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Characterization of Fans and Hereditarily Pythagorean Fields

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Introduction

Let K be a real field and T a preordering of K, i.e. a subset T of K^* such that $T+T\subset T$, $T\cdot T\subset T$ and $K^{*2}\subset T$. We denote with X the topological space ([19] S. 63) of all orderings P of K and X/T the subspace of all orderings P with $T\subset P$ (here ordering means a positive cone in the usual sense). Let $W(K)$ be the Wittring of non-degenerated bilinear forms over K and $J(T)$ the ideal of $W(K)$, which is generated by the forms $\langle 1\rangle\perp\langle -t\rangle$, $t\in T$. Generalizing Pfister's theorem [21] slightly [5] we get an exact sequence

$$0\to J(T)\to W(K)\xrightarrow{\sigma} C(X/T,\mathbb{Z})$$

where $C(X/T,\mathbb{Z})$ is the ring of continuous functions on X/T with values in $\mathbb{Z}$ (provided with discrete topology) and σ is the signature ranging over all $P\in X/T$.

Of special interest is the case, where $2C(X/T,\mathbb{Z})\subset\sigma W(K)$. This is equivalent to the *strong approximation property* (SAP) for X/T, i.e. for any two disjoint closed subsets A, $B\subset X/T$ there exists an $a\in K^*$ with $a\in P$ for all $P\in A$ and $a\notin P$ for all $P\in B$. A further equivalent condition is that any four orderings of X/T are independent [7]. Here we are mainly interested in the opposite situation, which is characterized by the following equivalent conditions (compare [5] or §2 below):

a) *For any three different orderings P_1, P_2, $P_3\in X/T$ there exists an ordering $P_4\neq P_1, P_2, P_3$, $P_4\in X/T$ such that P_1, P_2, P_3, P_4 are dependent, i.e.*

$$P_1\cap P_2\cap P_3=P_1\cap P_2\cap P_3\cap P_4.$$

b) *$W(K)/J(T)$ is a group ring $\mathbb{Z}[H]$. Then H is necessarily an elementary abelian 2-group* [27, 5].

c) *$T+aT\subset T\cup aT$ for all $a\in K^*\setminus T$.*

Following Becker and Köpping [5], a preordering T with the above equivalent properties is called a *fan (Fächer)*. Trivial examples of fans are orderings and intersections of two orderings. Furthermore, let T be a preordering which is compatible with a valuation ϕ such that the induced preordering $\bar{T}$ on the residue classfield K_ϕ is a fan: then T is a fan. In §2 we shall prove that all fans arise from the above principles (Theorem (2.7)). As it is remarked in [5], (SAP) holds for the space X of all orderings of K iff all fans of K are intersections of at most 2 orderings. Then $2C(X, \mathbb{Z}) \subset \sigma W(K)$. Anyway $S(K) := \operatorname{coker} \sigma$ is an abelian 2-group [16]. Call $\mathrm{st}_{\mathrm{red}}(K)$ the height of $S(K)$. $\mathrm{st}_{\mathrm{red}}(K)$ is thoroughly discussed in [7]. Here we derive a formula, which connects $\mathrm{st}_{\mathrm{red}}(K)$ with the existence of "small" fans, namely

(2.10) $\mathrm{st}_{\mathrm{red}}(K)+1 = \sup_{n\in\mathbb{N}} \{\text{ex. a fan } T \text{ of } K \text{ with } (K^*: T) \geqq 2^n\}$.

Of special interest are the fields K for which K^{*2} forms a fan. To my knowledge they firstly occur in the literature in [8] (1965). In [6] I proposed to call them "*strictly pythagorean*". They were rediscovered in [10] (1972) under the name "superpythagorean" and studied since then in many papers. With our characterization of fans we get:

(*) *Every strictly pythagorean field admits a 2-henselian valuation ϕ such that the residue classfield K_ϕ is an intersection of at most 2 euclidean closures.*

All known properties of strictly pythagorean fields can be derived from (*). In particular K is strictly pythagorean if and only if K is hereditarily pythagorean with respect to the 2-closure of K. In [4] Becker has developed an Artin-Schreier-theory of p-closed extensions Ω over K (for a definition of p-closed extensions see §1) and also a concept of Ω-pythagorean fields (hereditarily pythagorean for extensions in Ω). Using one of his main results (cited as Theorem (3.3) in §3) I generalize (*):

(3.5) *K is Ω-pythagorean if and only if K admits an Ω-henselian valuation ϕ such that K_ϕ is $\Omega_{\tilde{\phi}}$-pythagorean with at most two orderings ($\tilde{\phi}$ = unique extension of ϕ to Ω).*

The concept of Ω-henselian fields is developed in §1.

I wish to express my gratitude to E. Becker, who has influenced this work by many helpful remarks. After I had finished this manuscript, I learned of Ron Brown's valuation-theoretic characterization of superpythagorean fields (to appear in J. Algebra) which coincides partly with our Theorem (2.7).

§1. Valued Fields and Orderings

Let K be a field provided with a (Krull-)valuation ϕ. The objects: valuation ring, valuation ideal, group of units, residue class field, group of values and value will be denoted by R_ϕ, U_ϕ, K_ϕ, $|K|_\phi$ and $|\ |_\phi$ respectively. The group of values will be multiplicative. More generally, for a subset $A \subset K$ we denote by $|A|_\phi$ the set of values of A. An extension L, ψ over K, ϕ of valued fields is an extension of the

fields, where ψ is an extension of ϕ. Such an extension is called *immediate* iff the canonical imbeddings $K_\phi \to L_\psi$ and $|K|_\phi \to |L|_\psi$ are isomorphisms.

If L over K is an extension of fields, then any valuation of K can be extended to L. The valued field K, ϕ is called *henselian* iff there exists at most one extension of ϕ to any algebraic extension of K. More generally, let Ω be a normal algebraic extension of K (finite or infinite). Then K, ϕ will be called *Ω-henselian* iff ϕ admits at most on extension to Ω.

(1.1) Examples: a) Ω purely inseparable over $K \Rightarrow K$ is Ω-henselian ([24] F. Prop. 6).

b) Ω = separable or algebraic closure of K: Ω-henselian = henselian.

c) Ω = quadratic closure of K: Ω-henselian = 2-henselian.

(1.2) **Hensel's Lemma.** *Let K, ϕ be a valued field and Ω a normal algebraic extension of K. Then the following conditions are equivalent:*

a) *K, ϕ is Ω-henselian.*

b) *Let f be a polynomial of $R_\phi[x]$, $f \notin I_\phi |x|$, which splits over Ω. If $\bar{f}$ splits over K_ϕ in relative prime factors $\bar{g}$ and $\bar{h}$, then there exist polynomials g_1 and h_1 of $R_\phi[x]$ for which $f = g_1 h_1$; $\bar{g}_1 = \bar{g}$; $\bar{h}_1 = \bar{h}$ and $\deg g_1 = \deg \bar{g}$. (Here "bar" means factorizing by coefficients.)*

c) (Krasner). *Let f be a separable polynomial of $K[x]$ which splits over Ω. Then there exists an ε of $|K|_\phi$ such that any polynomial g of $K(x)$ splitting over Ω with $\|f-g\| < \varepsilon$ admits the same splitting field as f. (Here $\| \ \|$ means the maximal distance of coefficients with respect to $| \ |_\phi$.)*

Proof. The same arguments hold as for the case Ω = algebraic closure. Compare [24] F.

Let K, ϕ be a valued field, Ω a Galois extension of K and $\tilde{\phi}$ an extension of ϕ to Ω. The decomposition field $Z(\tilde{\phi})$, provided with the valuation ψ induced by $\tilde{\phi}$ has the following properties:

a) Z, ψ is Ω-henselian.

b) Z, ψ is an immediate extension of K, ϕ.

c) For $\Omega > L > K$, L, ζ a henselian extension of K, ϕ, there exists a monomorphism $K, \phi \to L, \zeta$ of valued fields. Such a field is called Ω-henselization of K. If there are no immediate extensions of K, ϕ between K and Ω, then K, ϕ is Ω-henselian. The converse is also true if $\operatorname{char} K_\phi$ is prime to $[\Omega : K]$ ([24] Theorem 2).

Let ϕ and ψ be valuations of K. ϕ is called finer then ψ (ψ coarser then ϕ), if $R_\phi \subset R_\psi$. For ϕ finer or coarser then ψ the two valuations are called comparable. Let ϕ be finer then ψ. ϕ induces a canonical valuation ϕ/ψ on K_ψ with isomorphism: $K_\phi \cong (K_\psi)_{\phi/\psi}$. Furthermore, we have

(1.3) **Lemma.** *K, ϕ is Ω-henselian if and only if K, ψ is Ω-henselian and $K_\psi, \phi/\psi$ is $\Omega_{\tilde{\psi}}$-henselian, where $\tilde{\psi}$ is the unique extension of ψ to Ω.*

For the proof compare [24] F. Prop. 9 and 10 and [20] Th. 10 Cor.

Two non-trivial valuations ϕ and ψ of K are called *independent* iff R_ϕ and R_ψ don't admit a common prime ideal $\neq \{0\}$, or equivalently, if R_ϕ and R_ψ generate K.

(1.4) **Proposition** (Schmidt [25]). *Let Ω over K be a prime-closed (Definition (1.7)) extension of fields with Ω-henselian independent valuations ϕ and ψ of K. Then Ω is purely inseparable over K.*

Proof. Let $f=x^n+a_{n-1}x^{n-1}+\cdots+a_0$ be separable of $K[x]$ and

$$g=x^n+b_{n-1}x^{n-1}+\cdots+b_0=(x-\beta_1)\ldots(x-\beta_n)$$

with arbitrary pairwise different elements $\beta_i\in K$. Independent valuations have the approximation property: For $0<\varepsilon$, $\delta\in|K|_\phi$ or $|K|_\psi$ there exists a polynomial $h=x^n+c_{n-1}x^{n-1}+\cdots+c_0$ of $K[x]$, such that $|c_i-a_i|_\phi<\varepsilon$ or $|c_i-b_i|_\psi<\delta$ respectively. By (1.2) f splits over Ω.

(1.5) **Corollary.** *Let Φ be a set of mutually incomparable Ω-henselian valuations of K. Suppose that $\#\Phi\geqq 2$ and that Φ is independent i.e. K is generated by all the valuation rings R_ϕ, $\phi\in\Phi$. Then Ω is purely inseparable over K.*

Proof. Case 1. For at least one $\phi\in\Phi$ there exists a non-trivial ψ of K, which is coarser then ϕ and of rank 1. By assumption we find a valuation $\zeta\in\Phi$ such that ψ and ζ are incomparable, hence independent. ψ is Ω-henselian, so we may apply (1.4).

Case 2. No $\phi\in\Phi$ can be coarsened by a non-trivial rank 1-valuation. Suppose, there exists a proper Galois extension L of K with $\Omega>L>K$. Say L is the decomposition field of the polynomial $f(x)=x^n+a_{n-1}x^{n-1}+\cdots+a_0\in K[x]$ with mutually different roots $\alpha_1,\ldots,\alpha_n\in L$. Consider a valuation $\phi\in\Phi$ with unique extension $\tilde{\phi}$ to Ω. We define $\alpha\in|\Omega|_{\tilde{\phi}}$ by $\alpha:=\max_{k,i\neq j}\{|a_k|_{\tilde{\phi}},|\alpha_i-\alpha_j|_{\tilde{\phi}}^{-1}\}$. There exists a $\beta\in|K|_\phi\subset|\Omega|_{\tilde{\phi}}$ with $\beta\geqq\alpha$. Let ψ be the coarsening of ϕ for which $|K|_\psi=|K|_\phi/\langle\beta\rangle$ where $\langle\beta\rangle$ is the convex subgroup generated by β. Since we are in case 2, ψ is not trivial. Let ψ_1 be the unique extension of ψ to L and $\tilde{\psi}$ the unique extension of ψ_1 to Ω. By construction L_{ψ_1} is separable over K_ψ, and hence $\Omega_{\tilde{\psi}}$ is not purely inseparable over K_ψ; but this leads to a contradiction:

There exists a $\zeta\in\Phi$ such that ζ and ψ are incomparable. Let ξ be the finest common coarsening of ζ and ψ with unique extension $\tilde{\xi}$ to Ω. Then ψ/ξ and ζ/ξ are independent $\Omega_{\tilde{\xi}}$-henselian valuations of K_ξ, thus by (1.4) $\Omega_{\tilde{\xi}}$ is purely inseparable over K_ξ.

We want to study how the property "henselian" behaves under algebraic extensions. Of course, we have:

(1.6) **Proposition.** *Let Ω over K be a normal extension of fields. Then for $\Omega>L>K$ and L, ψ over K, ϕ an extension of valuated fields L, ψ is Ω-henselian if K, ϕ is so.*

For the converse we need one more condition: Recall that a field K is called *p-closed* for a prime p if K admits no Galois p-extension. The following definition is fundamental for this paper:

(1.7) **Definition.** An extension Ω over K of fields is called (weakly) *prime-closed*, if it is algebraic Galois and if for p prime with $p|[\Omega:K]$ each separable polynomial of degree p splits over Ω (Ω admits no normal p-extension).

(1.8) *Examples.* Let Ω be the maximal p-extension for some prime p, or the algebraic closure or the maximal solvable extension of K, then Ω is weakly prime-

closed over K. The importance of the above definition arises from the following theorem, which generalizes a well known result of Artin [2].

(1.9) **Theorem** (Becker [3]). *If for some prime p the maximal p-extension K_p of a field K has finite degree over K, then $K=K_p$, or K is real, $p=2$ and $K_p=K(\sqrt{-1})$.*

In the latter case, K is a so called euclidean field, i.e. K is ordered and all positive elements are squares. If Ω over K is prime-closed of finite degree, then $\Omega=K$ or $\Omega=K(\sqrt{-1})$ with K euclidean.

(1.10) **Proposition.** *Let L, ψ over K, ϕ be a finite extension of valuated fields and suppose that Ω is prime-closed over K with $\Omega>L>K$. If L, ψ is Ω-henselian, then also K, ϕ, or for each extension $\tilde{\phi}$ of ϕ to Ω the field $\Omega_{\tilde{\phi}}$ is purely inseparable over K_ϕ.*

Proof. Let N be the normal closure of L over K in Ω, and let ϕ_1 be the unique extension of ψ to N. So ϕ_1 is Ω-henselian, and we have reduced the problem to the case where L over K is Galois.

If ψ is the only extension of ϕ to L, we are through. Otherwise, let $\psi=\psi_1, \psi_2, \ldots, \psi_r$, $r\geqq 2$, be the different extensions of ϕ to L. The Galois group $g(L, K)$ acts transitively on these ([24] F. Theorem 1). Therefore, all ψ_i are Ω-henselian and the finest common coarsening ζ of these too. For the restriction ξ of ζ on K ζ is the unique extension of ξ to L, since ζ is fixed under $g(L, K)$. Therefore K, ξ is Ω-henselian.

The ψ_i are pairwise incomparable ([24] F. Prop. 1 Cor. 5). So the ψ_i/ζ form an independent system of $\Omega_{\tilde{\zeta}}$-henselian valuations of L_ζ, where $\tilde{\zeta}$ is the unique extension of ζ to Ω. By (1.5) $\Omega_{\tilde{\zeta}}$ is purely inseparable over L_ζ. Let S be the separable closure of K_ξ in L_ζ. Then $\Omega_{\tilde{\zeta}}$ is purely inseparable over S, and S is prime-closed over K_ξ as can be easily proved with [11] (27.1). By (1.9) $K_\xi=S$ or $S=K_\xi(\sqrt{-1})$ and K_ξ is euclidean. The first case cannot occur, for then we would have only one extension of ϕ/ξ to L_ζ. In the second case, either ϕ/ξ is S-henselian, hence ϕ/ξ is $\Omega_{\tilde{\zeta}}$-henselian too, and by (1.3) ϕ is Ω-henselian, or ϕ/ξ is not S-henselian: Now let $\tilde{\phi}$ be an extension of ϕ to Ω. Then $\tilde{\phi}/\tilde{\zeta}$ is an extension of ϕ/ξ to $\Omega_{\tilde{\zeta}}$. Call σ its restriction to S. Then $(K_\xi)_{\phi/\xi}=S_\sigma$, and, of course, $(\Omega_{\tilde{\zeta}})_{\tilde{\phi}/\tilde{\zeta}}$ is purely inseparable over S_σ. Now we are in the latter case of the assertion.

(1.11) **Corollary.** *Let L, ψ over K, ϕ be a finite extension of fields, and suppose, that L, ψ is henselian. Then K, ϕ is henselian or K_ϕ is separably closed.*

(1.12) **Corollary.** *Let L, ψ over K, ϕ be a finite extension of fields, and suppose, that L, ψ is 2-henselian. Then K, ϕ is 2-henselian or L_ψ is a finite extension of a separably quadratic closed field.*

Proof. If K, ϕ is not 2-henselian, consider a chain K, $\phi<F$, $\zeta<H$, $\eta<L$, ψ of valuated fields such that H, η is 2-henselian but F, ζ fails to be so, and such that there is no field properly between F and H. A polynomial of second degree $f\in F[x]$, which fulfills the conditions of Hensels lemma but which does not split over F, splits over H. H over F is separable of second degree. Let $\tilde{\eta}$ be the unique extension of η to the separable 2-closure Ω of H which is also the separable

2-closure of F. By (1.10) $\Omega_{\bar{\eta}}$ is purely inseparable over F_ζ. On the other hand $\Omega_{\bar{\eta}}$ is separably quadratic closed, from which the assertion follows.

Let us consider some connections between orderings and valuations of fields. Recall that the ordering P (positive cone) and the valuation ϕ of K are called *compatible* if $(P \cap U_\phi) + I_\phi \subset P$. P is compatible with ϕ means that P induces obviously an ordering $\bar{P}$ on K_ϕ. Conversely, take an ordering $\bar{P}$ on K_ϕ and a set $A \subset K$ such that the elements $|a|_\phi$, $a \in A$ represent a base of $|K|_\phi / |K|_\phi^2$. Prescribe the elements $a \in A$ and $x \in U_\phi$ with $\bar{x} \in \bar{P}$ to be positive. Then these data define an ordering on K, compatible with ϕ, and all compatible ordering can be constructed like that.

(1.13) *Remark.* ϕ is compatible with P if and only if R_ϕ is convex with respect to P.

For other equivalent conditions see [16].

There exists a finest valuation π which is compatible with P. Its valuation ring R_π is the set $\{a \in K;$ ex. $x \in \mathbb{Q}$ with $x + a \in P$ and $x - a \in P\}$.

Thus the valuations compatible with P form a chain. Their valuation rings can be formed as above, if we replace $\mathbb{Q}$ by fields F with $\mathbb{Q} \leqq F \leqq K$. P is archimedean if and only if its finest compatible valuation is trivial. A valuation ϕ of a field K is called *real* if K_ϕ is a real field.

(1.14) *Remark* (compare [15] Theorem 1.6 and Theorem 2.6). Let L over K be an algebraic extension of fields, and let ϕ be a real valuation of K. Then ϕ admits a real extension to L if and only if at most one ordering of K which is compatible with ϕ can be extended to L. Recall that a preordering S of a field K is a subset of K for which $S + S \subset S$, $S \cdot S \subset S$, $K^{*2} \subset S$ and $0 \notin S$. It is well known, that a preordering is the intersection of all orderings which lie over it.

(1.15) **Proposition.** *Let ϕ be a valuation and S a preordering of the field K. The following conditions are equivalent:*

a) $(S \cap U_\phi) + I_\phi \subset S$.

b) *ϕ is compatible with each ordering $P \supset S$.*

c) *To each quadratic extension L over K there exists at most one extension ψ of ϕ or $L = K(\sqrt{s})$ with $s \in S$.*

Proof. Obviously a) $\leftrightarrow$ b).

a) $\rightarrow$ c): Let L over K be a quadratic extension, which admits two extensions ψ and η of ϕ. Then, for instance, L, ψ is an immediate extension of K, ϕ by the fundamental inequality ([24] G. Prop. 4). Hence, we may suppose $L = K(\sqrt{s})$ with $s \in U_\phi$ and s is a square mod I_ϕ. By a) we have $s \in S$.

c) $\rightarrow$ a): Consider an element $b \in I_\phi$ and the polynomial $x^2 + x - b$. It splits over $K(\sqrt{\frac{1}{4} + b})$. Now either $\frac{1}{4} + b$ and so $1 + b$ is in S, or there is at most one extension of ϕ to $K(\sqrt{\frac{1}{4} + b})$. But then $\frac{1}{4} + b$ is a square in K by (1.2), since the polynomial $x^2 + x$ splits over K_ϕ in prime factors. Anyway, we get $1 + b \in S$ from which a) follows.

We call the preordering S and the valuation ϕ *compatible*, if the above conditions hold for them. If S is compatible with ϕ, then S induces a preordering

$\bar{S}$ on K_ϕ; thus K_ϕ is real. The set of those preorderings which are compatible with ϕ can be constructed as for the orderings above.

(1.16) *Remark.* ϕ is compatible with all orderings of K if and only if ϕ is compatible with the sums of squares in K. This happens, for instance, if ϕ is henselian or at most 2-henselian. A pythagorean valuated field K, ϕ is 2-henselian, if and only if ϕ is compatible with all orderings of K.

We conclude this paragraph by stating some remarks which show that Ω-henselian fields and valuated fields, which are compatible with some preordering, behave similar.

(1.17) **Proposition.** *Let ϕ, ψ be valuations of the field K, ψ coarser then ϕ, and let S be a preordering of K. Then S is compatible with ϕ, if and only if S is compatible with ψ and $\bar{S}$ is compatible with ϕ/ψ, where $\bar{S}$ is the induced preordering on K_ψ.*

There exists a finest valuation compatible with a preordering S. Its valuation ring is generated by all valuation rings of the finest valuations of the orderings P over S. Thus the valuations compatible with S form a chain.

(1.18) **Proposition.** *A valuation ϕ of a real field K is 2-henselian, if and only if it is compatible with all orderings of K and if to each extension $K(\sqrt{q})$ of K, q a sum of squares, there exists an extension of ϕ with the same property.*

Proof. If ϕ is 2-henselian, the above statement holds. Conversely let q be a sum of squares. Then under the above condition all extensions of ϕ to $K(\sqrt{q})$ are compatible with all orderings of $K(\sqrt{q})$, so they are comparable, but this can only occur if there is at most one extension of ϕ to $K(\sqrt{q})$ ([24] F. Prop. 1 Cor. 5). Now the assertion follows from (1.16).

(1.19) **Proposition.** *Let L, ψ over K, ϕ a quadratic extension of fields. Suppose that ϕ is compatible with some preordering S of K and that $T=\{\sum l_i^2 s_i;\ l_i\in L,\ s_i\in S\}$ is a preordering of L. Then ψ is compatible with T or $L=K(\sqrt{s})$ with $s\in S$.*

Proof. Suppose L is not of the form $K(\sqrt{s})$ for some $s\in S$. By (1.15) c) ψ is the only extension of ϕ to L. We consider an element $1+b$, $b\in I_\psi$. In order to prove that this lies in T we must know that its norm and trace lie in S. But now R_ψ is fixed under the involution (denoted by "bar") of L over K, hence

$$N_{L/K}(1+b)=1+b+\bar{b}+b\bar{b}\in 1+I_\phi \quad \text{and} \quad \mathrm{Tr}_{L/K}(1+b)=1+1+b+\bar{b}\in 2+I_\phi,$$

but $1+I_\phi\subset S$ and $2+I_\phi\subset S$ by assumption.

§ 2. Characterization of Fans

Let K be a field, T a preordering of K and $W(K)$ the Witt ring of symmetric bilinear forms over K. The diagonalized forms $\sum_{i=1}^{n} a_i x_i^2$, $a_i\in K$ will be denoted by $\langle a_1, \ldots, a_n\rangle$. Let $J(T)$ be the ideal of $W(K)$ which is generated by forms of the type

$\langle 1, -t\rangle$ with $t\in T$. The quotient

$$W_T(K) := W(K)/J(T)$$

is called the reduced Witt ring modulo the preordering T. Let X/T be the space of orderings of K which lie over T, and let $C(X/T, \mathbb{Z})$ be the ring of continuous functions $X/T\to\mathbb{Z}$, $\mathbb{Z}$ provided with the discrete topology. Then the signature σ extended to all orderings of X leads to a sequence

$$(2.1)\quad J(T)\to W(K)\xrightarrow{\sigma} C(X/T,\mathbb{Z})$$

which is exact by [5]. If we take for T the set of sums of squares, then $J(T)$ is just the torsion part of $W(K)$. So exactness of (2.1) means a slight generalization of Pfister's famous theorem [21].

Two symmetric bilinear forms $\rho=\langle a_1,\ldots,a_n\rangle$ and τ are called chain equivalent mod T if ρ can be transformed into τ by a chain of transformations of the following type:

(i) equivalent transformation;
(ii) replacing $\rho=\langle a_1, a_2, \ldots, a_n\rangle$ by $\langle t a_1, a_2, \ldots, a_n\rangle$ for $t\in T$.

In [5] it is also proven:

(2.2) **Proposition.** *ρ and τ are chain equivalent modulo T if and only if $\rho\equiv\tau \bmod J(T)$ in $W(K)$ and $\dim\rho=\dim\tau$.*

For a symmetric bilinear form $\rho=\langle a_1,\ldots,a_n\rangle$ set

$$T(\rho) := \{t_1 a_1+\cdots+t_n a_n;\ t_i\in T\}.$$

$T(\rho)$ is invariant under chain equivalence [5]. As a consequence of the work in [5] we get

(2.3) **Proposition.** *Let $\rho=\langle a_1,\ldots,a_n\rangle$, $n\leqq 3$, be a symmetric bilinear form over K, and let T be a preordering of K. Then there exist elements $t_1,\ldots,t_n\in T$ such that $t_1a_1+\cdots+t_na_n=0$, if and only if ρ is indefinite for all orderings $P\supset T$.*

Proof. One direction and the cases $n=1$ and $n=2$ are trivial. Furthermore, we may assume $a_1=1$. The round form $\tau := \langle 1, a_2, a_3, a_2a_3\rangle$ has signature 0 with respect to all orderings $P\supset T$. Hence τ is chain equivalent mod T to a hyperbolic form. In particular the form $\tau_1 := \langle t_1, t_2a_2, t_3a_3, t_4a_2a_3\rangle$ is isotropic for suitable elements $t_i\in T$; thus $\tau_1\cong\langle t_4a_2a_3, -t_4a_2a_3, a, -at\rangle$ with suitable elements $a, t\in K$, and compairing discriminants one shows that $t\in T$. By Witt's cancellation law we get

$$\langle t_1, t_2a_2, t_3a_3\rangle\cong\langle -t_4a_2a_3, a, -at\rangle$$

and deduce the assertion from the left-hand side. We want to study a special kind of preordering which we call a fan. In order to understand fans we describe $W(K)$ by generators and relations. Consider the group ring $\mathbb{Z}[K^*]$ and the epimorphism $\kappa\colon \mathbb{Z}[K^*]\to W(K)$ defined by $a\mapsto\langle a\rangle$. Its kernel is generated by

the relations

W_1: $1 \oplus -a^2 = 0$; $\quad a \in K^*$,

W_2: $a \oplus b = (a+b) \oplus (a+b)ab$; $\quad a, b, a+b \in K^*$,

where $\oplus$ designs the addition in $\mathbb{Z}[K^*]$ ([29] Satz 7). Let T be a preordering of the field K. For the sequence

$$\mathbb{Z}[K^*] \xrightarrow{\kappa} W(K) \xrightarrow{\rho} W(K)/J(T) = W_T(K).$$

We obtain generators of the kernel of $\rho \circ \kappa$ if we replace the relations W_1 by

W_1^T: $1 \oplus -t = 0$; $\quad t \in T$.

We are interested in those preorderings T for which the relations W_2 are generated from the relations W_1^T:

(2.4) **Proposition and Definition** (Becker-Köpping [5], Compare also [10, 26, 27]). *The following properties of a preordering T of a field K are equivalent:*

a) $T + aT \subset T \cup aT$ *for each* $a \in K^* \setminus -T$.

b) *Each subgroup* $U \subset K^*$ *with* $(K:U) = 2$, $T \subset U$ and $-1 \notin U$ *is an ordering of* K.

c) *The kernel of* $\rho \circ \kappa$ *is generated by the relations of type* W_1^T.

If these equivalent conditions hold for T, then T is called a fan.

(2.5) *Remark.* For a fan T and an ordering $P \supset T$ of K condition c) shows, that $W_T(K)$ is isomorphic to $\mathbb{Z}[P/T]$. Conversely let T be a preordering of K and suppose there is an isomorphism $\phi: W_T(K) \to \mathbb{Z}[H]$ with a suitable group H; then by Ware [27, 28] and Becker-Köpping [5] T is a fan, and the elements $a \in K$ with $\phi(\langle a \rangle) \in H$ form an ordering $P \supset T$.

(2.6) *Examples.* Let K be a field. Then:

a) Each ordering of K is a fan of K.

b) Each intersection of two orderings of K forms a fan in K.

c) Let T be a preordering and ϕ a valuation of K such that ϕ and T are compatible. Then T is a fan of K if and only if the induced preordering $\bar{T}$ is a fan of K_ϕ.

The fans of type a) and b) are called trivial. The above examples of fans are most general, for we will prove:

(2.7) **Theorem.** *Each fan T of a field K is compatible with some valuation ϕ such that the induced fan $\bar{T}$ on K_ϕ is trivial. Moreover, ϕ can be chosen in such a way that $|T|_\phi$ contains no convex subgroups of $|K|_\phi$.*

Before proving (2.7) let us state some remarks and lemmata:

(2.8) *Remark.* Let T be a fan of K. Then each preordering S with $K^* \supset S \supset T$ is a fan too. Now let L over K be a finite extension of fields, and let P be an ordering of K. If $T := \{\sum l_i^2 p_i;\ l_i \in L, p_i \in P\}$ is a fan of L, then the number of extensions of P to L is a power of 2.

(2.9) **Lemma.** *Let T be a fan and $P_1 \supset T$ an ordering of the field K. Let π_1 be the finest valuation compatible with P_1. If $|K|_{\pi_1} = |T|_{\pi_1}$, then T is trivial.*

Proof. Suppose, there exist – besides P_1 – two further orderings P_2, P_3 of K with $P_i \supset T$; $i=2,3$. There exists an element $a \in K^*$ with $a \in P_3$ but $a \notin P_1 \cup P_2$. We will construct an element $b \in K^*$ with $b \in P$ for all orderings $P \supset T$ for which $a \in P$ and such that b lies just in one of the orderings P_1, P_2. But this leads to a contradiction:

The form $\rho := \langle 1, a, -b \rangle$ is indefinite with respect to all orderings $P \supset T$, but that is not true for the form $\langle 1, a \rangle$. Hence, by (2.3) there exist $t_1, t_2 \in T$ with $b = t_1 + t_2 a$. By the definition of a fan we get: $b \in T \cup aT$, which is impossible.

For the construction of b we may suppose $a \in U_{\pi_1}$. Let π_2 be the finest valuation compatible with P_2.

Case 1. $a \in U_{\pi_2}$. The residue classes $a + I_{\pi_1}$ and $a + I_{\pi_2}$ define Dedekind cuts α_1 and $\alpha_2 \in \mathbb{R}$ in the fields K_{π_1} and K_{π_2} with their archimedean orderings induced by P_1 and P_2. If $\alpha_1 \neq \alpha_2$, say $\alpha_1 < \alpha_2 < 0$, then, if q is an element of the prime field $\mathbb{Q}$ of K with $\alpha_1 < q < \alpha_2$, the element $b := a - q$ is the required one. For $\alpha_1 = \alpha_2$ we chose an element $c \in P_1 \setminus P_2$. By assumption we may suppose $c \in U_{\pi_1} \cap R_{\pi_2}$. Now for a sufficiently large $n \in \mathbb{N}$ the element $a(1 + c/n)^2$ defines different Dedekind cuts at P_1 and P_2.

Case 2. $a \notin U_{\pi_2}$. We may suppose: $a \notin R_{\pi_2}$ (take a^{-1} eventually). Now take $b = a + n$ for a sufficiently large $n \in \mathbb{N}$.

(2.10) **Lemma.** *Let T be a fan and ϕ a valuation of the field K. If $|T|_\phi$ contains no convex subgroups $\neq \{1\}$ of $|K|_\phi$, then T is compatible with ϕ.*

Proof. For $t \in U_\phi \cap T$ and $b \in I_\phi$ assume first that $|b|_\phi \notin |T|_\phi$. Then $b \notin -T$, hence $t + b \in T \cup bT$; but $|t+b|_\phi = 1 \notin |bT|_\phi$, thus $t + b \in T$. Now let be $|b|_\phi \in |T|_\phi$. We choose an element $c \in K$ with $|b|_\phi < |c|_\phi < 1$ and $|c|_\phi \notin T_\phi$. Such an element c exists, for otherwise $|T|_\phi$ would contain a convex subgroup of $|K|_\phi$. But now

$$t + b = (t + c) + (b - c) = s + i - j = r$$

with suitable elements s and $r \in T$.

Proof of (2.7). Take an arbitrary ordering $P \supset T$ and its finest compatible valuation π. Denote by V the union of all convex subgroups of $|K|_\pi$ which lie in $|T|_\pi$, i.e. V is the largest subgroup with that property. We coarsen π by ϕ such that $|K|_\phi = |K|_\pi / V$. The largest convex subgroup of K_ϕ which lies in $|T|_\phi$ is $\{1\}$; hence ϕ is compatible with T by (2.10), and so ϕ is compatible with P. For the induced preordering $\bar{T}$ and the induced ordering $\bar{P}$ of K_ϕ the conditions of (2.9) are fulfilled, from which the assertion follows.

Among the real fields K we get two extreme cases: On the one hand it happens that the sums of squares form a fan in K. The pythagorean fields of this type are just the so called superpythagorean fields or strictly pythagorean fields [6, 10, 27]. On the other hand many real fields admit only trivial fans. These are just the fields which fulfill the *strong approximation property* (SAP): For any two disjoint closed subsets A, B of the space of orderings X of K there exists an element $a \in K^*$ with $a \in P$ for all $P \in A$ and $a \notin P$ for all $P \in B$. A real field K fulfills (SAP) if and only if the image under the signature map $\sigma\colon W(K) \to C(X, \mathbb{Z})$ is as large as possible, i.e. if it consists of all functions with even or odd values anywhere.

In [7] we introduced the *reduced stability index*[1] $\mathrm{st}_{\mathrm{red}}(K)$ to be the height of the 2-group $C(X, \mathbb{Z})/\sigma(W(K))$. It measures how far the field K is from being an (SAP) field. Its connection with the existence of non-trivial fans is given by the following formula:

(2.11) $\mathrm{st}_{\mathrm{red}}(K)+1=\sup_{n\in\mathbb{N}}\{\text{ex. a fan } T \text{ of } K \text{ with } (K^*:T)\geqq 2^n\}$.

Proof. Let $T\subset K$ be a fan with $(K^*:T)=2^n$. Choose ϕ and $\bar{T}$ according to (2.7). We set $2^r:=(|K|_\phi:|K|_\phi^2)$ and $2^s=(K_\phi^*:\bar{T})$, hence $r+s=n$. Now by (2.7) $s=1$ or $s=2$, and $s=2$ only if K_ϕ admits at least 2 orderings. Thus in any case $s\leqq \mathrm{st}_{\mathrm{red}}(K_\phi)+1$. For the henselian closure $\tilde{K}_\phi$ we get by [7] (3.18):

$$\mathrm{st}_{\mathrm{red}}(\tilde{K}_\phi)=\mathrm{st}_{\mathrm{red}}(K_\phi)+r;$$

hence by the global stability formula [7], (3.19):

$$\mathrm{st}_{\mathrm{red}}(K)+1\geqq \mathrm{st}_{\mathrm{red}}(\tilde{K}_\phi)+1=\mathrm{st}_{\mathrm{red}}(K_\phi)+1+r\geqq s+r=n.$$

Conversely, first we consider the case where $s(K)=\infty$. Then there exist real valuations ϕ of K for which $(|K_\phi|:|K_\phi|^2)$ is arbitrarily large, and thus there exist also fans of arbitrarily large index in K^*. For $\mathrm{st}_{\mathrm{red}}(K)<\infty$ we proceed by induction by $\mathrm{st}_{\mathrm{red}}(K)$ the assertion for $\mathrm{st}_{\mathrm{red}}(K)=0$ (K admits just one ordering) being obvious. Let $\mathrm{st}_{\mathrm{red}}(K)$ be >0 and let Ω' be the set of real valuations ϕ of K, for which $|K|_\phi\neq|K|_\phi^2$. If $\Omega'=\varnothing$, then $\mathrm{st}_{\mathrm{red}}(K)\leqq 1$ by [7] (3.19). But since $\mathrm{st}_{\mathrm{red}}(K)>0$, K admits at least 2 orderings; hence there exists a fan of index 4 in K^*. Now suppose $\Omega'\neq 0$ and let ϕ be $\in\Omega'$ with $\mathrm{st}_{\mathrm{red}}(K)=\mathrm{st}_{\mathrm{red}}(\tilde{K}_\phi)=r+\mathrm{st}_{\mathrm{red}}(K_\phi)$, where

$$2^r=(|K|_\phi:|K|_\phi^2) \quad \text{and} \quad \mathrm{st}_{\mathrm{red}}(K_\phi)<\mathrm{st}_{\mathrm{red}}(K).$$

By induction there exists a fan $\bar{T}$ of K_ϕ with $(K_\phi^*:\bar{T})=2^s$ and $s\geqq \mathrm{st}_{\mathrm{red}}(K_\phi)+1$. The fan $T:=K^{*2}\{t\in U_\phi;\ t+I_\phi\in\bar{T}\}$ has index 2^{r+s} in K, where

$$r+s\geqq r+\mathrm{st}_{\mathrm{red}}(K_\phi)+1\geqq \mathrm{st}_{\mathrm{red}}(K)+1.$$

§ 3. Ω-pythagorean Fields

For this paragraph we make the following general assumption: K is a real field and Ω a prime-closed extension of K such that $2|[\Omega:K]$.

(3.1) **Definition.** K is called *Ω-pythagorean* if each real extension L of K with $\Omega>L>K$ is pythagorean.

(3.2) *Remark.* If Ω is the 2-closure of K, then K is Ω-pythagorean just means that K is strictly pythagorean in the sense of [6] i.e. superpythagorean in the sense of [10]. Fields of this kind I first found in [8]. For different characterizations of them compare [8, 6, 10, 27, 28]. The most profound characterization is given by a theorem of Becker ([4] III Satz 1 and § 5), which holds for our general assumptions about Ω:

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[1] Reduced stability index $\mathrm{st}_{\mathrm{red}}(K)=$ *Stabilitätsgrad* $s(K)$ in [7]

(3.3) **Theorem** (Becker). *The following conditions are equivalent:*

a) *K is Ω-pythagorean.*

b) *$g(\Omega, K(\sqrt{-1}))$ is abelian.*

c) *Each non-real extension of K in Ω contains $\sqrt{-1}$.*

d) *Each real extension of K is intersection of its maximal real extensions in Ω.*

Moreover, if K is Ω-pythagorean, for each prime p there is given a cardinality α_p such that

$$g(\Omega, K(\sqrt{-1})) \cong \prod_{\text{primes}} \mathbb{Z}_p^{\alpha_p},$$

where $\mathbb{Z}_p$ is the additive group of p-adic numbers. 2^{α_2} is the number of orderings of K.

Of course, $g(\Omega, K)$ is a split extension of $g(\Omega, K(\sqrt{-1}))$ with the group of 2 elements which acts by inversion on $g(\Omega, K(\sqrt{-1}))$.

As a consequence of Becker's results we get

(3.4) **Corollary.** *The following conditions are equivalent:*

a) *K is Ω-pythagorean and admits at most 2 orderings.*

b) *$g(\Omega, K(\sqrt{-1}))$ is abelian with cyclic 2-Sylow-subgroup.*

c) *Each real extension of K in Ω admits at most 2 orderings.*

This has been independently proved by the present author. To get from c) to a) we use the fact that a real field is pythagorean if and only if its pythagorean closure has finite degree over it [8].

Now let us state our characterization of Ω-pythagorean fields by means of valuations.

(3.5) **Proposition.** *The following conditions are equivalent:*

a) *K is Ω-pythagorean.*

b) *K admits an Ω-henselian valuation ϕ such that K_ϕ is $\Omega_{\tilde{\phi}}$-pythagorean with at most two orderings ($\tilde{\phi}$ = unique extension of ϕ to Ω).*

If $\Omega = 2$ closure, then K_ϕ is intersection of at most 2 euclidean fields. I do not know whether in general ϕ can be chosen such that K_ϕ is intersection of two in $\Omega_{\tilde{\phi}}$ maximally ordered fields.

In the proof of (3.5) we shall use

(3.6) **Lemma.** *For a real field K the following conditions are equivalent.*

a) *K is pythagorean and admits at most 2 orderings.*

b) *K is Ω-pythagorean for $\Omega = 2$-closure and admits at most two orderings.*

c) *K is intersection of at most 2 euclidean closures (in some 2-closure of K).*

Proof. a) $\rightarrow$ b): This is clear if K admits only one ordering (K being euclidean). Otherwise, let P_1 and P_2 be the orderings of K and say $p \in P_1 \setminus P_2$. Then $K(\sqrt{p})$ admits also two orderings, namely the extensions of P_1. Let $z := a + b\sqrt{p} \in K(\sqrt{p})$ be totally positive. Then $a \in P_1$ and $a^2 - b^2 p \in P_1 \cap P_2 = K^{*2}$. If $b = 0$, clearly

$z \in K(\sqrt{p})^2$. Otherwise $z \in K(\sqrt{p})^2$, if and only if the polynomial $x^4 - ax^2 + \frac{p}{4}b^2$ admits a root in K. We get $x^2 = \frac{1}{2}(a + \sqrt{a^2 - pb^2})$. Chosing that square root of $a - pb^2$, which lies in P_2, we get $a + \sqrt{a^2 - pb^2}$ to be totally positive in K_ϕ, hence it is a square in K_ϕ and thus $K(\sqrt{p})$ is pythagorean.

b) $\rightarrow$ a): Obvious. By Beckers theorem (3.3) we get b) $\leftrightarrow$ c). One can also prove directly a) $\leftrightarrow$ c) : c) $\rightarrow$ a) being obvious, we consider in some 2-closure Ω of K the Frattini field Φ of the extension Ω over K. It is easily seen that $[\Phi : K] = 4$; hence, $g(\Phi, K)$ is a 2-dimensional vector space over $\mathbb{F}_2$. Since K is pythagorean, it is intersection of its euclidean closures in Ω, thus $g(\Omega, K)$ is generated by involutions. Hence by a well-known theorem of Burnside ([14], p. 273) $g(\Omega, K)$ is generated by two involutions.

Proof of (3.5). Obviously b) $\rightarrow$ a).

a) $\rightarrow$ b): We proceed in 5 steps:

(1) We first assume that $\Omega = 2$-closure of K. Then K is pythagorean, and by [8] K^{*2} is a fan of K. We shall only use this fact, thus proving at the same time that "strictly pythagorean", "superpythagorean" and "Ω-pythagorean" for $\Omega = 2$-closure are equivalent properties of fields.

For the fan K^{*2} we chose a valuation ϕ according to (2.7). By (1.16) ϕ is 2-henselian. K_ϕ is a pythagorean field with at most two orderings, and for the unique extension $\tilde{\phi}$ of ϕ to Ω, the residue field $\Omega_{\tilde{\phi}}$ is the 2-closure of K_ϕ. Hence, by (3.6) K_ϕ is $\Omega_{\tilde{\phi}}$-pythagorean.

(2) Now suppose that the general assumptions of this paragraph hold for Ω and K. Let Z be a 2-Sylow field between Ω and K. Then Z is real and Ω is the 2-closure of Z. We chose a valuation ζ of Z according to the first part of the proof. Then ζ is Ω-henselian and Z_ζ admits at most two orderings.

Let κ be the restriction of ζ on K. We first assume, that κ is Ω-henselian too. Now set $\phi = \kappa$. Since K_ϕ has char 0, Z_ζ has odd degree over K_ϕ; hence K_ϕ admits at most two ordering too.

(3) Now we suppose that κ is not Ω-henselian. For $L = Z(\sqrt{-1})$ let Λ be the set of extensions of κ to L. Since, by (3.3), $g(\Omega, K(\sqrt{-1}))$ is abelian, L is normal over K. The unique extension of ζ to L lies in Λ and is Ω-henselian. Hence, all valuations $\in \Lambda$ are Ω-henselian too. Let ψ be the finest common coarsening of all $\lambda \in \Lambda$ and ϕ the restriction of ψ on K.

Of course (apply (1.3)), ψ is Ω-henselian, and by [24] F. Theorem 1 this is also true for φ. We have to show that K_ϕ admits at most two orderings. In fact we claim:

(4) K_ϕ is euclidean. In order to show this, consider the set $\{\lambda/\psi, \lambda \in \Lambda\}$ of valuations on L_ψ. It is independent, and by (1.3) all λ/ψ are $\Omega_{\tilde{\psi}}$-henselian, where $\tilde{\psi}$ is the unique extension of ψ to Ω. Since L_ψ has char 0, (1.5) tells us, that $L_\psi = \Omega_{\tilde{\psi}}$. In particular, L_ψ is 2-closed. Since ϕ is Ω-henselian, L_ψ has odd degree over $K_\phi(\sqrt{-1})$. Therefore $K_\phi(\sqrt{-1})$ is 2-closed too, and so K_ϕ admits at most two square classes, i.e. one ordering.

(5) K_ϕ is $\Omega_{\tilde{\phi}}$-pythagorean for the unique extension $\tilde{\phi}$ of ϕ to Ω.

We omit the trivial proof.

(3.7) *Remark.* In step (1) we didn't use Becker's theorem (3.3). In fact, if $\Omega =$ 2-closure of K and K is Ω-pythagorean, it can be deduced from (3.5) and (3.6), that $g(\Omega, K(\sqrt{-1}))$ is abelian: Let ϕ be the valuation of (3.5). Extend it to $K(\sqrt{-1})$ and consider the sequence of groups

$$\{1\} = V < T \lhd Z = g(\Omega, K(\sqrt{-1}))$$

where V is the ramifaction group, T the inertia group and Z the decomposition group of the Galois group $g(\Omega, K(\sqrt{-1}))$ with respect to ϕ. $V=1$ because $\operatorname{char} K_\phi = 0$ and $Z = g(\Omega, K)$ since ϕ is Ω-henselian. T/V is isomorphic to the topological group of characters of $|\Omega|_\phi / |K(\sqrt{-1})|_\phi$ with values in the quadratic closure of K_ϕ. In fact, the values are roots of unity of 2-power order, so they lie in $K_\phi(\sqrt{-1})$, since K_ϕ is pythagorean. But this means that the action of Z/T on T is trivial, i.e. T lies in the center of Z. $Z/T = g(\Omega_\phi, K_\phi(\sqrt{-1}))$ is cyclic by (3.6), hence T is abelian.

All known properties of superpythagorean fields [6, 8, 24, 25] can easily be derived from (3.3) and (3.5). We conclude this work with two further examples.

(3.8) **Proposition.** *Let Ω over K be a prime-closed extension of fields such that 2 divides $[\Omega : K]$. If there exists a finite extension L of K with $K < L < \Omega$, which is Ω-pythagorean, then K is Ω-pythagorean, too.*

This was independently proved by Becker [4] and the present author. We sketch a proof: The case: $[L : K]$ odd is easy, using the fact that a field is pythagorean if it admits a finite pythagorean extension [8]. If $[L : K]$ is even, consider the normal closure N of L over K. By (3.3) we have an Ω-pythagorean field M with $L < M < N$ and $[N : M] = 2$ and thus a 2-Sylow field S with $K < S < M < N$, hence a reduction to the case $L = K(\sqrt{a})$. Now let F be a real field with $K < F < \Omega$. If $F(\sqrt{a})$ is real, it is pythagorean by assumption and so F is pythagorean too. Otherwise, by (3.3) $g(\Omega, F)$ is abelian; hence, F is again pythagorean.

(3.9) **Proposition.** *Let K be a field with finitely many orderings, and let Ω over K be a prime-closed extension of fields, such that 2 divides $[\Omega : K]$. Then the following conditions are equivalent:*

a) *K is Ω-pythagorean.*

b) *Each finite real extension L with $K < L < \Omega$ admits exactly as many orderings as K.*

c) *Each extension L with $K < L < \Omega$ admits at most finitely many orderings.*

Proof. b) $\rightarrow$ c) and c) $\rightarrow$ a) are easy.

c) $\rightarrow$ b): For a real field K call $o(K)$ the number of orderings. Choose an Ω-henselian valuation ϕ of K according to (3.5). Then each ordering of K is compatible with ϕ; hence $o(K) = (|K|_\phi : |K|_\phi^2)\, o(K_\phi)$ where $o(K_\phi) \in \{1, 2\}$. Let ψ be the unique extension of ϕ to L. Then $(|L|_\psi : |L|_\psi^2) = (|K|_\phi : |K|_\phi^2)$ by [7], (3.2). On the other hand, by (3.4) and (1.9) we get $o(L_\psi) = o(K_\phi)$, which proves the assertion.

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