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Irreducible Components of the Chow Scheme of Space Curves

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Introduction

In this paper we investigate what we feel are several interesting examples of the Chow Scheme of curves in $\mathbb{P}^3$. Peskine and Szpiro in [5] have shown that the space curve consisting of a smooth plane cubic curve and a line (not lying in the plane of the cubic curve) intersecting at one point, cannot be flatly deformed to a smooth curve. In Section 1 we generalize this result to show that given $X \subset \mathbb{P}^3$ a curve whose irreducible components consist of a smooth plane curve of degree $d \geqq 3$ plus a line not lying in the same plane which intersects the plane curve in one point cannot be flatly deformed to a smooth curve. Moreover, we show that such a curve is contained in a unique component of the Chow Scheme.

In Section 2 we give a modern proof to an interesting example of Max Noether [4] in which he shows the Chow Scheme of space curves may have two irreducible components of the same dimension each with generic member a smooth irreducible curve of the same degree and genus. Moreover, we prove this phenomenon cannot occur for smooth rational or for smooth elliptic curves, i.e., all smooth rational curves of the same degree lie in one component of the Chow Scheme, and all smooth elliptic curves of the same degree lie in one component of the Chow Scheme.

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Notation and Terminology

(i) All our schemes will be projective algebraic defined over a fixed algebraically closed field k, $\mathrm{char}(k)=0$. By "point" of a scheme, unless explicitly stated otherwise, we mean "closed point".

(ii) By *curve* we mean 1-dimensional scheme (hence our curves are not necessarily reduced, irreducible). By *space curve* we mean a curve embedded in projective three space.

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(iii) For X a scheme of dimension n, the *arithmetic genus* $p_a(X)$ is defined to be:

$$p_a(X)=1-\chi(\mathcal{O}_X) \quad \text{where} \quad \chi(\mathcal{O}_X)=\sum_{i=0}^{n}(-1)^i \dim H^i(X,\mathcal{O}_X).$$

(iv) $\mathbb{P}^n=\mathbb{P}^n_k=\operatorname{Proj} k[X_0,\ldots,X_n]$.

(v) By *flat smoothing* of a curve $X\subset\mathbb{P}^n$, we mean a flat family $\mathcal{X}\subset\mathbb{P}^n_R$ parametrized by $T=\operatorname{Spec} R$, R a discrete valuation ring, with the generic fiber of $\mathcal{X}$ smooth, the special fiber isomorphic to X. This implies that X lies on the same irreducible component of the Hilbert scheme of curves in $\mathbb{P}^n$ as a smooth curve. (See [2] for definition and construction of the Hilbert scheme.)

(vi) Given $X\subset\mathbb{P}^n$ closed subscheme, let $\deg(X)=$degree of X.

(vii) Let X be a scheme of dimension r and let $X_1,\ldots,X_n$ be those irreducible components of X of dimension equal to r. Let $|X_i|$ be the underlying point set of X_i, let $x_i\in X_i$ be the generic point, and let $r_i=$length $\mathcal{O}_{X_i,x_i}$. Then the associated cycle of $X:=\sum_i r_i|X_i|$. Note that we may identify a reduced scheme of pure dimension r with its associated cycle. Hence under this identification we may speak of the "arithmetic genus" of the cycle.

(viii) Given $\mathcal{F}$ a coherent sheaf on a scheme X, $h^i(\mathcal{F})=\dim H^i(X,\mathcal{F})$.

(ix) $C_d=$Chow Scheme of 1-cycles of degree d in $\mathbb{P}^3$ (see [6]).

1. Non-Smoothable Reduced Curves Contained in a Unique Component of C_d

In this section we discuss a class of reduced space curves which cannot be flatly smoothed and moreover whose associated 1-cycles belong to one and only one irreducible component of the Chow Scheme C_d. We begin with the following remark on a classical result of Castelnuovo:

Remark (1.1). In [1] Griffiths and Harris give a proof (based on the original proof of Castelnuovo) of the following result of Castelnuovo: Let $X\subset\mathbb{P}^n$ be a smooth irreducible curve of degree d not contained in any hyperplane. Then $p_a(X)\leqq M\left(d-\frac{M+1}{2}(n-1)-1\right)$ where M is the greatest integer $\leqq\frac{d-1}{n-1}$. Note for $X\subset\mathbb{P}^3$ this result appears in Noether [4].

Now their proof goes over word for word when $X\subset\mathbb{P}^n$ is reduced, irreducible (of degree d not contained in any hyperplane). Indeed they first show that for H a hyperplane section of X we have

$$h^0(kH)-h^0((k-1)H)\begin{cases}\geqq k(n-1)+1 & k\leqq M\\ =d & k>M\end{cases}$$

where $M=$greatest integer $\leqq\frac{d-1}{n-1}$. Thus

$$h^0((M+m)H)\geqq\frac{M(M+1)}{2}(n-1)+M+1+md$$

and so taking $m \gg 0$ from Riemann-Roch we get the Castelnuovo bound. (Note in point of fact from Mumford [3] we have $h^1((M+m)H)=0$ when $\deg(M+m)H > 2p_a(X)-2$ exactly as in the smooth case.)

In particular, if $X \subset \mathbb{P}^3$ is of degree d, reduced, irreducible and not contained in any plane, then

$$p_a(X) \leqq \frac{(d-2)^2}{4} \qquad \text{if } d \text{ is even,}$$

$$p_a(X) \leqq \frac{(d-1)(d-3)}{4} \qquad \text{if } d \text{ is odd.}$$

Moreover it is easy to check that such a space curve of maximal genus exists and lies on a smooth quadric surface.

Proposition (1.2). *There exist no smooth space curves of degree $d+1$ and genus $\frac{(d-1)(d-2)}{2}$ for $d \geqq 4$.*

Proof. We first show that there are no such irreducible curves. So suppose to the contrary that $X \subset \mathbb{P}^3$ were a smooth irreducible curve of degree $d+1$, genus $\frac{(d-1)(d-2)}{2}$. Then X cannot lie in any plane $\left(\text{for then its genus would be } \frac{d(d-1)}{2}\right)$. So by Castelnuovo's bound, if $d+1$ is even, then $\frac{(d-1)^2}{4} \geqq \frac{(d-1)(d-2)}{2}$ implying that $d \leqq 3$, and if $d+1$ is odd, then $\frac{d(d-2)}{4} \geqq \frac{(d-1)(d-2)}{2}$ implying $d \leqq 2$, in both cases contradicting the hypothesis that $d \geqq 4$.

Finally we show there are no smooth reducible curves of degree $d+1$, genus $\frac{(d-1)(d-2)}{2}$. In point of fact we will show there are no non-connected reduced curves of degree $d+1$, genus $\frac{(d-1)(d-2)}{2}$. (A smooth reducible curve is, in particular, not connected.) Indeed let $Y = Y_1 \cup Y_2$, Y_i reduced (not necessarily irreducible), $\deg(Y_i) = d_i$ $(i=1,2)$, $d_1 + d_2 = d+1$, and such that $Y_1 \cap Y_2 = \varnothing$. Then $p_a(Y_i) \leqq \frac{(d_i-1)(d_i-2)}{2}$. So

$$\begin{aligned} p_a(Y) &= p_a(Y_1) + p_a(Y_2) - 1 \\ &\leqq \frac{(d_1-1)(d_1-2)}{2} + \frac{(d_2-1)(d_2-2)}{2} - 1 \\ &< \frac{(d-1)(d-2)}{2}. \quad \text{Q.E.D.} \end{aligned}$$

Example (1.3). Let $X \subset \mathbb{P}^3$ be a curve consisting of two irreducible components X_1 and X_2, X_1 a smooth degree d plane curve and X_2 a line not lying in the

plane of X_1 and intersecting X_1 at one point. Then $p_a(X)=\frac{(d-1)(d-2)}{2}$, $\deg X = d+1$ and thus for $d \geqq 4$ by Proposition (1.2) X cannot be flatly smoothed.

It is also true for $d=3$, X cannot be smoothed. This result is contained in Peskine-Szpiro [5]. For $d=3$, there do exist smooth irreducible curves of degree $d+1=4$, and genus $\frac{(d-1)(d-2)}{2}=1$, namely the intersection of two smooth quadric surfaces. The proof of Peskine-Szpiro shows that X (for $d=3$) cannot be deformed to such a complete intersection.

Next we show that if X is defined as in Example (1.3), then X belongs to a unique component of the Chow Scheme C_{d+1}.

Definition (1.4). Let X be any projective curve. Then the *geometric genus* of X is defined to be the arithmetic genus of the normalization of $(X)_{\text{red}}$ (where $(X)_{\text{red}}$ = reduction of X).

Lemma (1.5). *Let $\mathscr{X} \to T$ be a flat family of projective curves, where T is a smooth connected curve. Suppose that the generic fiber $\mathscr{X}_t$ $(t \in T)$ is a reduced curve of arithmetic genus p. Suppose moreover that the special fiber $\mathscr{X}_0$ is a generically reduced curve. Then the geometric genus of $\mathscr{X}_0 \leqq p$.*

Proof. Let $\tilde{\mathscr{X}} \to T$ be the normalization of $\mathscr{X}$. Then since $\tilde{\mathscr{X}}$ is a normal surface, $\tilde{\mathscr{X}}_0$ is Cohen-Macaulay, and therefore has no embedded components. Thus since $\mathscr{X}_0$ is generically reduced, $\tilde{\mathscr{X}}_0$ is reduced. Hence $p_a(\tilde{\mathscr{X}}_0) \geqq$ geometric genus of $\tilde{\mathscr{X}}_0$ = geometric genus of $\mathscr{X}_0$, since geometric genus is a birational invariant. Now by flatness $p_a(\tilde{\mathscr{X}}_0)=p_a(\tilde{\mathscr{X}}_t)$ and since $p_a(\tilde{\mathscr{X}}_t) \leqq p_a(\mathscr{X}_t)=p$ we get the geometric genus of $\mathscr{X}_0 \leqq p$ as required. Q.E.D.

Theorem (1.6). *Let $X=X_1 \cup X_2 \subset \mathbb{P}^3$ be such that X_1 is a smooth plane curve of degree $d \geqq 3$, X_2 is a line not lying in the plane of X_1 and such that X_1 and X_2 intersect at one point. Then X (or more precisely the 1-cycle associated to X) lies in one and only one component of C_{d+1} namely the component with generic member a line and smooth plane curve of degree d non-intersecting.*

Proof. Consider first the case when $d \geqq 4$. We note that a generic member of an irreducible component of C_{d+1} corresponds either to an irreducible cycle or a reducible cycle with distinct components (i.e., of multiplicity 1) in which none of the components intersect. Now the geometric genus of $X=\frac{(d-1)(d-2)}{2}-1$. Given any irreducible reduced curve Y of degree $d+1$ not lying in a plane by the Castelnuovo bound we have geometric genus of $X > p_a(Y)$ for $d \geqq 4$. Moreover from Example (1.3) if Y is not connected reduced of degree $d+1$, then

$$p_a(Y) \leqq \frac{(d-1)(d-2)}{2}-1$$

and it is easy to check Y reaches this upper bound precisely when Y is of the form, a line plus a plane curve of degree d non-intersecting. Hence the geometric genus of $X >$ arithmetic genus of the generic numbers of all the irreducible components of C_{d+1}, other than the component of plane curves and other than

the component with generic member a line and non-intersecting plane curve. (Note for a 1-cycle with components only of multiplicity one, we can give it the natural reduced scheme structure and then can compute its arithmetic genus.)

Now clearly X does not lie on the plane curve component. Suppose then that X were contained in some other component of C_{d+1} with generic member Y and such that geometric genus of $X >$ arithmetic genus of Y. Then there would exist a family of cycles $\mathscr{X} \hookrightarrow \mathbb{P}^3_R$, $\mathscr{X} \to \operatorname{Spec} R$, R a discrete valuation ring, with special fiber isomorphic to X, generic fiber isomorphic to Y. But if we take $\bar{\mathscr{X}} \hookrightarrow \mathbb{P}^3_R$ with the reduced scheme structure, we get $\bar{\mathscr{X}} \to \operatorname{Spec}(R)$ a flat family of curves with generic fiber isomorphic to Y and the special fiber isomorphic to X' where $(X')_{\text{red}} \approx X$ and such that X' and X only differ by embedded components so that X' is generically reduced. Thus we are in the situation of Lemma (1.5) from which we derive an immediate contradiction.

For $d=3$ (i.e. X as above with plane curve component of degree 3) checking the components of C_4 (see [4]) it is immediate that X has geometric genus greater than the arithmetic genus of the generic members of all the components except the components of plane curves, the component of complete intersections of two quadric surfaces, and the component with generic member line plus cubic plane curve non-intersecting. X does not lie on the plane curve component, and as noted in Example (1.3) it does not lie in the component of complete intersections of quadric surfaces. Q.E.D.

2. Smooth Curves of the Same Degree and Genus Belonging to Distinct Components of C_9

In this section we give a modern proof of a classical result of Max Noether in [4]. The result shows that there may exist two distinct irreducible components of the Chow Scheme C_d which have generic members corresponding to smooth irreducible curves of the same genus. However, we first prove that all smooth rational curves of given degree $d \geqq 2$ lie on the same irreducible component of C_d and all smooth elliptic curves of given degree $d \geqq 3$ lie on the same irreducible component of C_d.

Proposition (2.1). *All smooth elliptic curves of a given degree $d \geqq 3$ lying in the same projective space lie on the same component of the appropriate Hilbert Scheme. Hence they lie on the same component of the appropriate Chow Scheme.*

Proof. Note first that given Γ an elliptic curve and $P \in \Gamma$ a closed point, the linear system $|nP|$ defines an embedding of Γ into $\mathbb{P}^{n-1}$ as a degree n curve for all $n \geqq 3$. Also note that subsequently the image of Γ in $\mathbb{P}^{n-1}$ can be embedded in any $\mathbb{P}^m$ $(m \geqq 3)$ as a degree n elliptic curve via generic projection.

Let Q be the open subset of the Hilbert scheme corresponding to smooth elliptic curves of degree n. Then we must prove Q is irreducible. To do this we define:

$$\begin{aligned}\tilde{Q} = \{(\Gamma;\, P;\, \varphi_1, \ldots, \varphi_n \in H^0(|nP|)) \mid\ & \Gamma \text{ is a smooth elliptic curve, } P \in \Gamma, \text{ and} \\ & \varphi_1, \ldots, \varphi_n \text{ are independent sections which define an embedding} \\ & \text{of } \Gamma \text{ as a degree } n \text{ projective curve}\}.\end{aligned}$$

(Note $\tilde{Q}$ may be constructed as a finite covering space of Q.) Then there exists a natural morphism $p\colon \tilde{Q} \to M_1$ where M_1 is the moduli space of elliptic curves defined by $(\Gamma; P; \varphi_1, \ldots, \varphi_n) \mapsto \Gamma$. Since M_1 is reduced, irreducible (it is isomorphic to the affine line) and since p has reduced, irreducible fibers ($\approx \Gamma \times PGL(n-1)$), it follows $\tilde{Q}$ is reduced, irreducible. Hence Q is irreducible as required. Q.E.D.

Remark (2.2). A similar proof as in Proposition (2.1) shows that all smooth rational curves of a given degree lie on the same component of the appropriate Hilbert Scheme (and hence Chow Scheme) since given Γ rational $P \in \Gamma$, Γ has a natural embedding in $\mathbb{P}^n$ via $|nP|$ (for all $n \geqq 2$).

We now give the example of Noether of two irreducible components of C_9 each with generic member a smooth irreducible curve of the same genus. We begin with the following lemma:

Lemma (2.3). *Let $H_d \subset \mathbb{P}^3$ be a non-singular surface of degree $d \leqq 4$. Let $\Gamma \subset H_d$ be a reduced, irreducible curve of degree r and arithmetic genus p. Then the family of divisors which are linearly equivalent to Γ on H_d has dimension $(4-d)r+p+\delta_{4,d}-1$ where $\delta_{4,d}$ is the Kronecker delta.*

Proof. If K is the canonical divisor on H_d, then K is linearly equivalent to $(d-4)H$ where H is a hyperplane section of H_d, and $p_a(H_d) = \binom{d-1}{3} = \delta_{4,d}$ where $p_a(H_d)$ is the arithmetic genus of H_d. Then by the Riemann-Roch theorem for surfaces:

$$h^0(\mathcal{O}_{H_d}(\Gamma)) - h^1(\mathcal{O}_{H_d}(\Gamma)) + h^2(\mathcal{O}_{H_d}(\Gamma)) = \tfrac{1}{2}\Gamma\cdot(\Gamma - K) + p_a(H_d) + 1.$$

But

$$\begin{aligned}\tfrac{1}{2}\Gamma\cdot(\Gamma-K)+1 &= \tfrac{1}{2}\Gamma\cdot(\Gamma+K)+1-\Gamma\cdot K\\ &= p_a(\Gamma)-(d-4)r.\end{aligned}$$

From Serre Duality we have that

$$h^2(\mathcal{O}_{H_d}(\Gamma)) = h^0(\mathcal{O}_{H_d}(K-\Gamma)) = 0$$

since $K-\Gamma<0$. Thus we get that

$$h^0(\mathcal{O}_{H_d}(\Gamma)) = h^1(\mathcal{O}_{H_d}(\Gamma)) + (4-d)r + p + \delta_{4,d}$$

(recall $p = p_a(\Gamma)$).

We claim now that

$$h^1(\mathcal{O}_{H_d}(\Gamma)) = h^1(\mathcal{O}_{H_d}(K-\Gamma)) = 0.$$

Indeed we have that $-K+\Gamma$ is linearly equivalent to $(4-d)H+\Gamma$ and since $d \leqq 4$, $-K+\Gamma$ represents a reduced connected curve on H_d. Thus we see $H^0(\mathcal{O}_{-K+\Gamma}) = k$. From the exact sequence

$$0 \to \mathcal{O}_{H_d}(K-\Gamma) \to \mathcal{O}_{H_d} \to \mathcal{O}_{-K+\Gamma} \to 0$$

we derive

$$0 \to \underset{\substack{\| \\ k}}{H^0(\mathcal{O}_{H_d})} \to \underset{\substack{\| \\ k}}{H^0(\mathcal{O}_{-K+\Gamma})} \to H^1(\mathcal{O}_{H_d}(K-\Gamma)) \to H^1(\mathcal{O}_{H_d})=0$$

(since H_d is a non-singular surface in $\mathbb{P}^3$ the "irregularity" $H^1(\mathcal{O}_{H_d})=0$). Therefore $H^1(\mathcal{O}_{H_d}(K-\Gamma))=0$ and hence $H^1(\mathcal{O}_{H_d}(\Gamma))=0$. Thus $h^0(\mathcal{O}_{H_d}(\Gamma))=(4-d)r+p+\delta_{4,d}$. Q.E.D.

Proposition (2.4). *Let $\mathcal{F}$ be a maximal algebraic family of curves in $\mathbb{P}^3$ of degree $d>4$ and arithmetic genus p, with generic member lying on a non-singular quadric surface in $\mathbb{P}^3$. Then the dimension of $\mathcal{F}$ is $p+2d+8$.*

Proof. Fix a non-singular quadratic in $\mathbb{P}^3$, H_2. Then any two curves of $\mathcal{F}$ which lie on H_2 are linearly equivalent. Indeed let $\Gamma \in \mathcal{F}$, $\Gamma \subset H_2$. Then Γ is linearly equivalent to $mE+nF$ where E and F are generators of the Neron-Severi group of H_2. Therefore since $\deg \Gamma = d$ and $p_a(\Gamma)=p$ we have, using the usual formulae, that $p=(m-1)(n-1)$, $d=m+n$ and thus the numbers p and d determine m and n.

By Lemma (2.3) we have that $\dim|\Gamma|_{H_2}=p+2d-1$. But every $\Gamma \in \mathcal{F}$ belongs to a unique quadric since $d>4$. Since the dimension (as a projective space) of the space of quadrics in $\mathbb{P}^3$ is 9, the proposition immediately follows. Q.E.D.

Remark (2.5). Note that in the proof of Proposition (2.4) we use the hypothesis that the degree d of $\Gamma \in \mathcal{F}$ is >4 only to conclude that it is contained in a unique quadric surface. Hence the proposition is valid also for maximal algebraic families of space curves whose generic member is contained in a *unique* smooth quadric surface.

Example (2.6). We now give the example of Noether of two irreducible components of C_9 with generic members smooth irreducible curves of the same genus.

Let $\mathcal{F}_1$ be the algebraic family of complete intersections of cubic surfaces in $\mathbb{P}^3$. Then a generic member of $\mathcal{F}_1$ is a smooth curve of degree 9 and arithmetic genus 10. Next let $\mathcal{F}_2$ be the algebraic family of curves in $\mathbb{P}^3$ with generic member Γ of the form $H_2 \cap H_6 - 3L$ where H_2 is a smooth quadric surface, H_6 a smooth degree 6 surface, and "$3L$" represents three lines contained in the same linear system of H_2. Then Γ is linearly equivalent to $6H-3E=3E+6F$ where E and F are generators of the Neron-Severi group of H_2 and $H=E+F$ is a hyperplane section. Then Γ is a smooth, irreducible curve of degree 9 and arithmetic genus 10 (Bertini's theorem).

We claim that $\mathcal{F}_1$ and $\mathcal{F}_2$ are distinct algebraic families. To prove this we will show they have the same dimension. By Proposition (2.4) $\dim \mathcal{F}_2 = 36$. Thus we need only compute the dimension of the family $\mathcal{F}_1$ of complete intersections. We first show that if $\Gamma \in \mathcal{F}_1$ is a generic member than Γ is contained in exactly a two-dimensional family of cubic surfaces. Indeed, let $H=\mathcal{O}_{\mathbb{P}^3}(1)$, $H_\Gamma = \mathcal{O}_{\mathbb{P}^3}(1)|_\Gamma$. Then by Riemann-Roch,

$$\chi(3H_\Gamma)=\chi(\mathcal{O}_\Gamma)+\deg 3H_\Gamma = 1-10+27=18.$$

Since $\deg(K_\Gamma - 3H_\Gamma) < 0$ we have $h^1(3H_\Gamma) = 0$ so that $h^0(3H_\Gamma) = 18$. Since $h^0(3H) = 20$ we see that Γ is contained in a two-dimensional family of cubic surfaces as asserted.

Now by definition of $\mathscr{F}_1$, two distinct cubic surfaces of $\mathbb{P}^3$ intersect to give an element of $\mathscr{F}_1$. Let P_1 be the projective space of cubic surfaces in $\mathbb{P}^3$. Then $\dim P_1 = 19$, so $\dim P_1 \times P_1 = 38$. Therefore the open set U of $P_1 \times P_1$ minus the diagonals also has dimension 38. Since a generic member of $\mathscr{F}_1$ is contained in a two-dimensional family of cubic surfaces, $\dim \mathscr{F}_1 = 36$.

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