1}{ds}\right)$ must be perpendicular to S at all points of all curves on S, i.e. the component of this vector tangent to S must be zero. If we let a_3 be the surface normal vector field along C and $a_2 = a_3 \times e_1$ the tangent normal vector field along C, then the condition that $\frac{d}{ds}\left(e_1 \times \frac{de_1}{ds}\right)$ is perpendicular to S yields $\frac{d^2 e_1}{ds^2} \cdot a_3 = 0$ for all curves. Now for plane curves, $\frac{d^2 e_1}{ds^2} = \frac{d\kappa}{ds} N - \kappa^2 e_1$, where κ is the curvature and N the principal normal of the curve. Hence, for plane sections of our surface S, $\frac{d\kappa}{ds} N \cdot a_3 = 0$..Since $N \cdot a_3 \neq 0$ at almost all points of almost all curves, it follows that every plane section of S has constant curvature. Hence, S is part of a plane or sphere.

The Case $n > 1$

Let $x\colon M^n \to E^{n+2}$ be an immersion of an n-dimensional differentiable manifold into Euclidean $(n+2)$-space. Let C be a closed curve on M. Along C we define an orthonormal frame field $e_1, \ldots, e_{n+2}$ such that e_1 is tangent to $x(C)$, $e_2, \ldots, e_n$ are perpendicular to $x(C)$ and tangent to $x(M)$, and e_{n+1}, e_{n+2} are perpendicular to $x(M)$. Further, let $s = s_1, s_2, \ldots, s_n$ be local coordinates on M along C, so that

C is defined by $s_2=\cdots=s_n=0$ and that the coframe field to $e_1,\ldots,e_n$ along C, $\varphi_1,\ldots,\varphi_n$ satisfies $\varphi_i=ds_i$, $i=1,\ldots,n$.

We first study the behavior of the total twist $\int_C \varphi_{n+1\,n+2}$ under variations of M^n along C. Let A be the variation vector and t the variation parameter, then we shall prove:

Theorem 2.3. *Let* $B_i=\left(\frac{\partial e_i}{\partial s}\cdot e_{n+1}\right)e_{n+2}-\left(\frac{\partial e_i}{\partial s}\cdot e_{n+2}\right)e_{n+1}$. *Then*

$$\left.\frac{\partial}{\partial t}\right|_A \int_C \varphi_{n+1\,n+2}=\sum_{i=1}^{n}\int_C\left(\frac{\partial A}{\partial s_i}\cdot B_i\right)ds.$$

Remark. Since $s=s_1$, one may integrate by parts the first term in the sum on the right-hand side and rewrite the theorem:

$$\left.\frac{\partial}{\partial t}\right|_A \int_C \varphi_{n+1\,n+2}=\int_C-\left(A\cdot\frac{\partial B_1}{\partial s}\right)ds+\sum_{i=2}^{n}\int_C\left(\frac{\partial A}{\partial s_i}\cdot B_i\right)ds.$$

Before we present the proof of this theorem, we prove two important consequences. First, a curve C on M is called a critical curve if $\left.\frac{\partial}{\partial t}\right|_A \int_C \varphi_{n+1\,n+2}=0$ for all variations of M along C.

Theorem 2.4. *Let* $x\colon M^n\to E^{n+2}$ *be an immersion of an n-dimensional differentiable manifold M into Euclidean* $(n+2)$*-space. If all closed curves on M are critical, then M is immersed as a part of an n-plane or n-sphere in a hyperplane* E^{n+1} *of* E^{n+2}.

Proof of Theorem. Since $\left.\frac{\partial}{\partial t}\right|_A \int_C \varphi_{n+1\,n+2}=0$ for any variation vector A, it holds true for the particular variation vector $\bar{A}=\sum_{i=2}^{n} s_iB_i$. Clearly on C, $\bar{A}=0$. Furthermore, $\frac{\partial\bar{A}}{\partial s_i}=\sum_{i=2}^{n}\frac{\partial}{\partial s_i}(s_iB_i)$ so that on C, $\frac{\partial\bar{A}}{\partial s_i}=B_i$. Therefore, $0=\sum_{i=2}^{n}\int_C B_i\cdot B_i\,ds$. From this it follows that if C is critical $B_i=0$, and thus $\left.\frac{\partial}{\partial t}\right|_A \int_C \varphi_{n+1\,n+2}=\int_C-\left(A\cdot\frac{\partial B_1}{\partial s}\right)ds$. If we choose another variation vector $\bar{A}=\frac{\partial B_1}{\partial s}$, we conclude that if C is critical, $B_i=0$, $i=2,\ldots,n$ and $\frac{\partial B_1}{\partial s}=0$. Now, since $\frac{\partial B_1}{\partial s}=0$ along C, the vector

$$B_1=\left(\frac{\partial e_1}{\partial s}\cdot e_{n+1}\right)e_{n+2}-\left(\frac{\partial e_1}{\partial s}\cdot e_{n+2}\right)e_{n+1}$$

is constant along C. Hence, since B_1 is constant and $e_i\cdot B_1=0$ for $i=1,\ldots,n$ along C, the tangent plane to $x(M)$ along C is perpendicular to a constant direction. Thus C lies in a hyperplane. Finally, since all curves are critical it follows immediately that $x(M)$ lies in a hyperplane of E^{n+2}. Next, if we observe that $\frac{\partial e_1}{\partial s}$ is the curvature vector of the curve and that $B_1\cdot B_1$ is constant, we see the normal

curvature is constant along C. Since this is true for all C on M, $x(M)$ must be part of a plane or sphere.

Our second consequence generalizes the theorem of Ferus [7] by studying the problem via the conformally invariant secondary Euler form in the following manner. Let $x\colon N^{n+1}\to E^{n+2}$ be an immersion of an $(n+1)$-manifold onto a hypersurface in Euclidean $(n+2)$-space. Let C be a closed curve on N, let e_{n+2} be the unit normal vector field on C such that e_{n+2} is perpendicular to $x(N)$ along $x(C)$, and let e_{n+1} be an arbitrary unit tangent normal vector field on C, i.e., e_{n+1} is tangent to $x(N)$ along $x(C)$ but is perpendicular to $x(C)$. As usual we define the total twist $Tw(x)$ of e_{n+1} by $\int_C \varphi_{n+1\,n+2}$. This is the integral of the secondary Euler form of the two-plane bundle over C spanned by the unit tangent normal field e_{n+1} and the unit normal field e_{n+2}.

Theorem 2.5. *Let $x(N)$ be a hypersurface in Euclidean $(n+2)$-space. A necessary and sufficient condition for $x(N)$ to be part of a hypersphere or hyperplane is that along every closed curve $Tw(x)\tilde{}=0$ for every two-dimensional oriented subbundle of the normal bundle to C which contains the unit normal to the hypersurface along C.*

Proof. That $Tw(x)\tilde{}=0$ when S is part of a hypersphere or hyperplane is an easy generalization of the case $n=1$. Suppose, on the other hand, $Tw(x)\tilde{}=0$ for all curves and all subbundles. As in the case $n=1$, we consider variations of the total twist. By Theorem 2.3 we have

$$\frac{\partial}{\partial t}\bigg|_A \int_C \varphi_{n+1\,n+2} = \int_C -\left(A\cdot\frac{\partial B_1}{\partial s}\right)ds + \sum_{i=2}^{n}\int_C\left(\frac{\partial A}{\partial s_i}\cdot B_i\right)ds$$

where $e_1,\ldots,e_n$ are chosen so that e_1 is tangent to $x(C)$ and $e_1,\ldots,e_n$, e_{n+1}, e_{n+2} form an orthonormal frame along C.

Now by hypothesis any variation of $\int_C \varphi_{n+1\,n+2}$ in a direction tangent to $x(N)$ will be zero. Thus, e.g. if A is along e_{n+1}, $\dfrac{\partial}{\partial t}\Big|_A \int_C \varphi_{n+1\,n+2}=0$. Let B_i^T be the component of B_i tangent to $x(N)$, i.e. $B_i^T=\left(\dfrac{\partial e_i}{\partial s}\cdot e_{n+2}\right)e_{n+1}$. Set $\bar{A}=\sum_{i=2}^{n} s_i B_i^T$. Then $\bar{A}=0$ along C and $\dfrac{\partial \bar{A}}{\partial s_i}=B_i^T$ along C, $i=2,\ldots,n$. Thus

$$0=\frac{\partial}{\partial t}\bigg|_{\bar{A}}\int_C \varphi_{n+1\,n+2}=\int_C B_i^T\cdot B_i^T\,ds$$

so that $B_i^T=0$, $i=2,\ldots,n$. Thus $\dfrac{\partial e_i}{\partial s}\cdot e_{n+2}=0$, $i=2,\ldots,n$ and hence $\dfrac{\partial e_{n+2}}{\partial s}\cdot e_i=0$ for $i=2,\ldots,n$. Hence, $\dfrac{\partial e_{n+2}}{\partial s}=fe_1+g\,e_{n+1}$ for some functions f and g. We wish to show $g=0$, for then the curve will be a principal curve, i.e. $\dfrac{\partial e_{n+2}}{\partial s}$ is along e_1. This will imply N is part of a hyperplane or hypersphere. Let e_{n+1} and $\bar{e}_{n+1}$ be

two linearly independent tangent normal unit vector fields. Our result gives

$$\frac{\partial e_{n+2}}{\partial s} = f e_1 + g e_{n+1} = \bar{f} e_1 + \bar{g} \bar{e}_{n+1}.$$

But since e_{n+1} and $\bar{e}_{n+1}$ are linearly independent $g = \bar{g} = 0$. This completes the proof of the theorem.

Finally, we prove Theorem 2.3. We use the method of Chern in [4]. We apply the variation in the following manner. Let I be the interval $-\frac{1}{2} < t < \frac{1}{2}$. Let $F\colon M \times I \to E^{n+2}$ be a differentiable mapping such that its restriction to $M \times \{t\}$, $t \in I$, is an immersion and that $F(m, 0) = x(m)$, $m \in M$. Let $e_1, \ldots, e_{n+2}$ be an orthonormal frame field over $M \times I$ such that at every point $(m, t) \in M \times I$, $e_1(m, t), \ldots, e_n(m, t)$ are tangent to $F(M \times t)$ at (m, t) and $e_{n+1}(m, t)$, $e_{n+2}(m, t)$ are normal to $F(M \times t)$ at (m, t). If $\omega_1, \ldots, \omega_{n+2}$ is the dual frame to $e_1, \ldots, e_{n+2}$, then we may write

$$\begin{aligned} \omega_i &= \varphi_i + a_i dt && i = 1, \ldots, n \\ \omega_\alpha &= a_\alpha dt && \alpha = n+1, n+2 \\ \omega_{AB} &= \varphi_{AB} + a_{AB} dt && A, B = 1, \ldots, n+2 \end{aligned}$$

where φ_i and φ_{AB} are linear differential forms with coefficients which may depend on t such that at $t = 0$, they reduce to the forms with the same notation on M. The vector $A = \sum_{B=1}^{n+2} a_B e_B$ at $t = 0$ is the variation parameter. Finally, the operator d on $M \times I$ may be written as $d = d_M + dt \frac{\partial}{\partial t}$. We now compute the variation of the twist.

By the structure equations

$$d\omega_{n+1\,n+2} = \sum_{i=1}^{n} \omega_{n+1\,i} \wedge \omega_{in+2}.$$

Hence,

$$\left(d_M + dt \frac{\partial}{\partial t}\right)(\varphi_{n+1\,n+2} + a_{n+1\,n+2}\, dt) = \sum_i (\varphi_{n+1\,i} + a_{n+1\,i}\, dt) \wedge (\varphi_{in+2} + a_{in+2}\, dt).$$

Equating terms in dt, we get

$$\frac{\partial \varphi_{n+1\,n+2}}{\partial t} - d_M a_{n+1\,n+2} = \sum_i (a_{n+1\,i} \varphi_{in+2} + a_{in+2} \varphi_{in+1}).$$

If we write $\mu_{i\alpha} = \frac{\partial e_i}{\partial s} \cdot e_\alpha$, then this last equation restricted along the curve C becomes

$$\left.\frac{\partial \varphi_{n+1\,n+2}}{\partial t}\right|_C - \frac{\partial a_{n+1\,n+2}}{\partial s}\, ds = \sum_i (a_{n+1\,i} \mu_{in+2} + a_{in+2} \mu_{in+1})\, ds.$$

Integrating around the closed curve C, we obtain

$$\frac{\partial}{\partial t}\bigg|_A \int_C \varphi_{n+1\,n+2} = \sum_i \int_C (a_{n+1\,i}\mu_{in+2} + a_{in+2}\mu_{in+1})\,ds. \tag{2.2}$$

The left-hand side is the integral we want. We now compute the right-hand side.

The structure equations

$$d\omega_{n+1} = \sum_B \omega_B \wedge \omega_{Bn+1}$$

$$d\omega_{n+2} = \sum_B \omega_B \wedge \omega_{Bn+2}$$

yield after equating terms in dt,

$$\begin{aligned} d_M a_{n+1} &= \sum_B (a_{Bn+1}\varphi_B - a_B \varphi_{Bn+1}) \\ d_M a_{n+2} &= \sum_B (a_{Bn+2}\varphi_B - a_B \varphi_{Bn+2}). \end{aligned} \tag{2.3}$$

Along the curve C, we may write

$$d_M = \sum_i \frac{\partial}{\partial s_i} ds_i \quad \text{and} \quad \varphi_i = ds_i$$

with $\varphi_{n+1} = \varphi_{n+2} = 0$.

Equating terms in ds_i on both sides of Eq. (2.3) we obtain

$$\begin{aligned} \frac{\partial a_{n+1}}{\partial s_i} &= a_{in+1} - \sum_B a_B \mu^{(i)}_{Bn+1} \\ \frac{\partial a_{n+2}}{\partial s_i} &= a_{in+2} - \sum_B a_B \mu^{(i)}_{Bn+2} \end{aligned} \tag{2.4}$$

where $\mu^{(i)}_{AB} = \dfrac{\partial e_A}{\partial s_i} \cdot e_B$.

Putting Eqs. (2.4) into Eq. (2.2), we get

$$\begin{aligned} \frac{\partial}{\partial t}\bigg|_A \int_C \varphi_{n+1\,n+2} = \sum_i \int_C \Bigg\{ &\left(-\frac{\partial a_{n+1}}{\partial s_i} - \sum_B a_B \mu^{(i)}_{Bn+1} \right) \mu_{in+2} \\ &+ \left(\frac{\partial a_{n+2}}{\partial s_i} + \sum_B a_B \mu^{(i)}_{Bn+2} \right) \mu_{in+1} \Bigg\} ds. \end{aligned}$$

Now the variation vector $A = \sum_B a_B e_B$, so that

$$\frac{\partial A}{\partial s_i} = \sum_B \left(\frac{\partial a_B}{\partial s_i} e_B + \sum_C a_B \mu^{(i)}_{BC} e_C \right) \quad \text{and} \quad B_i = \mu_{in+1} e_{n+2} - \mu_{in+2} e_{n+1}.$$

Hence,

$$\frac{\partial A}{\partial s_i} \cdot B_i = \frac{\partial a_{n+2}}{\partial s_i} \mu_{in+1} - \frac{\partial a_{n+1}}{\partial s_i} \mu_{in+2} + \left(\sum_B a_B \mu^{(i)}_{Bn+2} \right) \mu_{in+1} - \left(\sum_B a_B \mu^{(i)}_{Bn+1} \right) \mu_{in+2}.$$

Hence,

$$\frac{\partial}{\partial t}\bigg|_A \int_C \varphi_{n+1\,n+2} = \sum_{i=1}^{n} \int_C \frac{\partial A}{\partial s_i} \cdot B_i \, ds.$$

3. Duality of Characteristic Forms

Useful background for this section is the article of Chern and Simons [5].

Given a connected Lie group G with Lie algebra $\mathfrak{g}$, there is a category whose objects are smooth principal G-bundles with connection, and whose morphisms are connection-preserving G-bundle maps. Thus an object in the category is a triple (P, M, φ), where P is a principal G-bundle over the smooth manifold M (with projection $\pi\colon P \to M$), and φ is a $\mathfrak{g}$-valued one form on P which satisfies the usual "connection conditions" [9, Vol. I, p. 64]. A morphism from (P, M, φ) to (P', M', φ') is a smooth G-bundle map

$$\begin{array}{ccc} P & \xrightarrow{\tilde{f}} & P' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ M & \xrightarrow[f]{} & M' \end{array}$$

such that $\tilde{f}^*(\varphi') = \varphi$.

Suppose that F is a symmetric invariant polynomial of degree k for G, i.e. $F \in I^k(G)$. (See [9, Vol. II, Chapter 12].) If (P, M, φ) is an object in our category with $\mathfrak{g}$-valued curvature form Φ, we have a "characteristic" differential $2k$-form $F(\Phi)$ on P. $F(\Phi)$ is exact on P, and in fact [5] describes a functorial procedure for constructing a $(2k-1)$-form $TF(\varphi)$ on P, called a "transgressed characteristic form," such that $d(TF(\varphi)) = F(\Phi)$. $F(\Phi)$ is the lift of a characteristic differential form on M, and for simplicity the differential form on M will also be denoted by $F(\Phi)$.

For a given object (P, M, φ), it may happen that $F(\Phi) = 0$. In this case, $TF(\varphi)$ represents a de Rham cohomology class on P, which may depend upon the connection φ.

Let $G(n, r, \mathbb{C})$ denote the Grassmann manifold of n-dimensional complex subspaces of $\mathbb{C}^{n+r}$; $G(n, r, \mathbb{C}) = U(n+r)/U(n) \times U(r)$. We wish to study the duality of characteristic forms and transgressed characteristic forms in the "universal" diagram of principal bundles over the Grassmann manifold:

$$\begin{array}{ccccc} & & U(n+r) & & \\ & \swarrow^{\tilde{\pi}} & & \searrow^{\tilde{\pi}^\perp} & \\ P = \dfrac{U(n+r)}{1 \times U(r)} & & & & P^\perp = \dfrac{U(n+r)}{U(n) \times 1} \\ & \searrow_{\pi} & & \swarrow_{\pi^\perp} & \\ & & G(n, r, \mathbb{C}) & & \end{array} \tag{3.1}$$

We recall some known facts regarding the cohomology of the spaces appearing in (3.1). (A standard reference for the integral cohomology is Borel [2], whereas Greub, Halperin, and Vanstone [8] give a detailed treatment of real cohomology.) $H^*(U(n+r);\mathbb{Z})$ is an exterior algebra on primitive generators $x_1, \ldots, x_{n+r}$, where $\dim x_i = 2i-1$; we write $H^*(U(n+r);\mathbb{Z}) = \bigwedge(x_1, \ldots, x_{n+r})$. $H^*(P;\mathbb{Z})$ is an exterior algebra $\bigwedge(y_{r+1}, \ldots, y_{n+r})$, where $\dim y_i = 2i-1$; moreover, the projection $\tilde{\pi}\colon U(n+r) \to P$ pulls y_i back to x_i. (Compare [2, Proposition 21.1].) Finally, $H^*(G(n,r,\mathbb{C});\mathbb{Z})$ is the quotient of a polynomial algebra on generators $c_1, \ldots, c_n$ (where $\dim c_i = 2i$) by certain relations. The relations are $c_k^\perp = 0$ for $k > r$, where $c_k^\perp$ is the polynomial in the c_i's defined inductively by the relations

$$(1 + c_1 + \cdots + c_n)(1 + c_1^\perp + \cdots + c_k^\perp + \cdots) = 1.$$

The real images of the cohomology classes described above can be represented by differential forms, and it is these differential form representatives which interest us. Since $U(n+r)$ is a compact Lie group, there is a unique biinvariant $(2i-1)$-form σ_i on $U(n+r)$ whose de Rham cohomology class is the image of x_i under the coefficient homomorphism $H^*(U(n+r);\mathbb{Z}) \to H^*(U(n+r);\mathbb{R})$. In fact, if $\tilde{\varphi}$ is the $\mathfrak{u}(n+r)$-valued Maurer-Cartan form of $U(n+r)$, it can be shown that $\sigma_i = (\text{constant})\ (\text{trace}\ (\tilde{\varphi}^{2i-1}))$. The $(\mathfrak{u}(n) \times 0)$-component of $\tilde{\varphi}$ is the lift of a connection φ on P, which we will call the canonical connection. Similarly, $\tilde{\varphi}$ determines a canonical connection $\varphi^\perp$ on $P^\perp$. Let Φ and $\Phi^\perp$ denote the curvatures of these connections.

In order to describe the canonical de Rham representatives for the cohomology of $G(n,r,\mathbb{C})$, we need to investigate the invariant polynomials $I^k(G)$, in the cases where $G = U(n)$ or $U(r)$. For simplicity, we consider $I^k(U) = \lim_{N\to\infty} I^k(U(N))$. For N large, define maps B_k, $C_k\colon \mathfrak{u}(N) \to \mathbb{R}$ by

$$\begin{aligned} B_k(A) &= \left(\frac{\sqrt{-1}}{2\pi}\right)^k \text{trace}\,(A^k), \\ \det\left(\lambda I + \frac{\sqrt{-1}}{2\pi} A\right) &= \sum_{k=0}^{N} C_k(A)\,\lambda^{n-k}. \end{aligned} \tag{3.2}$$

By polarization, we obtain invariant polynomials $B_k, C_k \in I^k(U(N))$, which restrict to elements of $I^k(U(n))$ for all $n \leqq N$. Any element of $I^k(U)$ is a polynomial in the C_k's ([9, Vol. II, p. 300]), and it can be shown that the B_k's are related to the C_k's by the Newton identities:

$$k\,C_k - C_{k-1}\,B_1 + \cdots + (-1)^k\,B_k = 0. \tag{3.3}$$

Using (3.3) and induction, one can express B_k as a polynomial in the C_i's. It will also be useful to consider the invariant polynomials $C_k^\perp$, defined inductively by the relation

$$(1 + C_1 + \cdots + C_k + \cdots)(1 + C_1^\perp + \cdots + C_k^\perp + \cdots) = 1. \tag{3.4}$$

Under the coefficient homomorphism $H^*(GL(n,r,\mathbb{C});\mathbb{Z}) \to H^*(GL(n,r,\mathbb{C});\mathbb{R})$, c_i goes to a de Rham class represented by either $C_i(\Phi)$ or $C_i^\perp(\Phi^\perp)$, whereas $c_i^\perp$

goes to a class represented by either $C_i(\Phi^\perp)$ or $C_i^\perp(\Phi)$. Since $G(n, r, \mathbb{C})$ is a compact symmetric space, each de Rham cohomology class on $G(n, r, \mathbb{C})$ possesses a unique $U(n+r)$-invariant representative. (See e.g. Wolf [15, § 8.5].) Hence

$$(3.5)\quad C_i(\Phi)=C_i^\perp(\Phi^\perp),\qquad C_i^\perp(\Phi)=C_i(\Phi^\perp).$$

More generally, any $F\in I^k(U)$ must be a polynomial in the C_i's (say $F=p(C_1,\ldots,C_k)$), and we let $F^\perp$ denote the corresponding polynomial in the $C_i^\perp$'s ($F^\perp=p(C_1^\perp,\ldots,C_k^\perp)$). Note that $(F^\perp)^\perp=F$. It follows immediately from (3.5) that

$$(3.6)\quad F(\Phi)=F^\perp(\Phi^\perp).$$

For example, by means of the Newton identities (3.3) it can be shown that $B_k^\perp=-B_k$, and hence it follows from (3.6) that $B_k(\Phi)+B_k(\Phi^\perp)=0$. (This can also be proven directly as in [6, § 3].) Note that if $F\in I^k(U)$,

$$(3.7)\quad F=m\,C_k+p'(C_1,\ldots,C_{k-1}),\qquad F^\perp=-m\,C_k+p''(C_1,\ldots,C_{k-1}),$$

where m is a real number, and p' and p'' are polynomials.

The Stiefel manifold P is not a symmetric space, and hence its cohomology cannot be identified with the algebra of $U(n+r)$-invariant forms. Nevertheless, the connection φ does pick out a de Rham representative for the real image of y_k when $r+1\leqq k\leqq n+r$, namely $-TC_k^\perp(\varphi)$; indeed, this is a corollary of the following theorem:

Theorem 3.1. *Suppose $F\in I^k(U)$. On $U(n+r)$, the differential form $TF(\varphi)-TF^\perp(\varphi^\perp)$ is closed and cohomologous to $m\sigma_k$, where m is the real number appearing in* (3.7). *In particular, $TB_k(\varphi)+TB_k(\varphi^\perp)$ is cohomologous to $(-1)^k k\sigma_k$.*

(To keep the notation from becoming cumbersome, we let $TF(\varphi)$ and $TF^\perp(\varphi^\perp)$ denote the lifts of $TF(\varphi)$ and $TF^\perp(\varphi^\perp)$ from P and $P^\perp$ to $U(n+r)$.)

The fact that $TF(\varphi)-TF^\perp(\varphi^\perp)$ is closed is an immediate consequence of (3.6).

To prove that $TF(\varphi)-TF^\perp(\varphi^\perp)$ is cohomologous to $m\sigma_k$, assume first that $F=C_k$ where $k\leqq n$, and consider the fibration

$$\begin{array}{ccc} U(n) & \longrightarrow & P \\ & & \downarrow \\ & & G(n,r,\mathbb{C}) \end{array}$$

As mentioned before, $H^*(U(n);\mathbb{Z})=\bigwedge(x_1,\ldots,x_n)$; it is well-known that x_i can be chosen to be the inverse transgression of $c_i\in H^{2i}(G(n,r,\mathbb{C});\mathbb{Z})$. (Compare Borel [2, p. 73].) Thus on the differential form level, $TC_k(\varphi)|$ fiber is cohomologous to σ_k, the biinvariant de Rham representative of x_k.

Now we make use of the commutative diagram

$$\begin{array}{ccc} U(n) & \subseteq & U(n+r) \\ & \searrow & \downarrow \\ & & P \end{array}$$

Note that the biinvariant form σ_k on $U(n+r)$ pulls back to σ_k on $U(n)$, i.e. $i^*(\sigma_k)=\sigma_k$, where $i\colon U(n)\to U(n+r)$ is the inclusion. The argument of the preceding paragraph shows that $i^*(TC_k(\varphi))=\sigma_k$. On the other hand, since the composition $U(n)\to U(n+r)\to P^\perp$ is constant, $i^*(TC_k^\perp(\varphi^\perp))=0$. Hence

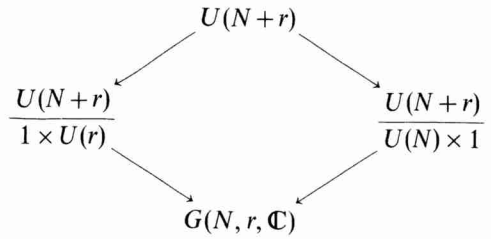
$$i^*(TC_k(\varphi)-TC_k^\perp(\varphi^\perp))=\sigma_k.$$

Since the inclusion $U(n)\to U(n+r)$ induces an isomorphism

$$H^{2k-1}(U(n+r);\mathbb{R})\cong H^{2k-1}(U(n);\mathbb{R}),\qquad TC_k(\varphi)-TC_k^\perp(\varphi^\perp)$$

must be cohomologous to σ_k on $U(n+r)$, so the theorem is proven in the case where $F=C_k$, $k\leqq n$.

Suppose next that $F=C_k$, where $k>n$. Choose $N\geqq k$. The inclusion $U(n+r)\to 1\times U(n+r)\subseteq U(N+r)$ induces a map from diagram (3.1) into the diagram



The preceding paragraphs show that $TC_k(\varphi)-TC_k^\perp(\varphi^\perp)$ is cohomologous to σ_k on $U(N+r)$; pull back to show that $TC_k(\varphi)-TC_k^\perp(\varphi^\perp)$ is cohomologous to σ_k on $U(n+r)$.

Finally, we treat the case where $F=C_i\,G$, where C_i and G have positive degrees. In this case $F^\perp=C_i^\perp\,G^\perp$. We need to show that $TF(\varphi)-TF^\perp(\varphi^\perp)$ is exact. According to Proposition 3.7 from [5],

$$TF(\varphi)=TC_i(\varphi)\wedge G(\Phi)+\text{exact},\qquad TF^\perp(\varphi^\perp)=TC_i^\perp(\varphi^\perp)\wedge G^\perp(\Phi^\perp)+\text{exact}.$$

It follows from (3.6) that $G(\Phi)=G^\perp(\Phi^\perp)$ and hence

$$\begin{aligned}TF(\varphi)-TF^\perp(\varphi^\perp)&=(TC_i(\varphi)-TC_i^\perp(\varphi^\perp))\wedge G(\Phi)+\text{exact}\\&=d[(TC_i(\varphi)-TC_i^\perp(\varphi^\perp))\wedge TG(\varphi)]+\text{exact}=\text{exact},\end{aligned}$$

and we are finished in this case.

Since any invariant polynomial is a linear combination of the polynomials we have considered, Theorem 3.1 is completely proven.

Corollary. *On the Stiefel manifold P, $-TC_k^\perp(\varphi)$ is a de Rham representative for the real image of $y_k\in H^{2k-1}(P;\mathbb{Z})$, when $r+1\leqq k\leqq n+r$.*

Indeed $TC_k(\varphi^\perp)=0$ for $r+1\leqq k\leqq n+r$, since $P^\perp$ is a $U(r)$-bundle. Thus $-TC_k^\perp(\varphi)$ is closed on $U(n+r)$, and is a de Rham representative for the real image of $x_k\in H^{2k-1}(U(n+r);\mathbb{Z})$. Thus the corollary follows from the commutative

diagram with exact rows:

$$\begin{array}{ccccc} 0 & \longrightarrow & H^*(P;\mathbb{Z}) & \longrightarrow & H^*(U(n+r);\mathbb{Z}) \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^*(P;\mathbb{R}) & \longrightarrow & H^*(U(n+r);\mathbb{R}) \end{array}$$

There are versions of Theorem 3.1 which hold when $U(n+r)$ is replaced by $GL(n+r,\mathbb{C})$, $GL(n+r),\mathbb{R})$, or $O(n+r)$. In the $GL(n+r,\mathbb{C})$ case, we replace diagram (3.1) by

$$\begin{array}{ccccc} & & GL(n+r,\mathbb{C}) & & \\ & \swarrow & & \searrow & \\ P=\dfrac{GL(n+r,\mathbb{C})}{1\times GL(r,\mathbb{C})} & & & & P^{\perp}=\dfrac{GL(n+r,\mathbb{C})}{GL(n,\mathbb{C})\times 1} \\ & \searrow & & \nearrow & \\ & & \hat{G}(n,r,\mathbb{C}) & & \end{array} \tag{3.8}$$

Here $\hat{G}(n,r,\mathbb{C})$ is the generalized Grassmann manifold of pairs $(V,V^{\perp})$, where V and $V^{\perp}$ are complementary complex subspaces of $\mathbb{C}^{n+r}$ of dimensions n and r respectively. Note that each space in diagram (3.8) is homotopy equivalent to the corresponding space in (3.1). As before, we give P and $P^{\perp}$ the canonical connections φ and $\varphi^{\perp}$ determined by the Maurer-Cartan form of $GL(n+r,\mathbb{C})$, and let Φ and $\Phi^{\perp}$ denote their curvatures.

Invariant polynomials for the complex general linear group: For N large, we define B_k, C_k: $\mathrm{gl}(N,\mathbb{C})\to\mathbb{C}$ by (3.2) and obtain *complex*-valued invariant polynomials B_k, C_k by polarization. Any complex-valued invariant polynomial is a polynomial with complex coefficients in the C_k's, or equivalently, a polynomial with complex coefficients in the B_k's. (Milnor and Stasheff [11, p. 299].) Since $\mathrm{gl}(N,\mathbb{C})$ is the complexification of $\mathrm{u}(N)$, it follows from (3.5) that

$$C_k(\Phi)=C_k^{\perp}(\Phi^{\perp}),$$

where $C_k^{\perp}$ is defined by (3.4). Let $I^k(GL(\mathbb{C}))$ denote the direct limit of the *complex*-valued invariant polynomials for $GL(N,\mathbb{C})$ as $N\to\infty$. If $F\in I^k(GL(\mathbb{C}))$, F is a polynomial over the complex numbers in the C_i's. Then as before

$$F(\Phi)=F^{\perp}(\Phi^{\perp}). \tag{3.9}$$

If $F\in I^k(GL(\mathbb{C}))$,

$$F=m\,C_k+p'(C_1,\ldots,C_{k-1})=a\,B_k+p''(B_1,\ldots,B_{k-1}), \tag{3.10}$$

where m and a are complex numbers, p' and p'' are complex polynomials. It follows from the Newton identities (3.3) that $m=(-1)^k k\,a$.

Theorem 3.2. *Suppose $F\in I^k(GL(\mathbb{C}))$. On $GL(n+r,\mathbb{C})$ the complex-valued differential form $TF(\varphi)-TF^{\perp}(\varphi^{\perp})$ is closed and cohomologous to $m\sigma_k$, where m is the complex*

number determined by (3.10), *and* σ_k *is a de Rham representative of the* $(2k-1)$-*dimensional primitive integral cohomology class* $x_k \in H^{2k-1}(GL(n+r, \mathbb{C}); \mathbb{Z})$.

Indeed $TF(\varphi) - TF^{\perp}(\varphi^{\perp})$ is closed because of (3.9), and since $U(n+r) \subseteq GL(n+r, \mathbb{C})$ is a homotopy equivalence, it suffices to show that the restriction of $TF(\varphi) - TF^{\perp}(\varphi^{\perp})$ to $U(n+r)$ is cohomologous to $m\sigma_k$; this follows from Theorem 3.1.

In the case of $GL(n+r, \mathbb{R})$, we consider the diagram of principal bundles

$$\begin{array}{ccccc} & & \hat{P} = GL(n+r, \mathbb{R}) & & \\ & \swarrow & & \searrow & \\ P = \dfrac{GL(n+r, \mathbb{R})}{1 \times GL(r, \mathbb{R})} & & & & P^{\perp} = \dfrac{GL(n+r, \mathbb{R})}{GL(n, \mathbb{R}) \times 1} \\ & \searrow & & \swarrow & \\ & & \hat{G}(n, r, \mathbb{R}) & & \end{array} \tag{3.11}$$

which maps into (3.8) in the obvious manner. $\hat{G}(n, r, \mathbb{R})$ is the generalized Grassmann manifold of complementary n- and r-dimensional subspaces of $\mathbb{R}^{n+r}$. We give P and $P^{\perp}$ the canonical connections φ and $\varphi^{\perp}$ with curvatures Φ and $\Phi^{\perp}$.

Let $I^k(GL(\mathbb{R})) = \lim_{N \to \infty} I^k(GL(N, \mathbb{R})) =$ the direct limit of the *real*-valued invariant polynomials for $GL(N, \mathbb{R})$ as $N \to \infty$. The complex-valued invariant polynomials $(\sqrt{-1})^k B_k$, $(\sqrt{-1})^k C_k$ for $GL(N, \mathbb{C})$ restrict to real-valued invariant polynomials Q_k, R_k for $GL(N, \mathbb{R})$. $(\sqrt{-1})^k B_k^{\perp}$ restricts to a real-valued polynomial $Q_k^{\perp}$, and in fact $Q_k^{\perp} = -Q_k$. $\sum_k I^k(GL(\mathbb{R}))$ is the polynomial ring generated by the Q_k's; hence if $F \in I^k(GL(\mathbb{R}))$, we can write

$$F = aQ_k + p(Q_1, \ldots, Q_{k-1}), \tag{3.12}$$

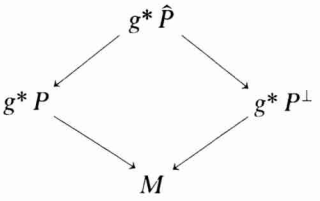
where a is a real number and p is a polynomial with real coefficients. Let $F^{\perp}$ denote the corresponding polynomial in the $Q_i^{\perp}$'s.

The inclusion map i: $GL(n+r, \mathbb{R}) \to GL(n+r, \mathbb{C})$ induces a homomorphism on cohomology which takes the primitive generator $x_{2i-1} \in H^{4i-3}(GL(n+r, \mathbb{C}); \mathbb{Z})$ to a torsion class, and $x_{2i} \in H^{4i-1}(GL(n+r), \mathbb{C})$ to an *even* integral class. (Compare the proof of Lemma 5.12 in [5].) Hence if $\tau_k = \frac{1}{2} i^*(\sigma_k)$, then τ_{2i-1} is an exact $(4i-3)$-form, and τ_{2i} is a closed $(4i-1)$-form on $GL(n+r, \mathbb{R})$ representing an integral cohomology class.

Theorem 3.3. *Suppose* $F \in I^k(GL(\mathbb{R}))$. *On* $GL(n+r, \mathbb{R})$, $TF(\varphi) - TF^{\perp}(\varphi^{\perp})$ *is a closed form, cohomologous to zero when* k *is odd, and cohomologous to* $(-1)^{(k/2)} ka(2\tau_k)$ *when* k *is even, where* a *is the real number determined by* (3.12).

This theorem follows from the preceding remarks and Theorem 3.2. The theorem also possesses a reformulation for the orthogonal group, which we leave to the reader. Theorems 3.2 and 3.3 can be regarded as an extension of Theorem 4 of Chern and White [6].

If we have a smooth map g: $M \to \hat{G}(n, r, \mathbb{R})$, we can pull diagram (3.11) back to a diagram of principal bundles over M:

(3.13)


For example, g might be the Gauss map of an immersion from an n-dimensional manifold M into $\mathbb{R}^{n+r}$. In this case, g^*P can be interpreted as the $GL(n, \mathbb{R})$-bundle of bases of the tangent spaces of M, $g^*P^\perp$ as the bundle of bases of the normal spaces. Theorem 3.3 pulls back to a theorem relating differential forms in diagram (3.13).

In particular, we can consider the i-th Pontryagin polynomials $P_i = R_{2i}$ and $P_i^\perp$. Note that

$$P_i^\perp = -\frac{1}{2i} Q_{2i} + p(Q_1, \ldots, Q_{2i-1});$$

hence according to Theorem 3.3, $\frac{1}{2}[TP_i^\perp(g^*\varphi) - TP_i(g^*\varphi^\perp)]$ is a closed form on $g^*\hat{P}$ lying in an integral cohomology class. (Compare Theorem 5.14 of [5].)

Corollary. *Suppose that* $\dim M = 2k-1$, *and* $s\colon M \to g^*\hat{P}$ *is a section. If* $F \in I^k(GL(\mathbb{R}))$,

$$\int_M s^*(TF(\varphi)) - \int_M s^*(TF^\perp(\varphi^\perp))$$

is zero if k *is odd, or an integral multiple of* $2ka$ *if* k *is even, where* a *is the real number determined by* (3.12).

This corollary can be regarded as a refinement of the duality theorem for Simons characters [3, §§ 4, 5] in the special case where $g^*\hat{P}$ possesses a section. The theory of Simons characters has been used successfully by Millson [10] to prove nonexistence of conformal immersions from certain constant positive curvature lens spaces into Euclidean space of low codimension.

Efficient application of the duality theorems for characteristic forms to conformal or isometric immersions of higher codimension seems to require an understanding of the second fundamental form structure, together with its implications for the connection and the curvature of the normal bundle. We will illustrate this with an application.

Suppose that M^3 is a compact orientable three-dimensional Riemannian manifold, B is the bundle of bases for the tangent spaces of M, and θ is the Levi-Civita connection on B. Thus (B, M, θ) is a principal $GL(3, \mathbb{R})$-bundle with connection. According to a theorem of Stiefel, M^3 is parallelizable, and hence there is a section $s\colon M^3 \to B$. Following Chern and Simons [5, § 6], we define $\Phi(M^3) \in \mathbb{R}/\mathbb{Z}$ by

$$\Phi(M^3) = \text{mod } \mathbb{Z} \text{ reduction of } \int_{M^3} s^*[\tfrac{1}{2} TP_1(\theta)],$$

where P_1 is the first Pontryagin polynomial. Chern and Simons show that $\Phi(M^3)$ is well-defined, and depends only on the conformal structure of M^3. This invariant can be calculated in many cases; for example, if P^3 is three-dimensional real projective space with the Riemannian metric of constant curvature one, $\Phi(P^3)=$ residue class of $\frac{1}{2}$ in $\mathbb{R}/\mathbb{Z}$.

If M^3 possesses an isometric immersion into E^5 $(=\mathbb{R}^5$ with the standard flat metric), and $g: M^3 \to \hat{G}(3, 2, \mathbb{R})$ is the corresponding Gauss map, then $B=g^*P$ and $\theta=g^*\varphi$. Our previous corollary yields:

Corollary. *Suppose that M^3 is a compact oriented three-dimensional Riemannian manifold, isometrically immersed in E^5 with trivial normal bundle. If e_4, e_5 are globally defined orthonormal sections of the normal bundle and $\varphi_{45}=de_4\cdot e_5$, then*

$$\Phi(M^3)+\left[\text{mod } \mathbb{Z} \text{ reduction of } \frac{1}{(8\pi^2)}\int_{M^3}\varphi_{45}\wedge d\varphi_{45}\right]=0.$$

The two-form $d\varphi_{45}$ is simply the curvature of the normal bundle. The hypothesis that M^3 have trivial normal bundle is always satisfied when M^3 has positive curvature, since in this case an easy argument shows that the mean curvature vector is nowhere zero.

The structure equations (equations of Gauss, Codazzi, Ricci) give strong conditions on φ_{45}. If M^3 has *constant* positive curvature, a study of the Gauss equation shows that the connection in the normal bundle has zero curvature [12, § 8]. Thus the last corollary implies that if M^3 is a three-dimensional spherical space form which immerses isometrically in E^5, then $\Phi(M^3)=0$. In particular, P^3 with constant curvature metric has no isometric immersion in E^5. (This can also be proven using Morse theory and Hodge's theory of harmonic forms [13].)

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