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A Differentiation Theorem for Additive Processes*

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1. Introduction

Let L_1 denote the usual Banach space of equivalence classes of real valued integrable functions on a σ-finite measure space $(X, \mathscr{F}, \mu)$ and let $L_1^+ = \{f \in L_1 \mid f \geqq 0\}$ be its positive cone. As it is customary, we shall not distinguish between the equivalence classes of functions and the individual functions.

By a *process* F we mean a family $F = \{F_t\} = \{F_t\}_{t>0}$, indexed with the positive real numbers $t \in (0, \infty)$, such that $F_t \in L_1$ for each $t > 0$. A process is called *positive* if $F_t \in L_1^+$ $(t > 0)$, *nondecreasing* if $F_t \leqq F_s$ $(0 < t \leqq s)$ and *strongly continuous* if $\|F_t - F_s\|_1 \to 0$ as $t \to s > 0$.

Let $T = \{T_t\} = \{T_t\}_{t>0}$ be a *strongly continuous semigroup of positive contractions* in L_1; this means that the T_t are linear operators in L_1 of norm not exceeding 1 such that $T_t L_1^+ \subset L_1^+$, $T_t T_s = T_{t+s}$ $(t, s > 0)$, and such that $Tf = \{T_t f\}$ is a strongly continuous process for each $f \in L_1$. A process $F = \{F_t\}$ is called an *additive (superadditive)* process (with respect to $\{T_t\}$) if $F_t + T_t F_s = (\leqq) F_{t+s}$ for all $t, s > 0$. Interesting examples of additive processes are processes of the form $F_t = \int_0^t T_s f \, ds$ $(f \in L_1)$ and processes of the form $F_t = (I - T_t) f$, where I is the identity operator.

We study the convergence a.e. of $t^{-1} F_t$ as $t \to 0+0$. As this is meaningless, when the F_t denote equivalence classes and t ranges through all positive reals, we either have to select suitable representatives or we let t range through a countable set only. We shall say that $q\text{-}\lim_{t \to 0} f_t$ exists a.e. if the limit exists a.e., when t approaches zero taking only strictly positive rational values. Under the assumptions of our results it will always be possible to select representatives of $t^{-1} F_t$ in such a way that for fixed $x \in X$ $(t^{-1} F_t)(x)$ has bounded variation in every finite interval. Therefore, the existence of $q\text{-}\lim_{t \to 0} (t^{-1} F_t)(x)$ will be equivalent to

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the existence of $\lim_{t\to 0}(t^{-1}F_t)(x)$ when $t>0$ approaches zero through all positive reals.

The principal difficulties appear already in Sect. 2, where positive additive processes are discussed. This is combined with a decomposition theorem in Sect. 3 to prove the convergence a.e. of $t^{-1}F_t$ for additive processes satisfying $\sup_{0<t\leq 1}\|t^{-1}F_t\|_1<\infty$. This contains the local ergodic theorems of Krengel-Ornstein [6], Akcoglu-Chacon [1], and Ornstein [8] as well as the Lebesgue-differentiation theorem for functions with bounded variation. The norm-convergence of $t^{-1}(I-T_t)f$ is usually studied in the analytical theory of semigroups in connection with infinitesimal generators and it seems of interest that a.e.-convergence holds under the above conditions. Another interesting application of the decomposition theorem is Theorem 3.3 which implies that a measurable function $f\colon \mathbb{R}\to\mathbb{R}$ is a.e. (with respect to Lebesgue-measure) equal to a function of bounded variation if and only if $\lim_{t\to 0} t^{-1}\int|f(x+t)-f(x)|\,dx$ is finite. Section 4 treats superadditive processes.

The proofs in Sect. 2 make use of ideas of Akcoglu-Chacon [1]. The function Ψ_E appearing there has been inspired by Brunel's lemma. In Sect. 4, ideas of Kingman [5], and of Akcoglu-Sucheston [2] have influenced the arguments.

A refinement of the arguments of this paper can be used to derive a differentiation theorem for additive processes and semigroups of positive linear operators in L_p $(1\leq p<\infty)$ generalising the local ergodic theorem of Kubokawa [7]. As this is considerably more complicated and the present methods are interesting in their own right, this will be discussed in a subsequent paper.

2. Positive Additive Processes

A function $\tilde{F}\colon (0,\infty)\times X\to\mathbb{R}$ is called a *representative* for the process $F=\{F_t\}$ if $\tilde{F}$ is measurable with respect to the product-σ-algebra coming from the Lebesgue-measurable sets in $(0,\infty)$ and $\mathscr{F}$ in X, and if for each fixed $t>0$ the function $\tilde{F}(t,\cdot)\colon X\to\mathbb{R}$ is a representative function for the equivalence class $F_t\in L_1$. It is well known (and easy to see) that every strongly continuous process has a representative. A representative for a positive and therefore nondecreasing process is called a regular representative if $0\leq\tilde{F}(t,x)\leq\tilde{F}(s,x)$ whenever $0<t\leq s$ and $x\in X$. It is easy to see that every strongly continuous positive process has a regular representative. Note that an additive process F with $\|F_s\|_1\to 0$ $(s\to 0)$ is necessarily strongly continuous.

The main result of this section is the following theorem.

(2.1) **Theorem.** *If $\tilde{F}$ is a regular representative for a positive additive process with $\|F_s\|_1\to 0$ $(s\to 0+0)$, then $\lim_{0<t\to 0}\frac{1}{t}\tilde{F}(t,x)$ exists for a.e. $x\in X$.*

This theorem is equivalent to the following one.

(2.2) **Theorem.** *If F is a positive additive process and $\|F_s\|_1\to 0$ $(s\to 0+0)$, then $q\text{-}\lim_{t\to 0}\frac{1}{t}F_t$ exists a.e.*

To see this equivalence it is enough to observe that if $\varphi(t)$ is a monotone function then the existence of $\lim_{0<t\to 0} \frac{1}{t}\varphi(t)$ is equivalent to the existence of $q\text{-}\lim_{t\to 0} \frac{1}{t}\varphi(t)$.

The proof of Theorem (2.2) will be obtained after several lemmas. In this section F will always be a positive additive process with $\|F_s\|_1 \to 0$ as $s\to 0+0$.

(2.3) *Definition of* $\Psi_E(F)$. For each $t>0$ let $\mathscr{P}_t(F)\subset L_1^+$ consist exactly of those $f\in L_1^+$ for which there exists an $r>0$ and an integer $n\geqq 0$ and $f_0, f_1, \ldots, f_n \in L_1^+$ such that $(n+1)r<t$, $f=f_0+f_1+\cdots+f_n$ and such that $\sum_{i=0}^{k} T_r^{k-i} f_i \leqq T_r^k \frac{1}{r} F_r$ for all $k=0,1,\ldots,n$. We then let $\Psi_E^t(F)=\sup\{\int_E f\,d\mu \mid f\in\mathscr{P}_t(F)\}$ and $\Psi_E(F)=\lim_{t\to 0+0} \Psi_E^t(F)$. Here $E\in\mathscr{F}$, and the existence of the last limit follows from the obvious fact that $0\leqq \Psi_E^t(F)\leqq \Psi_E^s(F)$ whenever $0<t\leqq s$. Note that if $\alpha\geqq 0$ then $\Psi_E(\alpha F)=\alpha\,\Psi_E(F)$, where $\alpha F=\{\alpha F_t\}$.

(2.4) *Definitions of* S_t *and* R_t^m. Let $f\in L_1^+$. Then $Tf=\{T_t f\}$ is a strongly continuous positive process and $S_t f=\int_0^t T_\alpha f\,d\alpha$ exists for each $t>0$, defined as the strong limit of $R_t^m f=\sum_{i=0}^{m-1} \frac{t}{m} T_{t/m}^i f$ as the integer $m\to\infty$. The process $Sf=\{S_t f\}$ is a positive additive process. If $\tilde{T}f$ is a positive representative for the process Tf then $(\tilde{S}f)(t,x)=\int_0^t (\tilde{T}f)(\alpha,x)\,d\alpha$ defines a regular representative for Sf.

(2.5) *Properties of* T *and* S. Let $h'\in L_1$ with $h'>0$ a.e. and let $h=S_1 h'$. Then $\{S_t h\}$ is a positive additive process and also $q\text{-}\lim_{t\to 0} \frac{1}{t} S_t h=h$ a.e. If C is the support of h then $S_t h>0$ on C for all $t>0$. If $D=X-C$ then $T_t f=0$ on D for each $t>0$ and for each $f\in L_1$. From this it follows that if F is an additive process then $F_t=0$ on D for each $t>0$. Hence the existence of $q\text{-}\lim_{t\to 0} \frac{1}{t} F_t$ a.e. on D is trivial. The proofs of these facts are not too difficult; see [1] and [6] for more details.

For each real number $t>0$ let $Q(t)$ be the set of rational numbers r such that $0<r<t$. If $g\in L_1$ then, for example, $\{g>0\}$ will denote the set on which (a representative of) g is strictly positive.

(2.6) **Lemma.** *If* $f\in\mathscr{P}_s(F)$ *then* $S_t f\leqq F_{t+s}$ *for each* $t>0$.

Proof. Let $f=f_0+\cdots+f_n$ with $\sum_{i=0}^{k} T_r^{k-i} f_i \leqq T_r^k \frac{1}{r} F_r$ for $k=0,1,\ldots,n$ and $(n+1)r<s$. First we note that

$$\int_0^{t+nr} T_\alpha \frac{1}{r} F_r\,d\alpha = \int_0^{t+nr} \frac{1}{r}(F_{\alpha+r}-F_\alpha)\,d\alpha$$

$$=\frac{1}{r}\int_r^{t+(n+1)r} F_\alpha\,d\alpha - \frac{1}{r}\int_0^{t+nr} F_\alpha\,d\alpha \leqq \frac{1}{r}\int_{t+nr}^{t+(n+1)r} F_\alpha\,d\alpha \leqq F_{t+(n+1)r} \leqq F_{t+s}.$$

Now we also have that

$$\begin{aligned}
&\int_0^{t+nr} T_\alpha \frac{1}{r} F_r \, d\alpha \\
&= \sum_{k=0}^{n-1} \int_{kr}^{(k+1)r} T_\alpha \frac{1}{r} F_r \, d\alpha + \int_{nr}^{t+nr} T_\alpha \frac{1}{r} F_r \, d\alpha \\
&= \sum_{k=0}^{n-1} \int_0^r T_\alpha \left(T_r^k \frac{1}{r} F_r \right) d\alpha + \int_0^t T_\alpha \left(T_r^n \frac{1}{r} F_r \right) d\alpha \\
&\geqq \sum_{k=0}^{n-1} \int_0^r T_\alpha \sum_{i=0}^{k} T_r^{k-i} f_i \, d\alpha + \int_0^t T_\alpha \sum_{i=0}^{n} T_r^{n-i} f_i \, d\alpha \\
&= \sum_{i=0}^{n} \int_0^{t+(n-i)r} T_\alpha f_i \, d\alpha \geqq \sum_{i=0}^{n} \int_0^t T_\alpha f_i \, d\alpha = (Sf)_t.
\end{aligned}$$

These two results together give the desired inequality.

(2.7) **Lemma.** *Let F be a positive additive process and let $g \in L_1^+$. If $\sup_{t \in Q(t_0)} (F_t - S_t g) > 0$ on $E \in \mathscr{F}$ then $\Psi_E^{t_0}(F) \geqq \int_E g \, d\mu$.*

Proof. Let $\nu = g \cdot \mu$ and let $\varepsilon > 0$ be given. We can find finitely many $t_i \in Q(t_0)$, $1 \leqq i \leqq n$, and an $\alpha > 0$ such that, if

$$E_i = E \cap \{F_{t_i} - S_{t_i} g > \alpha\}$$

then $\nu\left(E - \bigcup_{i=1}^{n} E_i\right) < \varepsilon$. Let $\delta > 0$ be a number such that $\nu(A) < \frac{\varepsilon}{n}$ whenever $\mu(A) < \delta$, $A \in \mathscr{F}$. Choose an integer M such that $\|R_{t_i}^m g - S_{t_i} g\|_1 < \alpha \delta$ for all $m \geqq M$ and for all $i = 1, \ldots, n$. Therefore, if

$$E_i'(m) = E_i \cap \{F_{t_i} - R_{t_i}^m g > 0\}$$

then $\nu(E_i - E_i'(m)) < \frac{\varepsilon}{n}$ for all $m \geqq M$.

We now find an $r > 0$ such that $m_i = t_i / r$ is an integer with $m_i \geqq M$ for each $i = 1, \ldots, n$. Then

$$\begin{aligned}
F_{t_i} - R_{t_i}^{m_i} g &= F_{m_i r} - R_{m_i r}^{m_i} g \\
&= \sum_{j=0}^{m_i - 1} T_r^j F_r - \sum_{j=0}^{m_i - 1} r T_r^j g \\
&= r \sum_{j=0}^{m_i - 1} T_r^j \left(\frac{1}{r} F_r - g \right) > 0
\end{aligned}$$

on $E_i'(m_i)$. Hence, letting $E' = \bigcup_{i=1}^{n} E_i'(m_i)$ and $K = \max(m_1, \ldots, m_n)$, we have that $K r < t_0$, $\nu(E - E') < 2\varepsilon$ and that

$$\sup_{0 < k \leqq K} \sum_{j=0}^{k-1} T_r^j \left(\frac{1}{r} F_r - g \right) > 0 \quad \text{on } E'.$$

We now apply the Chacon-Ornstein filling scheme [4, Lemma 1] $\left(\text{with } \frac{1}{r} F_r\right.$ instead of f^+ and g instead of $\left.f^-\right)$ to obtain functions $d_0, \ldots, d_{K-1}$ such that

$$\sum_{j=0}^{k} T_r^{k-j} d_j \leqq T_r^k \frac{1}{r} F_r \quad \text{for all } k=0, \ldots, K-1,$$

and such that $d := d_0 + \cdots + d_{K-1} = g$ on E'. Since $d \in \mathscr{P}_{t_0}(F)$ this shows that

$$\Psi_E^{t_0}(F) \geqq \int_E d\, d\mu \geqq \int_{E'} d\, d\mu = \int_{E'} g\, d\mu \geqq \int_E g\, d\mu - 2\varepsilon$$

and completes the proof.

(2.8) **Lemma.** *Let F and G be two positive additive processes and assume that $\sup_{t \in Q(t_0)} (F_t - G_t) > 0$ on a set E with $\mu(E) < \infty$. Then for each $\varepsilon > 0$ there exists a set $E' \subset E$ and a number $\delta > 0$ such that $\mu(E - E') < \varepsilon$ and such that*

$$\sup_{t \in Q(t_0)} (F_t - S_t g) > 0 \quad \text{on } E' \quad \text{for all } g \in \mathscr{P}_\delta(G).$$

Proof. We find finitely many $t_i \in Q(t_0)$, $1 \leqq i \leqq n$, and an $\alpha > 0$ such that if

$$E_i = E \cap \{F_{t_i} - G_{t_i} > \alpha\}$$

then $\mu\left(E - \bigcup_{i=1}^{n} E_i\right) < \frac{\varepsilon}{2}$.

Since G is strongly continuous and G_{t_i+s} decreases to G_{t_i} when $s \in Q(t_0)$ decreases to zero there exists a $\delta > 0$ such that the sets $B_i = \{G_{t_i+\delta} - G_{t_i} > \alpha\}$ have measure $\mu(B_i) < \frac{\varepsilon}{2n}$ $(i = 1, \ldots, n)$. Now, if $g \in \mathscr{P}_\delta(G)$, then, by Lemma 2.6

$$F_{t_i} - S_{t_i} g \geqq F_{t_i} - G_{t_i+\delta} = (F_{t_i} - G_{t_i}) - (G_{t_i+\delta} - G_{t_i}) > 0$$

on $E_i \setminus B_i$. Therefore $E' = \bigcup_{i=1}^{n} (E_i \setminus B_i)$ has the desired property.

(2.9) **Lemma.** *Let F and G be two positive additive processes and assume that $\sup_{t \in Q(t_0)} (F_t - G_t) > 0$ on E for all $t_0 > 0$. Assume that $\mu(E) < \infty$. Then, for each $\varepsilon > 0$ there exists a set $E' \subset E$ such that $\mu(E - E') < \varepsilon$ and such that $\Psi_{E''}(F) \geqq \Psi_{E''}(G)$ whenever $E'' \in \mathscr{F}$ and $E'' \subset E'$.*

Proof. We choose $\varepsilon_i > 0$ with $\sum_{i=1}^{\infty} \varepsilon_i < \varepsilon$ and also $t_i > 0$ with $t_i \downarrow 0$. The previous Lemma shows that for each i we can find a set $E_i \subset E$ and a number $\delta_i > 0$ such that $\mu(E - E_i) < \varepsilon_i$ and such that $\sup_{t \in Q(t_i)} (F_t - S_t g) > 0$ on E_i for all $g \in \mathscr{P}_{\delta_i}(G)$. Let $E' = \bigcap_{i=1}^{\infty} E_i$. Then $\mu(E - E') < \varepsilon$.

Let $E'' \subset E'$ be a fixed set. Given an $\eta > 0$ we find an i such that $\Psi_{E''}^{t_i}(F) \leqq \Psi_{E''}(F) + \eta$. Hence, if $g \in \mathscr{P}_{\delta_i}(G)$ then, by Lemma (2.8), $\Psi_{E''}^{t_i}(F) \geqq \int_{E''} g\, d\mu$ and consequently $\Psi_{E''}(F) + \eta \geqq \Psi_{E''}^{\delta_i}(G)$.

(2.10) **Lemma.** *If* $h=S_1 h'$ *us defined is in* (2.5), *then* $\Psi_E(Sh) \geqq \int_E h\,d\mu$.

Proof. For any $0<s<t<1$ one has $h_s := \int_s^1 T_\alpha h'\,d\alpha \in \mathscr{P}_t(Sh)$ because of $h_s \leqq t^{-1} \int_0^t T_\alpha h\,d\alpha$. Hence $\Psi_E^t(Sh) \geqq \lim_{s\to 0} \int_E h_s\,d\mu = \int_E h\,d\mu$ for all $t>0$.

(2.11) **Lemma.** *If* F *is a positive additive process then, with the notations of* (2.5), $q\text{-}\lim_{t\to 0} \frac{F_t}{S_t h}$ *exists* a.e. *on* C.

Proof. If this limit does not exist a.e. on C then there is a set $E \subset C$ with $0<\mu(E)<\infty$ and two numbers α, β with $0 \leqq \alpha < \beta$ such that

$$q\text{-}\liminf_{t\to 0} \frac{F_t}{S_t h} < \alpha < \beta < q\text{-}\limsup_{t\to 0} \frac{F_t}{S_t h} \quad \text{on } E.$$

Therefore, $\sup_{t\in Q(t_0)} (F_t - \beta S_t h) > 0$ on E and $\sup_{t\in Q(t_0)} (\alpha S_t h - F_t) > 0$ on E for each $t_0 > 0$. Then, using Lemma (2.9), we can find a set $E'' \subset E$ such that $0<\mu(E'')$ and $\Psi_{E''}(F) \geqq \Psi_{E''}(\beta Sh) = \beta \Psi_{E''}(Sh)$ and $\alpha \Psi_{E''}(Sh) = \Psi_{E''}(\alpha Sh) \geqq \Psi_{E''}(F)$. This implies that $\alpha \Psi_{E''}(Sh) \geqq \beta \Psi_{E''}(Sh)$, which is a contradiction, since $\alpha<\beta$ and $0<\int_E h\,d\mu \leqq \Psi_{E''}(Sh) \leqq \int_X h\,d\mu < \infty$. The proof of Theorem (2.2) now follows directly from Lemma (2.11) and from the observations in (2.5).

3. Decomposition of an Additive Process

Let $\{T_t\}_{t>0}$ be a strongly continuous semigroup of positive linear operators in L_1 with $\sup_{0<t\leqq 1} \|T_t\|_1 < \infty$. We now consider general additive processes $F=\{F_t\}_{t>0}$ satisfying

(3.1) $\sup_{0<t\leqq 1} \|t^{-1} F_t\|_1 < \infty.$

It is easy to see that the additivity and (3.1) together imply the strong continuity of F.

Note that (3.1) is a consequence of the additivity if F is positive and $\{T_t\}_{t>0}$ is Markovian, i.e. $\|T_t f\|_1 = \|f\|_1$ for all $f \in L_1^+$.

(3.2) **Theorem.** *If* $\{T_t\}$ *and* F *are as above there exist two positive additive processes* $\{F_t^{(i)}\}_{t>0}$ *with the properties* (3.1) *such that* $F_t = F_t^{(1)} - F_t^{(2)}$.

Proof. For $t=k\cdot 2^{-n}$ put $F_t^{(n,1)} = \sum_{j=0}^{k} T_{2^{-n}}^j (F_{2^{-n}})^+$ and $F_t^{(n,2)} = \sum_{j=0}^{k} T_{2^{-n}}^j (F_{2^{-n}})^-$. From $(F_{2^{-n}})^+ = (F_{2^{-(n+1)}} + T_{2^{-(n+1)}} F_{2^{-(n+1)}})^+ \leqq F_{2^{-(n+1)}}^+ + T_{2^{-(n+1)}} F_{2^{-(n+1)}}^+$ it follows that $F_t^{(n,1)} \leqq F_t^{(n+1,1)}$. For dyadic rationals t $F_t^{(1)} = \lim_{m\to\infty} F_t^{(m,1)}$ exists in L_1 and a.e. and $F_t^{(2)}$ is defined in the same way. The additivity and the property (3.1) can be verified in a straightforward way for dyadic rationals t, s. By the strong continuity we can define $F_t^{(t)} = \lim_{s \to t} F_s^{(i)}$ for all $t > 0$ where s ranges through the dyadic rationals, and then the desired properties follow in general by continuity.

If for some $f \in L_1$ $F_t = S_t f$, (3.1) is an easy consequence of the strong continuity of $\{T_t\}_{t>0}$. If in this case $\{T_t\}_{t \geqq 0}$ is even strongly continuous at 0 and T_0 is the identity one easily shows that $F_t^{(1)} = \int_0^t T_\alpha f^+ \, d\alpha$. In general only $\leqq$ would be true.

It is much less obvious that the above decomposition essentially also gives the decomposition of a function of bounded variation into a difference of two monotone functions – except that above we have dealt with equivalence classes of functions. Actually, condition (3.1) in this case leads to a simple necessary and sufficient condition for a measurable function $f: \mathbb{R} \to \mathbb{R}$ to be a.e. equal to a function of bounded variation. This application of the above decomposition theorem seems of independent interest.

For a measurable $f: \mathbb{R} \to \mathbb{R}$ we define F_t on $\mathbb{R}$ (with Lebesgue-measure λ) by $F_t(x) = f(x+t) - f(x)$. This is additive when T_t denotes the translation operator $(T_t f)(x) = f(x+t)$. T_t is a contraction in L_1 and for contractions the supremum in (3.1) is a limit (as $t \to 0+0$) because the additivity implies $\|F_{t+s}\|_1 \leqq \|F_t\|_1 + \|F_s\|_1$. We define the essential total variation $\|f\|_{\text{ess.t.v.}}$ of f by

$$\|f\|_{\text{ess.t.v.}} = \lim_{t \to 0+0} \frac{1}{t} \int |f(x+t) - f(x)| \, dx.$$

The total variation of f is denoted by $\|f\|_{\text{t.v.}}$. Call f_1 *equivalent* to f_2 if $f_2 - f_1$ is constant and *λ-equivalent* if $f_2 - f_1$ is constant mod λ. $\|\cdot\|_{\text{t.v.}}$ is a norm in a space of equivalence-classes and $\|\cdot\|_{\text{ess.t.v.}}$ a norm in a space of λ-equivalence classes.

(3.3) **Theorem.** *If $f: \mathbb{R} \to \mathbb{R}$ is measurable and $f' = f$ λ-a.e., then $\|f\|_{\text{ess.t.v.}} \leqq \|f'\|_{\text{t.v.}}$. There exists an $f' = f$ λ-a.e. with $\|f\|_{\text{ess.t.v.}} = \|f'\|_{\text{t.v.}}$.*

Proof. As the essential total variation is not changed, when f is changed on a λ-nullset we may assume $f = f'$ and $\|f\|_{\text{t.v.}} < \infty$ for the proof of the first statement. As f has a representation $f = u - v$ as the difference of two increasing functions u, v with $\|f\|_{\text{t.v.}} = \|u\|_{\text{t.v.}} + \|v\|_{\text{t.v.}}$ it suffices to prove $\|u\|_{\text{ess.t.v.}} \leqq \|u\|_{\text{t.v.}}$. But for increasing functions the essential total variation coincides with the total variation because

$$\begin{aligned} \frac{1}{t} \int |u(x+t) - u(x)| \, dx &= \lim_{\substack{a \to -\infty \\ b \to +\infty}} \frac{1}{t} \int_a^b (u(x+t) - u(x)) \, dx \\ &= \lim_{\substack{a \to -\infty \\ b \to +\infty}} \left\{ \frac{1}{t} \int_b^{b+t} u(x) \, dx - \frac{1}{t} \int_a^{a+t} u(x) \, dx \right\} \\ &= \sup u(x) - \inf u(x) = \|u\|_{\text{t.v.}} \end{aligned}$$

To prove the second statement we use the decomposition $F_t = F_t^{(1)} - F_t^{(2)}$ described in Theorem 3.1. It is easy to check that

$$\|f\|_{\text{ess.t.v.}} = \lim_{t\to 0} \|t^{-1} F_t^{(1)}\|_1 + \lim_{t\to 0} \|t^{-1} F_t^{(2)}\|_1$$

The essential step of the proof is the following lemma:

(3.4) **Lemma.** *If $F=\{F_t\}$ is a positive additive process on $(\mathbb{R}, \lambda)$ satisfying (3.1) for the translations T_t there exists an increasing $u\colon \mathbb{R}\to\mathbb{R}^+$ such that for all $t>0$ the equation $F_t(x)=u(t+x)-u(x)$ holds λ-a.e.; u is determined uniquely* mod λ *up to an additive constant.*

Proof. Note that this lemma would be very simple if we dealt with actual functions F_t instead of equivalence classes and $F_{t+s}=F_t+T_tF_s$ would hold everywhere. Then $u(t)=F_t(0)$ would do.

For any $j\geqq 0$ we define

$$u^{(j)}(x) = \sum_{k=1}^{\infty} F_{2^{-j}}(x-ks^{-j})$$

The sums converge a.e. since for any interval J of the form

$$J=[r2^{-j}, (r+1)2^{-j}[\ (r\in\mathbb{Z}) \ \|u^{(j)}\cdot 1_J\|_1 \leqq \|F_{2^{-j}}\|_1 < \infty.$$

The equation

$$F_{i2^{-j}}(x) = u^{(j)}(x+i2^{-j}) - u^{(j)}(x) \qquad \text{(a.e.)}$$

follows for $i=1$ from the definition of $u^{(j)}$ and for general i by induction using the additivity. Again, using the additivity, we see that

$$F_{2^{-j}}(x-k2^{-j}) = F_{2^{-(j+1)}}(x-2k\,2^{-(j+1)}) + F_{2^{-(j+1)}}(x-(2k+1)\,2^{-(j+1)})$$

which implies that all $u^{(j)}$ are equal to $u^{(1)}$ λ-a.e. It follows that

(3.5) $\quad F_t(x) = u^{(1)}(x+t) - u^{(1)}(x) \qquad \lambda\text{-a.e.}$

holds for dyadic rationals t. As F_t converges to F_s $(t\to s)$ and $u^{(1)}$ restricted to any finite interval is integrable (so that the restriction of $T_t u^{(1)}$ converges to that of $T_s u^{(1)}$), (3.5) follows for all $t>0$. Because of (3.5) and $F_t\geqq 0$ the sequences

$$2^n \int_0^{2^{-n}} u^{(1)}(x+\alpha)\, d\alpha$$

are decreasing and converge for all x to a function $u(x)$ which is increasing. It coincides a.e. with $u^{(1)}$. This proves the desired representation.

Now assume that u' is another increasing function with $F_t(x)=u'(x+t)-u'(x)$ λ-a.e. Then $w=u-u'$ has the property $T_t w = w$ λ-a.e. for all t. It follows easily from the Lebesgue-density theorem that w must be a.e. constant.

Now we apply Lemma 3.4 to $(F_t^{(1)}, t>0)$ and $(F_t^{(2)}, t>0)$ and find increasing functions u_1, u_2 with $F_t^{(i)}(x) = u_i(x+t) - u_i(x)$ λ-a.e. Clearly $\|u_i\|_{\text{ess.t.v.}}$

$=\lim_{t\to 0}\|t^{-1}F_t^{(i)}\|_1$. It follows that $f'=u_1-u_2$ is a function of bounded variation with $\|f'\|_{\text{t.v.}}\leqq\|f\|_{\text{ess.t.v.}}$. For all $t>0$ $T_t f'-f'=F_t=T_t f-f$ λ-a.e. It follows as above that $f'=f$ λ-a.e. up to an additive constant and we may assume that this constant is zero by changing u_1.

The above proof also shows that the decomposition in Theorem 3.2 corresponds exactly to the decomposition of f' into a difference u_1-u_2 of two increasing functions with $\|u_1\|_{\text{t.v.}}+\|u_2\|_{\text{t.v.}}=\|f'\|_{\text{t.v.}}$. In particular $\lim_{t\to 0}\frac{1}{t}\int(f(x+t)-f(x))^+dx$ is the "essential positive variation". It may be of interest to note the following corollary:

(3.6) **Corollary.** *A measurable f is λ-a.e. equal to a constant if and only if* $\lim_{t\to 0}\frac{1}{t}\int|f(x+t)-f(x)|\,dx=0$; *it is λ-a.e. equal to an increasing function if and only if*

$$\lim_{t\to 0}\frac{1}{t}\int(f(x+t)-f(x))^-\,dx=0.$$

The principal application of Theorem 3.2 is the following differentiation theorem for non-positive additive processes.

(3.7) **Theorem.** *Let $\{T_t\}_{t>0}$ be a strongly continuous semigroup of positive contractions in L_1 and F an additive process satisfying* (3.1), *then* q-$\lim_{t\to 0}t^{-1}F_t$ *exists for a.e. $x\in X$.*

Again, an equivalent version of the result can be stated using representatives: By Theorem (3.2) and the existence of regular representatives for positive additive strongly continuous processes an additive process F satisfying (3.1) always admits a representative $\tilde{F}:(0,\infty)\times X\to\mathbb{R}$ such that $\tilde{F}(\cdot,x)$ has bounded variation in every finite interval for all x. For such a representative $\lim_{t\to 0+0}t^{-1}\tilde{F}(t,x)$ exists a.e.

Theorem 3.7 contains the Lebesgue differentiation theorem and the local ergodic theorems of Krengel-Ornstein, Akcoglu-Chacon, and Ornstein. We remark that Ornstein [8] has stated his result (Theorem 3, p. 109) for processes of the form $F_t=\int_0^t f_\alpha\,d\alpha$ with $(T_t f_s=f_{t+s})$ assuming only $\|f_t\|_1\leqq 1$, but the proof seems to use also $f_t\geqq 0$.

(3.8) *Identification of the limit in the local ergodic theorem:* It may be of interest to add a few words on the identification of the limit of $t^{-1}S_t f$. In the "initially dissipative part" D the limit is zero. Now assume that X coincides with the "initially conservative part" C. If h is defined as in (2.5) we have $\lim_{t\to 0}t^{-1}S_t h=h$. Akcoglu and Chacon [1] have proved that $T_0 f:=\lim_{t\to 0}t^{-1}S_t f$ defines a positive contraction such that $\{T_t\}_{t\geqq 0}$ is now defined also for $t=0$ and also strongly continuous at $t=0$. If we use $\nu=h\cdot\mu$ as the reference measure instead of μ the semigroup $\{T_t\}_{t\geqq 0}$ corresponds to a new semigroup $\{T_t'\}$ in $L_1(\nu)$ defined by

$$T_t'g=h^{-1}T_t(h\cdot g)\qquad g\in L_1(\nu).$$

This new semigroup has the property $T_0' 1 = 1$. By a theorem of Ando [3] a positive contraction in $L_1(\nu)$ with $T_0' 1 = 1$ and $(T_0')^2 = T_0'$ is a conditional expectation operator. Call $A \in \mathscr{F}$ initially invariant if $T_0' 1_A = 1_A$. The initially invariant sets form a σ-algebra $\mathscr{J} \subset \mathscr{F}$ and we have $T_0' g = E_\nu(g|\mathscr{J})$. It is not hard to show that A is initially invariant if and only if there is some $f \in L_1^+(\mu)$ such that $\{f>0\} = A$ and $\int_A T_t f \, d\mu \to \|f\|_1$. This characterization has the advantage of being independent of the new reference measure ν. We may say that using a suitable reference measure (invariant under T_0) the identification of the limit in the local ergodic theorem is given by a conditional expectation—just like in Birkhoff's theorem.

There does not seem to be any canonical description of the influence of the initially dissipative part. In fact, if we assume $h = 1_C$ for simplicity, the map $f \to \lim_{t\to 0} t^{-1} S_t f$ from $L_1(D) = \{f \in L_1 | f = 0 \text{ in } C\}$ to $L_1(C)$ can be any positive contraction U_0 with $U_0 L_1(D) \subset L_1(C, \mathscr{J}, \nu)$.

4. Superadditive Processes

Recall that a process F is called superadditive (with respect to $T = \{T_t\}$) if $F_t + T_t F_s \leqq F_{t+s}$ for all $t, s > 0$. Note that F is not assumed to be strongly continuous. We shall, however, assume that there exists a $t_0 > 0$ such that $\sup_{0<t\leqq t_0} \frac{1}{t} \|F_t\|_1 = K < \infty$, in order to prove a differentiation theorem for F. (If F is additive, this is equivalent to (3.1).) Also, the semi-group $T = \{T_t\}$ will be assumed to be Markovian; i.e. that $\int T_t f \, d\mu = \int f \, d\mu$ for each $t > 0$ and $f \in L_1$, in addition to the previous strong continuity and positivity conditions. Under these assumptions, note that $\int F_t \, d\mu + \int F_s \, d\mu \leqq \int F_{t+s} \, d\mu$, which shows that $\alpha = \lim_{0<t\to 0} \frac{1}{t} \int F_t \, d\mu$ exists and is finite. From now on $F = \{F_t\}$, K, t_0, and α will be as defined above and we shall show that $q\text{-}\lim_{t\to 0} \frac{1}{t} F_t$ exists a.e. First we observe that we may assume that $0 \leqq F$ and $\alpha = 0$:

(4.1) **Lemma.** *There is an additive process G such that $G \leqq F$, $\sup_{t>0} \frac{1}{t} \|G_t\|_1 \leqq K$ and such that $\frac{1}{t} \int G_t \, d\mu = \alpha$ for each $t > 0$.*

Proof. Let B be the set of strictly positive binary rational numbers. If $t \in B$ with $t = k 2^{-n}$ we let, as in the proof of Theorem (3.2), $G_t^n = \sum_{j=0}^{k-1} T_{2^{-n}}^j F_{2^{-n}}$. Then $G_t^n \geqq G_t^{n+1}$, since $F_{2^{-n}} \geqq F_{2^{-(n+1)}} + T_{2^{-(n+1)}} F_{2^{-(n+1)}}$, and also $\int G_t^n \, d\mu = k \int F_{2^{-n}} \, d\mu = t 2^n \int F_{2^{-n}} \, d\mu \geqq -K$ if $2^{-n} \leqq t_0$. This shows that G_t^n converges in L_1 to a function G_t, as $n \to \infty$, and also that $\int G_t \, d\mu = t\alpha$, for each $t \in B$. If $t, s \in B$ and if n is sufficiently large we see that $\|G_t^n - G_s^n\|_1 \leqq K|t-s|$ and that $G_t^n + T_t G_s^n = G_{t+s}^n$. Hence the same is also true for G. Therefore it is clear that G_t can be defined for all real numbers $t > 0$, and that G satisfies the required conditions.

(4.2) **Lemma.** *Let F be a positive superadditive process with respect to a Markovian semi-group T and assume that* $\lim_{0<t\to 0} \frac{1}{t}\int F_t\,d\mu=0$. *Then* $q\text{-}\lim_{t\to 0}\frac{1}{t}F_t=0$ *a.e.*

Proof. Given $\varepsilon>0$ we fix a t_0 such that $\int F_{t_0}\,d\mu\leqq\varepsilon t_0$ and define a process $H=\{H_t\}$ as $H_t=(I-T_t)\frac{1}{t_0}\int_0^{t_0} F_s\,ds+\frac{1}{t_0}\int_0^t T_s F_{t_0}\,ds$ for each $t>0$. I again denotes the identity operator. Note that there is no difficulty with the definition of $\int_0^{t_0} F_s\,ds$ as the strong limit of Riemann sums, since F is a non-decreasing positive process. Now it is clear that H is an additive process and also that $\int H_t\,d\mu=\frac{1}{t_0}\int d\mu\int_0^t T_s F_{t_0}\,ds\leqq\varepsilon t$. We now show that $H_t\geqq\left(1-\frac{t}{t_0}\right)F_t$ for each t, $0<t<t_0$. In fact,

$$\begin{aligned} t_0 H_t &= \int_0^{t_0} F_s\,ds-\int_0^{t_0} T_t F_s\,ds+\int_0^t T_s F_{t_0}\,ds\\ &=\int_0^t F_s\,ds+\int_t^{t_0}(F_s-T_tF_{s-t})\,ds+\int_0^t (T_sF_{t_0}-T_tF_{t_0-t+s})\,ds\\ &\geqq \int_0^t F_s\,ds+\int_t^{t_0}F_t\,ds+\int_0^t T_sF_{t-s}\,ds\geqq(t_0-t)F_t. \end{aligned}$$

Hence $H_t\geqq 0$ if $0<t<t_0$, and consequently $H_t\geqq 0$ for all $t>0$, because of the additivity. Therefore $q\text{-}\lim_{t\to 0}\frac{1}{t}H_t=h$ a.e. and $\int h\,d\mu\leqq\varepsilon$. This shows that $q\text{-}\limsup_{t\to 0}\frac{1}{t}F_t\leqq h$ has an arbitrarily small integral and therefore $q\text{-}\lim_{t\to 0}\frac{1}{t}F_t=0$ a.e.

Combining these two lemmas we have

(4.3) **Theorem.** *If $\{F_t\}$ is a superadditive process with respect to a Markovian semi-group $\{T_t\}$ and if there is a $t_0>0$ such that* $\sup_{0<t\leqq t_0}\frac{1}{t}\|F_t\|_1<\infty$ *then* $q\text{-}\lim_{t\to 0}\frac{1}{t}F_t$ *exists a.e.*

Again there is an equivalent formulation of Theorem 4.3: $F=(F_t)$ is a sum of a process G_t as treated in section 3 and a process H_t which is positive and superadditive and therefore increasing. The set J_D of discontinuities of $\|H_t\|_1$ in $[0,t_0]$ is at most countable. Let $Q'(t_0)=Q(t_0)\cup J_D$. After an elimination of a nullset H_t is increasing for $t\in Q'(t_0)$. For $s<t_0$ with $s\in Q'(t_0)$ put $\tilde{H}(s,x)=H_t(x)$, for $s<t_0$ with $s\notin Q'(t_0)$ put

$$\tilde{H}(s,x)=\lim_{\substack{t\to s\\ s<t\in Q'(t_0)}} H_t(x).$$

Then $\tilde{H}$ is a representative of H_t in $(0,t_0)$ having variation $\leqq H_{t_0}(x)$. It follows that F_t admits a representative $\tilde{F}$ with bounded variation in $[0,t_0]$ and for such a representation the limit $\lim_{t\to 0} t^{-1}\tilde{F}(t,x)$ exists a.e.

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