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Normal Fitting Pairs and Lockett's Conjecture

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This paper is centered around a rather simple technique (Definition (2.3)) for extending certain **X**-normal Fitting pairs from a Fitting class **X** to the class **F** of all finite groups. This technique allows one to consider Lockett's conjecture in various forms. We apply the technique to give simple proofs of two such theorems.

In answer to a question of Laue [8], we prove that $\mathbf{S}\cap\mathbf{F}_* = \mathbf{S}_*$ where $\mathbf{S}\subseteq\mathbf{F}$ is the class of finite solvable groups.

We also give a simple proof to a result of Bryce and Cossey [3], namely, that if $\mathbf{X}\subseteq\mathbf{S}$ is a primitive saturated formation then Lockett's conjecture holds for **X**.

I. Preliminary Definitions and Results

Let **F** be the class of finite groups and **S** the class of finite solvable groups. *All classes considered will be assumed in* **F**. A *Fitting class* **X** is a nonempty class of groups closed under forming normal subgroups and normal products. The Fitting class of groups of order 1 is called *trivial;* all others are *nontrivial.* Let **X** be a Fitting class. In a group $G\in\mathbf{F}$, the join $G_{\mathbf{X}}$ of all subnormal **X**-subgroups of G is a characteristic **X**-subgroup of G called the **X**-*radical of* G. If **Y** is also a Fitting class then we define a product

$$\mathbf{XY} = \{G\in\mathbf{F} \mid G/G_{\mathbf{X}}\in\mathbf{Y}\}.$$

Then **XY** is a Fitting class, and the product is associative.

We turn now to a discussion of the Lockett $*$-operation. For a Fitting class **X**, define

$$\mathbf{X}^* = \{G\in\mathbf{F} \mid G \text{ is subdirect in } (G\times G)_{\mathbf{X}}\}.$$

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We mass together the relevent facts about the $*$-operation. Throughout this section we quote results in a form often more general than the original. In all cases, the more general result is proven by minor alterations of the original proof.

(1.1) **Theorem** [10, Theorem 2.2, Proposition 2.3, Lemma 2.1]. *Let* $\mathbf{X}$ *be a Fitting class. The class* $\mathbf{X}^*$ *is a Fitting class. Let* $\mathbf{X}_*$ *be the intersection over all Fitting classes* $\mathbf{Y}$ *such that* $\mathbf{Y}^*=\mathbf{X}^*$.

(a) $\mathbf{X}^{**}=\mathbf{X}^*$; $(\mathbf{X}^*)_*=\mathbf{X}_*$; $(\mathbf{X}_*)^*=\mathbf{X}^*$; $\mathbf{X}_{**}=\mathbf{X}_*$.

(b) *If* $\mathbf{Y}$ *is a Fitting class, then*

$$\mathbf{X}^*=\mathbf{Y}^* \Leftrightarrow \mathbf{X}_*=\mathbf{Y}_* \Leftrightarrow \mathbf{X}_* \subseteq \mathbf{Y} \subseteq \mathbf{X}^*.$$

(c) *If* $\mathbf{X}\subseteq\mathbf{Y}$ *for a Fitting class* $\mathbf{Y}$ *then* $\mathbf{X}^*\subseteq\mathbf{Y}^*$ *and* [3, Corollary 3.5] $\mathbf{X}_*\subseteq\mathbf{Y}_*$.

(d) *For all* $G\in\mathbf{F}$, $[G_{\mathbf{X}^*}, \mathrm{Aut}(G)]\leqq G_{\mathbf{X}_*}$.

(e) [3, Theorem 3.6]. *If* $\mathbf{X}\subseteq\mathbf{Y}$ *for a Fitting class* $\mathbf{Y}$, *and* $N_{\mathbf{Y}}(\mathbf{X})$ *is the intersection over all Fitting classes* $\mathbf{U}\supseteq\mathbf{X}$ *such that* $\mathbf{Y}_*\subseteq\mathbf{U}\subseteq\mathbf{Y}^*$, *then* $\mathbf{X}^*\cap\mathbf{Y}_*=\mathbf{X}_*$ *implies* $\mathbf{X}^*\cap N_{\mathbf{Y}}(\mathbf{X})=\mathbf{X}$.

If $\mathbf{X}\subseteq\mathbf{S}$ then Lockett [10] asked if the following equality always held:

(1.2) $\mathbf{X}=\mathbf{X}^*\cap N_{\mathbf{S}}(\mathbf{X})$.

It does not [1]. In fact, there is a Fitting class $\mathbf{X}$ such that $\mathbf{X}_*\neq\mathbf{X}\subseteq\mathbf{S}_*$. However, the identity (1.2) is of some interest. In fact, because of part (e) of the theorem, a more interesting identity might be

(1.3) $\mathbf{X}^*\cap\mathbf{Y}_*=\mathbf{X}_*$

for Fitting classes $\mathbf{X}\subseteq\mathbf{Y}$. The identity (1.2) is called *Lockett's conjecture.*

We turn next to normal Fitting classes and normal Fitting pairs. A Fitting class $\mathbf{Y}\subseteq\mathbf{X}$ is called an $\mathbf{X}$-*normal Fitting class* if (1) $\mathbf{Y}\subseteq\mathbf{X}$, (2) for any $G\in\mathbf{X}$, $G_{\mathbf{Y}}\geqq[G, \mathrm{Aut}(G)]$, and (3) if $\mathbf{X}$ is nontrivial then $\mathbf{Y}$ is nontrivial. We call a pair (f, A) an $\mathbf{X}$-*normal Fitting pair* if A is an abelian group, and f assigns to each $G\in\mathbf{X}$ a homomorphism f_G: $G\to A$ such that

(1.4) (1) if $H, G\in\mathbf{X}$, φ: $H\to G$ is an isomorphism, and $\varphi(H)\lhd G$, then $f_H=f_G\,\varphi$; and

(2) $A=\langle f_G(G)\,|\,G\in\mathbf{X}\rangle$.

Let (f, A) be an $\mathbf{X}$-normal Fitting pair and $B\leqq A$ a subgroup. If for any $x\in G\in\mathbf{X}$ we set

$$\bar{f}_G(x)=f_G(x)\,B\in A/B$$

then $(\bar{f}, A/B)$ is an $\mathbf{X}$-normal Fitting pair and is said to be *induced* by (f, A). Assume now that $\mathbf{Y}\subseteq\mathbf{X}$ is a Fitting class; (f, A) is an $\mathbf{X}$-normal Fitting pair; and (g, B) is a $\mathbf{Y}$-normal Fitting pair. If $B\leqq A$ and $f_G=g_G$ for all $G\in\mathbf{Y}$ then we call (f, A) an *extension of* (g, B), and we call (g, B) the *restriction of* (f, A). There is only one restriction, though there may be many extensions. With respect to Fitting pairs and classes, the word "*normal*" means "**S**-normal."

We lump together the results we will need on normal Fitting pairs and classes.

(1.5) **Theorem.** *Let* $\mathbf{Y} \subseteq \mathbf{X}$ *be Fitting classes.*

(a) [2, Satz 5.1]. *If* $\mathbf{Y}$ *is an* $\mathbf{X}$*-normal Fitting class then* $\mathbf{Y}$ *contains all simple groups in* $\mathbf{X}$.

(b) [2, Satz 6.2] *An intersection of* $\mathbf{X}$*-normal Fitting classes is* $\mathbf{X}$*-normal. Thus, there is a unique minimal* $\mathbf{X}$*-normal Fitting class.*

(c) [3, Theorem 3.4] $\mathbf{X}_*$ *is the unique minimal* $\mathbf{X}$*-normal Fitting class.*

(d) [2, Satz 3.1] *If* (f, A) *is an* $\mathbf{X}$*-normal Fitting pair then* $\mathbf{X}_f = \{G \in \mathbf{X} \mid f_G(G) = 1\}$ *is an* $\mathbf{X}$*-normal Fitting class.*

(e) [9, Theorem 2.4] *If* $\mathbf{Y}$ *is* $\mathbf{X}$*-normal then there is an* $\mathbf{X}$*-normal Fitting pair* (f, A) *such that* $\mathbf{X}_f = \mathbf{Y}$.

A proof is omitted for the following easy lemma.

(1.6) **Lemma.** *If* $\mathbf{X}$ *is a Fitting class,* (f, A) *an* $\mathbf{X}$*-normal Fitting pair, and* $(f^p, A/O^p(A))$ *the induced pair for the prime p, then* $\mathbf{X}_f$ *is the intersection of* $\mathbf{X}_{f^p}$ *over all primes p.*

Next we discuss a particular Fitting class. For a prime p, let $\mathbf{F}^{p^*}$ be the class of all groups in $\mathbf{F}$ which have no composition factor of order p. It is straightforward to verify that $\mathbf{F}^{p^*}$ is a Fitting formation (i.e. both a Fitting class and a formation). (For definitions and facts about formations see [7, Chapter VI].) Notice that $\mathbf{F}^{p^*} \cap \mathbf{S} = \mathbf{S}_{p'}$, the class of *p'-groups in* $\mathbf{S}$. The class $\mathbf{F}^{p^*}$ is not saturated. For $p > 3$ this can be seen by considering $G = PSL(2, \mathbf{Z}_{p^2})$, the projective unimodular matrices over the integers modulo p^2. Here $\Phi(G)$ is of order p^3 and $G/\Phi(G) \simeq PSL(2, p) \in \mathbf{F}^{p^*}$. If $G \in \mathbf{F}$, we denote the $\mathbf{F}^{p^*}$ residual as $O^{p^*}(G)$.

II. The Extension Theorem

(2.1) *Hypothesis. Assume the following:*

(1) *p is a prime:*

(2) $\mathbf{K} \supseteq \mathbf{X} \supseteq \mathbf{Y}$ *are Fitting classes;*

(3) *if* $G \in \mathbf{X}$ *then* $O^{p^*}(G) \in \mathbf{Y}$;

(4) *if* $G \in \mathbf{K}$ *and P is a Sylow p-subgroup of G, then* $PG_{\mathbf{Y}} \in \mathbf{Y}$; *and*

(5) (f, A) *is an* $\mathbf{X}$*-normal Fitting pair for which A is a p-group.*

(2.2) **Extension Theorem.** *If Hypothesis* (2.1) *holds then there is a* $\mathbf{K}$*-normal Fitting pair* $(\hat{f}, A)$ *whose restriction to* $\mathbf{X}$ *is* (f, A).

Throughout the proof, G will denote a group in $\mathbf{K}$. We now define explicitly the desired extension.

(2.3) **Definition.** Let $G \in \mathbf{K}$. Fix $H \leqq G$ such that $H/G_{\mathbf{Y}}$ is a Sylow p-subgroup of $G/G_{\mathbf{Y}}$. Let $n = n(G)$ be a positive integer chosen such that

$$n[G:H] \equiv 1 \pmod{|G|_p}.$$

If $y \in G$ then set

$$\hat{f}_G(y) = f_H(V_{G \to H}(y))^n$$

where $V_{G \to H}$ is the transfer of G into H/H'.

(2.4) **Lemma.** *$\hat{f}_G$ is a homomorphism of G into A.*

Since $H/G_{\mathbf{Y}}$ is a p-group and H contains a Sylow p-group of G, $H = PG_{\mathbf{Y}}$ for some Sylow p-subgroup P of G and $[G:H]$ is prime to p. By hypothesis, $H = PG_{\mathbf{Y}} \in \mathbf{Y} \subseteq \mathbf{X}$. In particular, f_H and n both exist. Since f_H maps H into an abelian group, H' is in the kernel of f_H. Therefore, f_H determines a unique homomorphism of H/H' into A. It follows that $f_H(V_{G \to H}(y))$ defines a homomorphism. Since A is abelian, raising to the n^{th} power does not change the fact that we have a homomorphism.

(2.5) **Lemma.** *$\hat{f}_G$ is independent of the choice of $n = n(G)$.*

Since A is a p-group, the exponent of the image $\hat{f}_G(G)$ of G in A divides $|G|_p$. Thus by the congruence condition on n, $\hat{f}_G$ is independent of the choice of $n = n(G)$.

(2.6) **Lemma.** *If $\psi: G \to G_1$ is an isomorphism of G onto G_1 then for any $y \in G$,*

$$\hat{f}_G(y) = \hat{f}_{G_1}(y^{\psi}).$$

Suppose that H_1 and n_1 are chosen in the definition of $\hat{f}_{G_1}$ to correspond to H and n respectively. Since $G_{\mathbf{Y}}$ is the join of all normal $\mathbf{Y}$-subgroups of G, $G_{\mathbf{Y}}^{\psi} = G_{1\mathbf{Y}}$. Therefore, $H^{\psi}/G_{1\mathbf{Y}}$ is a Sylow p-subgroup of $G_1/G_{1\mathbf{Y}}$. By Sylow's Theorem there is a $u \in G_1$ so that $H_1^u = H^{\psi}$. Since A is abelian, G_1' is in the kernel of $\hat{f}_{G_1}$. Thus if $y \in G$ then

$$\begin{aligned}
\hat{f}_{G_1}(y^{\psi}) &= f_{H_1}(V_{G_1 \to H_1}(y^{\psi}))^{n_1} \\
&= f_{H_1}(V_{G_1 \to H_1}(y^{\psi u^{-1}}))^{n_1} \\
&= f_{H_1}([V_{G_1 \to H_1^u}(y^{\psi})]^{u^{-1}})^{n_1} \\
&= f_{H_1^u}(V_{G_1 \to H_1^u}(y^{\psi}))^{n_1} \\
&= f_{H^{\psi}}(V_{G^{\psi} \to H^{\psi}}(y^{\psi}))^{n_1} \\
&= f_H(V_{G \to H}(y))^{n_1} \\
&= f_H(V_{G \to H}(y))^{n} \\
&= \hat{f}_G(y).
\end{aligned}$$

Since $|G|_p = |G_1|_p$ and $[G:H] = [G_1:H_1]$, n and n_1 satisfy the same congruence condition. Lemma (2.5) allows us to change the exponent. Several times in the proof we used the fact that if $w \in H$ and $\varphi\colon H \to H_1$ is an isomorphism onto H_1 then $f_H(w) = f_{H^\varphi}(w^\varphi)$.

Remarks. (1) If ψ is taken to be conjugation by $x \in G$ or as an element of $\mathrm{Aut}(G)$ then we see that

$$\hat{f}_G(y^\psi) = \hat{f}_G(y).$$

(2) If $H_1/G_{\mathbf{Y}}$ is a Sylow p-subgroup of $G/G_{\mathbf{Y}}$ then $H_1 = H^u$ for some $u \in G$. We now have

$$\begin{aligned}\hat{f}_G(y) = \hat{f}_G(y^{u^{-1}}) &= f_H(V_{G\to H}(y^{u^{-1}}))^n \\ &= f_H([V_{G\to H^u}(y)]^{u^{-1}})^n \\ &= f_{H^u}(V_{G\to H^u}(y))^n \\ &= f_{H_1}(V_{G\to H_1}(y))^n.\end{aligned}$$

Thus $\hat{f}_G$ is independent of the choice of H in the definition.

(2.7) **Lemma.** *If $G \in \mathbf{K}$ is given, $\hat{f}_G$ is independent of all choices made in its definition.*

The only choices were H and n.

(2.8) **Lemma.** *If $N \lhd G$ then $\hat{f}_G|_N = \hat{f}_N$.*

Let $y \in N$. It is an easy computation and a well-known result that if $y \in G$ then $V_{G\to H}(y) = wH'$ where $w = y^{[G:H]}v$, $v \in G'$. The group A is a p-group, so that all p'-elements of N are mapped to 1 by both $\hat{f}_G$ and $\hat{f}_N$. In particular, we may assume that y is a p-element of N. Let P be a Sylow p-subgroup of H (hence of G). Now $\hat{f}_N(y^x) = \hat{f}_N(y)$ for all $x \in G$, so that we may assume $y \in P \cap N$. Since wH' is a p-element, we may take $w \in P$. Now both w and $y^{[G:H]}$ lie in P so that v does also. In particular, $v \in G' \cap P$, the Focal Subgroup of G. We show now that since $v \in H$, $f_H(v) = 1$. Let $x, t \in P$ and assume that $x^b = t$ for some $b \in G$. Since $H/G_{\mathbf{Y}}$, is a p-group, both $L = \langle x\rangle G_{\mathbf{Y}}$ and $L^b = \langle x^b\rangle G_{\mathbf{Y}} = \langle t\rangle G_{\mathbf{Y}}$ are subnormal in H. Therefore, $f_H(t) = f_H(x^b) = f_{L^b}(x^b) = f_L(x) = f_H(x)$, or $f_H(xt^{-1}) = 1$. By the Focal Subgroup Theorem [5, Theorem (7.3.4)], these elements xt^{-1} generate the focal subgroup $G' \cap P$. Therefore, f_H contains $G' \cap P$ in its kernel, and $f_H(v) = 1$. We conclude that

$$\begin{aligned}\hat{f}_G(y) = f_H(V_{G\to H}(y))^n = f_H(w)^n = f_H(y^{[G:H]}v)^n \\ = f_H(y)^{[G:H]n}.\end{aligned}$$

Now $G_{\mathbf{Y}} \cap N = N_{\mathbf{Y}}$ and $H \geqq G_{\mathbf{Y}}$ so that $N_{\mathbf{Y}} = G_{\mathbf{Y}} \cap (H \cap N)$. Thus

$$(H\cap N)/N_{\mathbf{Y}} = (H\cap N)/G_{\mathbf{Y}} \cap (H\cap N) \simeq (H\cap N)\,G_{\mathbf{Y}}/G_{\mathbf{Y}} \leqq H/G_{\mathbf{Y}},$$

a p-group. But $H \cap N \geqq P \cap N$, a Sylow p-subgroup of N. We conclude that if $K = H \cap N$ then $K/N_{\mathbf{Y}}$ is a Sylow p-subgroup of $N/N_{\mathbf{Y}}$. Since $K = H \cap N$, $[HN:H] = [N:K]$.

Computing, we now have

$$\begin{aligned} n[G:H] &= n[G:HN][HN:H] \\ &= (n[G:HN])[N:K] \\ &\equiv 1 \pmod{|G|_p} \\ &\equiv 1 \pmod{|N|_p}. \end{aligned}$$

By Lemma (2.7), we may take K (as H) and $m=n[G:HN]$ (as $n(N)$) in the definition of $\hat{f}_N$. Let $P_0=P\cap N$ so that P_0 is a Sylow p-subgroup of N. By an argument exactly as for G, we conclude that $V_{N\to K}(y)=u\,K'$ where $u=y^{[N:K]}b$ and $b\in N'\cap P_0$. Since, as before, $f_K(b)=1$, $\hat{f}_N(y)=f_K(y)^{[N:K]m}$. But $[N:K]m=[G:H]n$. Since $y\in K \lhd H$,

$$\hat{f}_G(y)=f_H(y)^{[G:H]n}=f_K(y)^{[G:H]n}=\hat{f}_N(y),$$

completing the proof of the lemma.

(2.9) **Lemma.** *If* $G\in\mathbf{X}$ *then* $\hat{f}_G=f_G$. *In particular*, $A=\langle \hat{f}_G(G)\,|\,G\in\mathbf{K}\rangle$.

Since $G\in\mathbf{X}$, $O^{p^*}(G)\in\mathbf{Y}$. Thus $O^{p^*}(G)\leqq G_{\mathbf{Y}}$. We now know that $G/G_{\mathbf{Y}}$ has no composition p-factors. That is, $G_{\mathbf{Y}}$ covers all composition p-factors of G. Thus $G=O^p(G)\,G_{\mathbf{Y}}$. Let $y\in G$ so that $y=x\,w$ where $x\in O^p(G)$ and $w\in G_{\mathbf{Y}}$. Both f_G and $\hat{f}_G$ map G into A, a p-group, so that $O^p(G)$ is in the kernel of both maps. Thus $\hat{f}_G(y)=\hat{f}_G(w)$ and $f_G(y)=f_G(w)$. In particular, $\hat{f}_G=f_G$ if $\hat{f}_{G_{\mathbf{Y}}}=f_{G_{\mathbf{Y}}}$. This latter equality is obvious from the definition of $\hat{f}_G$, since $G_{\mathbf{Y}}\in\mathbf{Y}$. The lemma follows.

These lemmas together prove the Extension Theorem.

III. Applications

A. *The Conjecture of Laue*

(3.1) **Theorem.** $\mathbf{F}_*\cap\mathbf{S}=\mathbf{S}_*$.

Choose an $\mathbf{S}$-normal Fitting pair (f,A) such that $\mathbf{S}_f=\mathbf{S}_*$. For a prime p, let (f^p,A^p) be the induced pair with $A^p=A/O^p(A)$. Then by Lemma (1.6), $\cap\,\mathbf{S}_{f^p}=\mathbf{S}_*$ where p runs over all primes. For the Extension Theorem we take $\mathbf{K}=\mathbf{F}$ and $\mathbf{X}=\mathbf{Y}=\mathbf{S}$. If $G\in\mathbf{X}=\mathbf{S}$ then $O^{p^*}(G)=O^{p'}(G)\in\mathbf{Y}=\mathbf{S}$. If $G\in\mathbf{F}$ and P is a Sylow p-subgroup of G, then $PG_{\mathbf{Y}}\in\mathbf{Y}=\mathbf{S}$ since $G_{\mathbf{Y}}\in\mathbf{Y}=\mathbf{S}$. Hypothesis (2.1) holds for (f^p,A^p) and each prime p. By the Extension Theorem there is an extension $(\hat{f}^p,A^p)$ of (f^p,A^p) to $\mathbf{F}$. Let $\mathbf{F}^0$ be the intersection over the classes $\mathbf{F}_{\hat{f}^p}$ for all primes p. Then clearly $\mathbf{F}^0\cap\mathbf{S}=\mathbf{S}_*$. Since by Theorem (1.1)(b) and Theorem (1.5)(c), $\mathbf{F}^0\supseteq\mathbf{F}_*$, we have $\mathbf{F}_*\cap\mathbf{S}\subseteq\mathbf{S}_*$. By Theorem (1.1)(c), $\mathbf{F}_*\supseteq\mathbf{S}_*$ so that we have $\mathbf{F}_*\cap\mathbf{S}=\mathbf{S}_*$, completing the proof.

B. *The Theorem of Bryce and Cossey*

Primitive saturated formations are defined by Hawkes in [6]. Bryce and Cossey prove the following.

(3.2) **Theorem** [4, Section 4]. *The primitive saturated formations in* $\mathbf{S}$ *are precisely the subgroup closed Fitting formations in* $\mathbf{S}$.

For a set of primes π we let $\mathbf{S}_\pi$ be the class of π-groups in $\mathbf{S}$. If $\pi=\{p\}$ then we set $\mathbf{S}_\pi=\mathbf{S}_p$.

(3.3) **Theorem** (3, Lemma 2.3). *If* $\mathbf{Y}\subseteq\mathbf{S}$ *is a primitive saturated formation of bounded Fitting height then* $\mathbf{Y}=\cap\mathbf{Y}_i$, $i=1,2,\ldots,\infty$ *where each* $\mathbf{Y}_i$ *is a finite product of* $\mathbf{S}_\pi$*'s for various sets of primes* π.

Fix a primitive saturated formation $\tilde{\mathbf{X}}\subseteq\mathbf{S}$. For an integer $n>0$ we let $\mathbf{N}^n$ be the class of groups in $\mathbf{S}$ of Fitting height at most n. Set $\tilde{\mathbf{X}}_n=\tilde{\mathbf{X}}\cap\mathbf{N}^n$ so that $\tilde{\mathbf{X}}=\cup\tilde{\mathbf{X}}_i$, $i=1,2,\ldots,\infty$. Each of the classes $\tilde{\mathbf{X}}_n$ is a primitive saturated formation of bounded Fitting height. By Theorem (2.3), $\tilde{\mathbf{X}}_n=\bigcap_{i=1}^{\infty}\tilde{\mathbf{Y}}_{ni}$ where $\tilde{\mathbf{Y}}_{ni}$ is a product

(3.4) $\mathbf{S}_{\pi(n,i,1)}\mathbf{S}_{\pi(n,i,2)}\cdots\mathbf{S}_{\pi(n,i,t_i)}$

where $t_i\geqq 1$ and all $\pi(n,i,j)$ are sets of primes.

Let $\mathbf{K}\subseteq\mathbf{S}$ be a subgroup closed Fitting class, and (f,A) a $(\mathbf{K}\cap\tilde{\mathbf{X}})$-normal Fitting pair such that $(\mathbf{K}\cap\tilde{\mathbf{X}})_f=(\mathbf{K}\cap\tilde{\mathbf{X}})_*$. We prove now that

(3.5) $(\mathbf{K}\cap\tilde{\mathbf{X}})_*=\mathbf{K}_*\cap\tilde{\mathbf{X}}$.

Let $\mathbf{X}=\mathbf{K}\cap\tilde{\mathbf{X}}$, $\mathbf{X}_n=\mathbf{K}\cap\tilde{\mathbf{X}}_n$, and $\mathbf{Y}_{ni}=\mathbf{K}\cap\mathbf{Y}_{ni}$ so that $\mathbf{X}_n=\bigcap_i\mathbf{Y}_{ni}$ and $\mathbf{X}\cup_n\mathbf{X}_n$.

Fix a prime p and let (f^p,A^p) be the Fitting pair induced by (f,A) in $A^p=A/O^p(A)$. Fix an integer $n>0$, and let $(f^{p,n},A^{p,n})$ be the restriction of (f^p,A^p) to $\mathbf{X}_n$. We wish to obtain an extension $(\hat{f}^{p,n},A^{p,n})$ of $(f^{p,n},A^{p,n})$ to $\mathbf{K}$. If there is an integer $i>0$ such that $p\notin\pi(n,i,j)$ for all sets $1\leqq j\leqq t_i$, then $\mathbf{X}_n$ is a class of p'-groups. Since $A^{p,n}$ is a p-group, the pair $(f^{p,n},A^{p,n})$ is trivial and has a trivial extension to $\mathbf{K}$. Therefore, we may assume that for each i there is a unique largest integer s_i, $1\leqq s_i\leqq t_i$, such that $p\in\pi(n,i,s_i)$. Let $\tilde{\mathbf{A}}_{ni}$ be the product of the first s_i factors in (3.4) and $\tilde{\mathbf{B}}_{ni}$ the product of the remaining factors. Since $p\in\pi=\pi(n,i,s_i)$, $\mathbf{S}_\pi\mathbf{S}_p=\mathbf{S}_\pi$ so that $\tilde{\mathbf{A}}_{ni}\mathbf{S}_p=\tilde{\mathbf{A}}_{ni}$. Further, $\tilde{\mathbf{B}}_{ni}$ is a class of p'-groups by the choice of s_i, and

$$\tilde{\mathbf{Y}}_{ni}=\tilde{\mathbf{A}}_{ni}\tilde{\mathbf{B}}_{ni}.$$

Let $\hat{\mathbf{Y}}_n=(\bigcap_i\mathbf{A}_{ni})\cap\mathbf{K}$ so that $\mathbf{Y}_n\subseteq\mathbf{X}_n\subseteq\mathbf{K}$.

We verify the hypotheses of the Extension Theorem for the classes $\mathbf{K}$, $\mathbf{X}_n$, and $\tilde{\mathbf{Y}}_n$. Let $G\in\mathbf{X}_n$. Since $O^{p^*}(G)=\mathbf{O}^{p'}(G)\lhd G$ and $\mathbf{X}_n\subseteq\mathbf{K}$, $O^{p^*}(G)\in\mathbf{K}$. Since $G\in\mathbf{X}_n\subseteq\tilde{\mathbf{X}}_n$, $G\in\tilde{\mathbf{Y}}_{ni}$ for all i. Now $G/G_{\tilde{A}_{ni}}\in\tilde{\mathbf{B}}_{ni}\subseteq\mathbf{S}_{p'}$ so that $O^{p^*}(G)=O^{p'}(G)\leqq G_{\tilde{\mathbf{A}}_{ni}}$ for all i. Thus $O^{p^*}(G)\in(\bigcap_i\tilde{\mathbf{A}}_{ni})\cap\mathbf{K}=\hat{\mathbf{Y}}_n$, verifying (3) of the hypotheses.

Next let $G\in\mathbf{K}$. Now $G_{\hat{\mathbf{Y}}_n}\in\tilde{\mathbf{A}}_{ni}$ for all i and $\tilde{\mathbf{A}}_{ni}\mathbf{S}_p=\tilde{\mathbf{A}}_{ni}$ so that $PG_{\hat{\mathbf{Y}}_n}\in\tilde{\mathbf{A}}_{ni}$ for all i where P is a Sylow p-subgroup of G. Since $\mathbf{K}$ is subgroup closed, $PG_{\hat{\mathbf{Y}}_n}\in(\bigcap_i\tilde{\mathbf{A}}_{ni})\cap\mathbf{K}=\hat{\mathbf{Y}}_n$, verifying (4) of the hypotheses.

Since all other parts of the hypotheses hold, by the Extension Theorem there is a pair $(\hat{f}^{n,p},A^{n,p})$ on $\mathbf{K}$ whose restriction to $\mathbf{X}_n$ is $(f^{n,p},A^{n,p})$.

Let $G \in \mathbf{K}_* \cap \mathbf{X} = \mathbf{K}_* \cap \tilde{\mathbf{X}}$. Since G has some Fitting height, we assume this to be n. Now $G \in \mathbf{K}_*$ so that $G \leqq \ker \hat{f}_G^{n,p}$ by Theorem (1.5)(c). Since $G \in \mathbf{K} \cap \tilde{\mathbf{X}} \cap \mathbf{N}^n = \mathbf{K} \cap \tilde{\mathbf{X}}_n = \mathbf{X}_n$, and $(\hat{f}^{n,p}, A^{n,p})$ extends $(f^{n,p}, A^{n,p})$, $\hat{f}_G^{n,p} = f_G^{n,p}$. But $(f^{n,p}, A^{n,p})$ is the restriction of (f^p, A^p) to $\mathbf{X}_n$ so that $\hat{f}_G^{n,p} = f_G^{n,p} = f_G^p$. In particular, $G \in \mathbf{X}_{f^p}$ for every prime p. By Lemma (1.6), $G \in \mathbf{X}_*$. We now have $\mathbf{K}_* \cap \mathbf{X} \subseteq \mathbf{X}_*$. By Theorem (1.1)(c), $\mathbf{X}_* \subseteq \mathbf{K}_*$ so that $\mathbf{K}_* \cap \tilde{\mathbf{X}} = \mathbf{K}_* \cap \mathbf{X} = \mathbf{X}_* = (\mathbf{K} \cap \tilde{\mathbf{X}})_*$. This proves the identity (2.5).

(3.6) **Theorem** (3, Theorem 4.17). *Assume that* $\mathbf{X} \subseteq \mathbf{S}$ *is a primitive saturated formation and that* $\mathbf{K} \subseteq \mathbf{S}$ *is a subgroup closed Fitting class. Then*

$$\mathbf{K}_* \cap \mathbf{X} = (\mathbf{K} \cap \mathbf{X})_*.$$

Applying the theorem with $\mathbf{K} = \mathbf{S}$ and Theorem (1.1)(e) we have:

(3.7) **Corollary** [3, Theorem 1.3]. *If* $\mathbf{X}$ *is a primitive saturated formation then* $\mathbf{X} = \mathbf{X}^* \cap N_{\mathbf{S}}(\mathbf{X})$, i.e. *Lockett's conjecture holds for* $\mathbf{X}$.

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