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Riemann's Mapping Theorem for n-Analytic Functions

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1. Introduction and Results

A function w is called "n-analytic" ($n \in \mathbb{N}$) in the unit disc $\mathscr{D} := \{z \mid |z| < 1\}$ if it has a representation[1]

$$w = F_n g := \frac{\Delta^n[g(z)(1-|z|^2)^{-1}]}{\Delta^n[(1-|z|^2)^{-1}]} \tag{1.1}$$

with $g(z)$ analytic in $\mathscr{D}$. Since[2]

$$\Delta^n[(1-|z|^2)^{-1}] = (n!)^2 P_n\left(\frac{1+|z|^2}{1-|z|^2}\right)(1-|z|^2)^{-n-1} \neq 0, \qquad z \in \mathscr{D},$$

these functions are well defined in $\mathscr{D}$. A n-analytic function is called "n-conformal" (in $\mathscr{D}$) if it is a (global) homeomorphism of $\mathscr{D}$. In the present paper we shall exhibit the main properties of n-conformal mappings.

The study of n-conformal mappings has been motivated by at least two extraordinary relations between n-analytic and analytic functions. The first one arises from the identity (generalisation of Poisson's integral)

$$\Delta^n[g(z)(1-|z|^2)^{-1}] = \frac{(n!)^2}{2\pi}\int_{-\pi}^{\pi} \frac{g(e^{i\phi})}{|e^{i\phi}-z|^{2n+2}}\,d\phi,$$

which holds if g is continuous in $\operatorname{cl}\mathscr{D}$ (and analytic in $\mathscr{D}$). The second one is as follows: The general solution of the equation

$$\Delta w + \left[\frac{1}{2|z|}\frac{d}{d|z|}\log P_n\left(\frac{1+|z|^2}{1-|z|^2}\right)\right](z w_z + \bar{z} w_{\bar{z}}) = 0, \qquad z \in \mathscr{D}, \tag{1.2}$$

is $w = w_1 + \bar{w}_2$ where w_1 and w_2 are n-analytic in $\mathscr{D}$. These functions are called n-harmonic and obviously the 0-harmonic functions are the usual harmonic

[1] $\Delta^0 := \mathrm{Id}$; $\Delta := \frac{\partial^2}{\partial_z \partial_{\bar{z}}}$, $\Delta^k = \Delta(\Delta^{k-1})$, $k \in \mathbb{N}$

[2] P_n is the Legendre polynomial of degree n

functions. We believe that our main result holds for a large class of equations of the same type as (1.2).

It should be remarked that w is n-harmonic if and only if

$$W(z) := w(z)\, P_n\left(\frac{1+|z|^2}{1-|z|^2}\right) \tag{1.3}$$

is a solution of

$$(1-|z|^2)^2\, \Delta W - n(n+1)\, W = 0, \qquad z \in \mathscr{D}. \tag{1.4}$$

The simple but very useful representation (1.1) of the solutions of (1.4) via (1.3) is essentially due to Bauer [1] and was the beginning of a detailed study of those partial differential equations whose solutions admit representations by means of "differential operators".

Mapping properties of n-analytic functions have been considered for the first time in [8, 9] where the leading role of "convexity" has already been discovered. Our Main Theorem below which in a certain sense is Riemann's mapping theorem for n-conformal mappings unifies previous results in this field and settles (in part) a conjecture made in [8].

Main Theorem. i) *Let w be a n-conformal mapping in $\mathscr{D}$. Then $w(\mathscr{D})$ is a convex domain in $\mathbb{C}$.*

ii) *Let $\mathscr{G}$ be an arbitrary convex domain in $\mathbb{C}$, $\mathscr{G} \neq \mathbb{C}$, and let $w_0 \in \mathscr{G}$. Then there exists a unique determined n-conformal mapping w with*

1) $w(\mathscr{D}) = \mathscr{G}$,
2) $w(0) = w_0$.
3) $w_z(0) > 0$.

Actually we have stronger results, in particular concerning the interaction of a n-conformal mapping $w = F_n g$ and its "associate function" g, more technical in their nature. We mention two of them.

Theorem 1. *$w = F_n g$ is n-conformal in $\mathscr{D}$ if and only if g is an univalent analytic function in $\mathscr{D}$ which maps $\mathscr{D}$ onto a convex domain. In this case $g(\mathscr{D}) = w(\mathscr{D})$.*

For $R > 0$ let $\mathscr{D}_R := \{z \mid |z| < R\}$.

Theorem 2. *Let w be n-conformal in $\mathscr{D}$. Then, for $0 < R \leqq 1$, the sets $w(\mathscr{D}_R)$ are convex domains.*

n-conformal mappings in $\mathscr{D}$ need not be quasiconformal in $\mathscr{D}$ as the 1-conformal mapping $w = \dfrac{z}{1+|z|^2}$ shows. It seems not unlikely, however, that n-conformal mappings are locally uniformly quasiconformal in $\mathscr{D}$. We shall not treat this problem here.

Finally we should mention one result concerning the de la Vallée Poussin polynomials which is needed for the proof of our Main Theorem but may deserve some interest for itself: It is a well known fact (see Pólya-Schoenberg [3]) that the functions

$$V_n(z) = \sum_{k=1}^{n} \binom{2n}{n+k} z^k \tag{1.5}$$

are univalent in $\mathscr{D}$ and map $\mathscr{D}$ onto convex domains. Furthermore these polynomials possess some important extremal properties within the class of convex univalent analytic functions. We shall prove

Theorem 3. *Let $0<x<\infty$, $n\in\mathbb{N}$. Then there exists $\rho=\rho(x,n)>1$, such that the function*

$$f(z):=\sum_{k=1}^{n}\binom{n}{k}x^k V_k(z)$$

is univalent in $\mathscr{D}_\rho$ and maps $\mathscr{D}_\rho$ onto a convex domain.

For the proof of Theorems 1, 2 and the Main Theorem in Section 6 we require quite a lot of Lemmata, appearing in Sections 2–5. Theorem 3 will be proved in Section 7.

In the sequel we shall denote by $\mathfrak{A}$ the set of all analytic functions in $\mathscr{D}$. clco $\mathscr{M}$ is defined to be the closed convex hull of a plane set $\mathscr{M}$.

2. Different Representations

For $g\in\mathfrak{A}$ we have

$$F_n g=\sum_{k=0}^{n} g^{(k)}(z)\frac{n!}{(n-k)!}\frac{\Delta^{n-k}[z^k(1-|z|^2)^{-k-1}]}{\Delta^n[(1-|z|^2)^{-1}]}.$$

Using the fact that for every real-valued $h(t)\in C^\infty[0,1)$ there exists a real-valued $\tilde{h}(t)\in C^\infty[0,1)$ such that

$$\Delta z^k h(|z|^2)=z^k\tilde{h}(|z|^2)$$

we deduce

$$\frac{n!}{(n-k)!}\frac{\Delta^{n-k}[z^k(1-|z|^2)^{-k-1}]}{\Delta^n[(1-|z|^2)^{-1}]}=z^k(1-|z|^2)^k B_{nk}(|z|^2),$$

where $B_{nk}(t)$ is a rational function without any pole in $[0,1]$. Hence

$$F_n g=g(z)+\sum_{k=1}^{n} B_{nk}(|z|^2)(1-|z|^2)^k z^k g^{(k)}(z). \tag{2.1}$$

(2.1) implies the existence of real-valued rational functions $d_{nk}(t)$, $k\in\mathbb{N}$, such that with $z=re^{i\phi}$

$$F_n z^k=e^{ik\phi}d_{nk}(r). \tag{2.2}$$

A straightforward calculation gives

$$\lim_{r\to 0}\frac{d_{nk}(r)}{r^k}=\binom{n+k}{k},\qquad k\in\mathbb{N}. \tag{2.3}$$

Lemma 2.1. *Near $r=1$ we have*

$$d_{nk}(r)=1-\frac{k^2}{4n-2}(1-r)^2+o[(1-r)^2]. \tag{2.4}$$

Proof. $d_{nk}(1)=1$ is a consequence of (2.1). Since d_{nk} is analytic at $r=1$, we can put

$$d_{nk}(r)=1+C(1-r)^s+(1-r)^{s+1}P(1-r), \tag{2.5}$$

where $C\in\mathbb{R}\setminus\{0\}$, $s\in\mathbb{N}$ and P represents a power series of its argument. Since $e^{ik\phi}d_{nk}(r)$ is n-analytic, it solves (1.2), and thus we are led to the ordinary differential equation

$$d''_{nk}(r)+\left[\frac{1}{r}+2A(r)\right]d'_{nk}(r)-\frac{k^2}{r^2}d_{nk}(r)=0, \tag{2.6}$$

where

$$A(r)=\frac{d}{dr}\log P_n\left(\frac{1+r^2}{1-r^2}\right)=\frac{n}{1-r}+\tilde{P}(1-r),$$

with a certain power series $\tilde{P}$. Inserting (2.5) into (2.6) and equating coefficients we deduce $s=2$, $C=k^2/(2-4n)$ and hence the assertion.

Let $g(z)=\sum_{k=0}^{\infty}a_k z^k\in\mathfrak{A}$. The absolute convergence of this series and the linearity of the operator F_n shows

$$F_n g=\sum_{k=0}^{\infty}a_k F_n z^k=\sum_{k=0}^{\infty}a_k d_{nk}(t)\,z^k.$$

For each fixed $t\in(0,1)$ we associate with $F_n g$ the function

$$f_g(z,t):=\sum_{k=0}^{\infty}a_k d_{nk}(t)\,z^k. \tag{2.7}$$

Obviously, we have $f_g(e^{i\phi},t)=(F_n g)(e^{i\phi}t)$, and from (2.1)

$$f_g(z,t)=g(zt)+\sum_{k=1}^{n}B_{nk}(t^2)(1-t^2)^k(tz)^k g^{(k)}(tz). \tag{2.8}$$

Lemma 2.2. *Let $g\in\mathfrak{A}$. Then for $t\in(0,1)$ the function $f_g(z,t)$ is analytic in $\mathscr{D}_{1/t}$. Furthermore $\lim_{t\to 1-0}f_g(z,t)=g(z)$ locally uniformly in $\mathscr{D}$.*

Finally, we mention a relation which has been proved in [8, Satz 2, 1] in a slightly different form.

Lemma 2.3. *For $g\in\mathfrak{A}$ we have*

$$\Delta^n[g(z)(1-|z|^2)^{-1}]=\frac{n!}{(1-|z|^2)^{2n+1}}\frac{\partial^n}{\partial x^n}\left[(x+z)^n(1+\bar{z}x)^n g\left(\frac{x+z}{1+\bar{z}x}\right)\right]\Bigg|_{x=0}.$$

3. Application of the Maximum Principle

Nonconstant real-valued n-harmonic functions cannot take a maximum in a domain of $\mathscr{D}$ since they are solutions of the elliptic equation (1.2) and thus subject to Hopf's maximum principle (compare [6, p. 61]). The second half of Lemma 3.1

is an immediate consequence of this fact and the invariance of the set of n-analytic functions against translations and rotations of the images.

Lemma 3.1. *Let $g \in \mathfrak{A}$, $w = F_n g$ and $f_g(z, t)$ be as in (2.7). Then, for $t \in (0, 1)$, we have*

$$f_g(\mathrm{cl}\,\mathscr{D}, t) \subset w(\mathrm{cl}\,\mathscr{D}_t), \tag{3.1}$$

$$w(\mathscr{D}_t) \subset \mathrm{int}\,\mathrm{clco}\, f_g(\mathscr{D}, t). \tag{3.2}$$

Proof. It remains to show (3.1). Let $w_0 \in f_g(\mathscr{D}, t) \setminus f_g(\mathrm{Bd}\,\mathscr{D}, t)$. Then

$$0 \neq \int_{|z|=1} d \arg[f_g(z, t) - w_0] = \int_{|z|=t} d \arg[w(z) - w_0],$$

since f_g and w take the same values on $\mathrm{Bd}\,\mathscr{D}$ and $\mathrm{Bd}\,\mathscr{D}_t$ respectively. But w is continuous in $\mathrm{cl}\,\mathscr{D}_t$ and thus we can conclude that w takes w_0 in $\mathscr{D}_t$ at least once (compare [11, p. 375]).

(3.1) and (3.2) show the significance of convex sets $f_g(\mathscr{D}, t)$ in our theory. We have to study this fact more closely.

A function $h \in \mathfrak{A}$ maps $\mathscr{D}$ univalently onto a convex domain if and only if $h'(0) \neq 0$ and

$$\mathrm{Re}\left(\frac{z h''(z)}{h'(z)} + 1\right) \geqq 0, \qquad z \in \mathscr{D}. \tag{3.3}$$

We shall denote the class of these functions by $\mathfrak{K}$.

Lemma 3.2. *Let $g \in \mathfrak{A}$, $w = F_n g$ and assume $f_g(z, t) \in \mathfrak{K}$ for $t \in (0, 1)$. Then w is n-conformal in $\mathscr{D}$ and $g(\mathscr{D}) = w(\mathscr{D})$ is a convex domain.*

Proof. 1) Let $\gamma_t := w(\mathrm{Bd}\,\mathscr{D}_t) = f_g(\mathrm{Bd}\,\mathscr{D}, t)$. Since $f_g(z, t) \in \mathfrak{K}$ and analytic in $\mathrm{cl}\,\mathscr{D}$ it becomes clear that γ_t are analytic Jordan curves for $t \in (0, 1)$. Hence w is univalent on the circles $\mathrm{Bd}\,\mathscr{D}_t$, $t \in (0, 1)$, and it remains to show $\gamma_{t_1} \cap \gamma_{t_2} = \varnothing$ for $t_1 < t_2$. But (3.2) yields

$$\gamma_{t_1} \subset w(\mathrm{clco}\,\mathscr{D}_{t_1}) \subset w(\mathscr{D}_{t_2}) \subset \mathrm{int}\,\mathrm{clco}\, f_g(\mathscr{D}, t_2),$$

and the assertion follows from the convexity of $f_g(\mathscr{D}, t_2)$ and the relation

$$\gamma_{t_2} = \mathrm{Bd}\,\mathrm{clco}\, f_g(\mathscr{D}, t_2).$$

2) The above considerations show $f_g(\mathscr{D}, t_1) \subset f_g(\mathscr{D}, t_2)$, $0 < t_1 < t_2 < 1$. Since $\lim_{t \to 1-0} f_g(z, t) = g(z)$ locally uniformly in $\mathscr{D}$ Carathéodory's kernel theorem (compare [5, p. 297]) implies

i) $f_g(\mathscr{D}, t) \subset g(\mathscr{D})$, $t \in (0, 1)$,
ii) $\lim_{t \to 1} f_g(\mathscr{D}, t) = g(\mathscr{D})$.

Hence $g \in \mathfrak{K}$ and the relation $f_g(\mathscr{D}, t) = w(\mathscr{D}_t)$ proves the assertion.

4. Application of Convolution Properties

The *Hadamard product* or *convolution* of two functions

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \qquad g(z) = \sum_{k=0}^{\infty} b_k z^k$$

in $\mathfrak{A}$ is defined to be

$$(f*g)(z)=\sum_{k=0}^{\infty} a_k b_k z^k$$

and obviously represents a member of $\mathfrak{A}$. $u(z)=\dfrac{1}{1-z}$ is the unity of this operation and hence

$$f_g(z,t)=(g*f_u(\cdot,t))(z) \tag{4.1}$$

for each $g\in\mathfrak{A}$, $t\in(0,1)$.

A function $g\in\mathfrak{A}$ is called *close-to-convex* if and only if there exists a function $h\in\mathfrak{K}$ such that

$$\operatorname{Re}\frac{g'(z)}{h'(z)}>0, \qquad z\in\mathscr{D}. \tag{4.2}$$

Close-to-convex functions are known to be univalent in $\mathscr{D}$. We denote the set of these functions by $\mathfrak{C}$.

Let $\mathfrak{A}_0$ be the class of all conformal automorphisms of $\mathscr{D}$, i.e.

$$\mathfrak{A}_0=\left\{\phi\in\mathfrak{A}\,|\,\phi(z)=e^{i\alpha}\frac{z-z_0}{1-\bar{z}_0 z},\ \alpha\in\mathbb{R},\ z_0\in\mathscr{D}\right\}.$$

From the geometrical facts and since $\phi_0(z):=z$ is an element of $\mathfrak{A}_0$ we see that $g\in\mathfrak{K}$ if and only if $g\circ\phi\in\mathfrak{K}$ for each $\phi\in\mathfrak{A}_0$. This combined with (4.2) shows that $\mathfrak{C}$ has the same invariance property: $g\in\mathfrak{C}$ if and only if $g\circ\phi\in\mathfrak{C}$ for each $\phi\in\mathfrak{A}_0$

Lemma 4.1 (Extended Pólya-Schoenberg conjecture [10]). *Let $g\in\mathfrak{A}$. Each one of the following two conditions is necessary and sufficient for g to be a member of $\mathfrak{K}$:*

$$g*h\in\mathfrak{K} \quad \textit{for each } h\in\mathfrak{K}, \tag{4.3}$$

$$(g*h)'(z)\neq 0, \quad z\in\mathscr{D}, \quad \textit{for each } h\in\mathfrak{C}. \tag{4.4}$$

Lemma 4.2. *Let $g\in\mathfrak{C}$. Then we have $F_n(zg'(z))\neq 0$ for $0<|z|<1$.*

Proof. Lemma 2.3 gives

$$F_n(zg')=\frac{n!(1-|z|^2)^{-(2n+1)}}{\Delta^n[(1-|z|^2)^{-1}]}\frac{\partial^n}{\partial x^n}\left[(x+z)^{n+1}(1+\bar{z}x)^{n-1}g'\left(\frac{x+z}{1+\bar{z}x}\right)\right]\Bigg|_{x=0}.$$

Since $\phi(x):=\dfrac{x+z}{1+\bar{z}x}\in\mathfrak{A}_0$, we can conclude that $g\in\mathfrak{C}$ if and only if $h:=g\circ\phi\in\mathfrak{C}$, and therefore it will be enough to prove

$$\frac{\partial^n}{\partial x^n}[(x+\zeta)^{n+1}(1+\bar{\zeta}x)^{n+1}h'(x)]|_{x=0}\neq 0, \qquad h\in\mathfrak{C}, \qquad 0<|\zeta|<1. \tag{4.5}$$

If we use the fact that $h(x)\in\mathfrak{C}$ implies $h(zx)\in\mathfrak{C}$ for $0<|z|\leqq 1$, we can rewrite (4.5):

$$zh'(z)*\sum_{k=1}^{n+1} R_{nk}(\zeta)\,z^k\neq 0, \qquad 0<|z|\leqq 1, \qquad 0<|\zeta|<1, \qquad h\in\mathfrak{C}, \tag{4.6}$$

where $R_{nk}(\zeta)=\frac{1}{(n+1-k)!}\frac{\partial^{n-k+1}}{\partial x^{n-k+1}}[(x+\zeta)^{n+1}(1+\bar{\zeta}x)^{n+1}]|_{x=0}$. Now let $\zeta=te^{i\alpha}$, $y=t(1-t)^{-2}$. Then, with the notation of (1.5) we obtain

$$\sum_{k=1}^{n+1} R_{nk}(\zeta)\, z^k=(1-t)^{2n+2}\sum_{k=1}^{n+1}\frac{z^k}{(n+1-k)!}\frac{\partial^{n+1-k}}{\partial x^{n+1-k}}[x+ye^{-i\alpha}(e^{i\alpha}+x)^2]^{n+1}|_{x=0}$$
$$=(1-t)^{2n+2}\sum_{k=1}^{n+1}\binom{n+1}{k}y^k V_k(e^{i\alpha}z).$$

(4.4) and (4.6) show that our Lemma is settled if we can find a number $\rho=\rho(n,y)>1$ such that

$$\sum_{k=1}^{n+1}\binom{n+1}{k}y^k V_k(\rho z)\in\mathfrak{K},\qquad 0<y<\infty.$$

Exactly this is the content of Theorem 3, which will be proved in Section 7.

Lemma 4.3. *Let $u(z)=(1-z)^{-1}$. Then we have $f_u(z,t)\in\mathfrak{K}$ for $t\in(0,1)$.*

Proof. By (4.1) and (4.4) it is obvious that it is enough to prove $f_{zg'}(z,t)\neq 0, 0<|z|\leqq 1$, for each $g\in\mathfrak{C}$ and $t\in(0,1)$. This is true on Bd $\mathscr{D}$ by Lemma 4.2. But since $h(z):=g(rz)\in\mathfrak{C}$ for $r\in(0,1]$ and $f_{zh'}(z,t)=f_{zg'}(rz,t)$, the assertion follows in full.

5. Application of Subordination Chains

Let $f,g\in\mathfrak{A}$. f is called "subordinate" to g ($f\prec g$) if and only if there exists $\phi\in\mathfrak{A}$ with $\phi(0)=0$, $|\phi(z)|<1$ for $z\in\mathscr{D}$, such that $f=g\circ\phi$. Following Pommerenke [4] we call a function $f(z,t)$ ($z\in\mathscr{D}, t\in E\subset\mathbb{R}$) a "subordination chain" over E if

i) $f(z,t)\in\mathfrak{A}$ for each $t\in E$,

ii) $\frac{\partial}{\partial z}f(z,t)|_{z=0}\neq 0,\qquad t\in E$,

iii) $f(z,t_1)\prec f(z,t_2)$ for $t_1,t_2\in E$, $t_1<t_2$.

Lemma 5.1 is one direction of a result of Pommerenke [4].

Lemma 5.1. *Let $f(z,t)$ be a subordination chain over an interval $I\subset\mathbb{R}$, locally absolutely continous, locally uniformly with respect to $\mathscr{D}$. Let $f(0,t)=0$, $\frac{\partial}{\partial z}f(0,t)>0$. Then*

$$\operatorname{Re}\frac{\frac{\partial}{\partial t}f(z,t)}{z\frac{\partial}{\partial z}f(z,t)}>0,\qquad z\in\mathscr{D},\tag{5.1}$$

for almost all $t\in I$.

Lemma 5.2. *Let $w=F_n g$ be n-conformal in $\mathscr{D}$. Then we have $g\in\mathfrak{K}$.*

Proof. Obviously we may assume $g(0)=0$. Let $\gamma_t=w(\text{Bd}\,\mathscr{D}_t)$, $t\in(0,1)$. It follows from the univalence of w that γ_t are analytic Jordan curves. This implies the

univalence of the functions $f_g(z,t)$ in $\mathscr{D}$, $t\in(0,1)$. Since $f_g(0,t)=g(0)=0$ and $f_g(\mathscr{D},t_1)\subset f_g(\mathscr{D},t_2)$ for $t_1<t_2$, it follows that $f_g(z,t_1)\prec f_g(z,t_2)$, $t_1<t_2$. The univalence of $f_g(z,t)$ gives $\frac{\partial}{\partial z}f_g(0,t)=g'(0)\,d_{n1}(t)\neq 0$, and since $d_{n1}(t)>0$, $t\in(0,1)$, there is no loss of generality to assume $\frac{\partial}{\partial z}f_g(0,t)>0$, $t\in(0,1)$. Hence $f_g(z,t)$ is a subordination chain over $(0,1)$, and we deduce from (5.1) for almost all $t\in(0,1)$:

$$\operatorname{Re}\frac{\sum_{k=1}^{\infty} a_k d'_{nk}(t)\,z^k}{\sum_{k=1}^{\infty} k\,a_k\,d_{nk}(t)\,z^k}>0,\qquad z\in\mathscr{D},\tag{5.2}$$

where $g=\sum_{k=1}^{\infty} a_k z^k$. Now divide (5.2) by $(1-t)$ and let $t\to 1$. Lemma 2.1 gives

$$\frac{1}{2n-1}\operatorname{Re}\frac{\sum_{k=1}^{\infty} a_k k^2 z^k}{\sum_{k=1}^{\infty} k\,a_k\,z^k}=\frac{1}{2n-1}\operatorname{Re}\left(\frac{zg''(z)}{g'(z)}+1\right)\geqq 0,\qquad z\in\mathscr{D},$$

and (3.3) proves our assertion.

6. Proof of the Main Theorem

Let $w=F_n g$ be n-conformal. Then (Lemma 5.2) we have $g\in\mathfrak{K}$. Since $f_n(z,t)\in\mathfrak{K}$, $t\in(0,1)$ (Lemma 4.3), we can use Lemma 4.1 and (4.1) to deduce $f_g(z,t)\in\mathfrak{K}$, $t\in(0,1)$. Hence $w(\mathscr{D}_t)(=f_g(\mathscr{D},t)$ by Lemma 3.1) are convex domains which proves Theorem 2.

Furthermore, $g(\mathscr{D})=w(\mathscr{D})$ follows from Lemma 3.2, and thus the first part of Theorem 1. If, conversely, $g\in\mathfrak{K}$ we can use Lemmata 4.3, 4.1 and (4.1) to show that $f_g(z,t)\in\mathfrak{K}$, $t\in(0,1)$, which implies that $F_n g$ is n-conformal (Lemma 3.2). Theorem 1 is settled.

Part i) of the Main Theorem is contained in Theorem 1. Now let $\mathscr{G}$, $\mathscr{G}\neq\mathbb{C}$, be a convex domain in $\mathbb{C}$, $w_0\in\mathscr{G}$. Let g be the unique determined conformal mapping of $\mathscr{D}$ onto $\mathscr{G}$ such that $g(0)=w_0$, $g'(0)>0$ (Riemann's mapping theorem). Hence $g\in\mathfrak{K}$ and $w=F_n g$ is n-conformal with $w(\mathscr{D})=g(\mathscr{D})=\mathscr{G}$ by Theorem 1. Furthermore, $w(0)=g(0)=w_0$ and

$$w_z(0)=g'(0)\,d'_{n1}(0)=(n+1)\,g'(0)>0,$$

by (2.3).

Now assume that $\tilde{w}=F_n\tilde{g}$ is a second n-conformal mapping with $\tilde{w}(\mathscr{D})=\mathscr{G}$, $\tilde{w}(0)=w_0$, $\tilde{w}_z(0)>0$. This would imply $\tilde{g}\in\mathfrak{K}$, $\tilde{g}(\mathscr{D})=\mathscr{G}$, $\tilde{g}(0)=w_0$, $\tilde{g}'(0)>0$. By the uniqueness part of Riemann's mapping theorem this implies $\tilde{g}\equiv g$ and hence $\tilde{w}\equiv w$. The Main Theorem is proven.

7. Proof of Theorem 3

Let $0<x<\infty$, $n\in\mathbb{N}$, and

$$g(z):=\sum_{k=1}^{n}\binom{n}{k}x^k V_k(z).$$

Since $V_k\in\mathfrak{K}$ with real MacLaurin coefficients, one can use a result of Robertson [7] to show $\operatorname{Re}(1-z^2)\,V_k'(z)>0$, $z\in\mathscr{D}$, $k\in\mathbb{N}$. Hence $\operatorname{Re}(1-z^2)\,g'(z)>0$ and in particular $g'(z)\neq 0$, $z\in\mathscr{D}$. In the sequel we shall prove

$$\operatorname{Re}(zg'(z))'\overline{g'(z)}>0 \qquad z\in\operatorname{Bd}\mathscr{D}. \tag{7.1}$$

This implies, by continuity, that $g'(z)\neq 0$ in $\mathscr{D}_{1+\varepsilon_1}$ for a certain $\varepsilon_1>0$ and thus, by continuity and the minimum principle for harmonic functions

$$\operatorname{Re}\frac{(zg'(z))'}{g'(z)}=\operatorname{Re}\left(\frac{zg''(z)}{g'(z)}+1\right)>0$$

in $\mathscr{D}_{1+\varepsilon_2}$ for a certain $\varepsilon_2>0$. Hence $h(z):=g((1+\varepsilon_2)\,z)\in\mathfrak{K}$ and the theorem follows.

It remains to prove (7.1) which may be transformed into

$$\operatorname{Im}\ddot{g}(e^{i\phi})\,\overline{\dot{g}(e^{i\phi})}>0, \qquad \phi\in[0,2\pi), \tag{7.2}$$

where $\dot{}:=\dfrac{\partial}{\partial\phi}$. We shall require the following identity

Lemma 7.1. *Let $s, k\in\mathbb{N}$, $1\leqq k\leqq s$, and*

$$a_\mu(k,s)=\begin{cases}0 & \mu<k-1,\ \mu>s-1,\\ 2^{2\mu+1}(\mu+1)(2\mu+1)\dbinom{2(s+k-\mu-2)}{s+k-\mu-2} & k-1=\mu=s-1,\\ 2^{2\mu}(\mu+1)(2\mu+1)\dbinom{2(s+k-\mu-2)}{s+k-\mu-2} & k-1=\mu<s-1,\\ -2^{2\mu}\dfrac{3ks(s-k)}{(s+k-\mu)(s+k-\mu-1)}\dbinom{2(s+k-\mu-2)}{s+k-\mu-2} & k-1<\mu<s-1,\\ 2^{2\mu}\dfrac{(\mu+1)(\mu+2)(2(s+k-\mu)-3)}{s+k-\mu}\dbinom{2(s+k-\mu-2)}{s+k-\mu-2} & k-1<\mu=s-1.\end{cases}$$

Then we have

$$\operatorname{Im}[\overline{\dot{V}_k(e^{i\phi})}\,\ddot{V}_s(e^{i\phi})+\overline{\dot{V}_s(e^{i\phi})}\,\ddot{V}_k(e^{i\phi})]=\sum_{\mu=k-1}^{s-1}a_\mu(k,s)\cos^{2\mu}\frac{\phi}{2}.$$

Proof. From the well known relation

$$V_k(e^{i\phi})=-1/2\binom{2k}{k}+2^{2k-1}\cos^{2k}\frac{\phi}{2}+i\sum_{p=1}^{k}\binom{2k}{k+p}\sin p\phi \tag{7.3}$$

we obtain

$$\dot{V}_k(e^{i\phi}) = -2^{2k-1}\, k \sin\frac{\phi}{2} \cos^{2k-1}\frac{\phi}{2} + i \sum_{p=1}^{k} \binom{2k}{k+p} p \cos p\phi .$$

Using a certain representation of Chebycheff polynomials [2, p. 257], we get

$$\cos p\phi = \sum_{\nu=0}^{p} (-1)^{p-\nu} \prod_{m=0}^{\nu-1} (p^2 - m^2) \frac{2^{2\nu}}{(2\nu)!} \cos^{2\nu}\frac{\phi}{2}$$

where the empty product is understood to equal one, and thus

$$\dot{V}_k(e^{i\phi}) = -k\, 2^{2k-1} \sin\frac{\phi}{2} \cos^{2k-1}\frac{\phi}{2}$$
$$+ i \sum_{\nu=0}^{k} \left[(-1)^{\nu} \frac{2^{2\nu}}{(2\nu)!} \sum_{p=1}^{k} \binom{2k}{k+p} p(-1)^{p} \prod_{m=0}^{\nu-1} (p^2 - m^2) \right] \cos^{2\nu}\frac{\phi}{2} .$$

The following identity may be proved by induction with respect to ν:

$$\sum_{p=1}^{k} (-1)^{p}\, p \binom{2k}{k+p} \prod_{m=0}^{\nu-1} (p^2 - m^2) = (-1)^{\nu-1} (2\nu)!\, \frac{k}{4k-4\nu-2} \binom{2k-2\nu}{k-\nu},$$

$0 \leqq \nu \leqq k$, $k \in \mathbb{N}$. Hence

$$\dot{V}_k(e^{i\phi}) = -2^{2k-1}\, k \sin\frac{\phi}{2} \cos^{2k-1}\frac{\phi}{2} - i \sum_{\nu=0}^{k} \frac{k\, 2^{2\nu-1}}{2k-2\nu-1} \binom{2k-2\nu}{k-\nu} \cos^{2\nu}\frac{\phi}{2} . \tag{7.4}$$

Similarly

$$\ddot{V}_k(e^{i\phi}) = 2^{2k-2} k(2k-1) \cos^{2k-2}\frac{\phi}{2} - 2^{2k-1} k^2 \cos^{2k}\frac{\phi}{2}$$
$$+ i \sin\frac{\phi}{2} \cos\frac{\phi}{2} \sum_{\mu=0}^{k-1} 2^{2\mu+1} \frac{k(\mu+1)}{2k-2\mu-3} \binom{2(k-\mu-1)}{k-\mu-1} \cos^{2\mu}\frac{\phi}{2} . \tag{7.5}$$

Straightforward computations using (7.4) and (7.5) lead to our assertion.

Obviously

$$\operatorname{Im} \ddot{g}(e^{i\phi})\, \overline{\dot{g}(e^{i\phi})} = \sum_{\lambda=2}^{2n} \left[\sum_{k=1}^{\lambda-1} \binom{n}{k} \binom{n}{\lambda-k} \operatorname{Im} \overline{\dot{V}_k(e^{i\phi})}\, \ddot{V}_{\lambda-k}(e^{i\phi}) \right] x^{\lambda} .$$

From Lemma 7.1 we deduce that

$$\sum_{k=1}^{\lambda-1} \binom{n}{k} \binom{n}{\lambda-k} \operatorname{Im} \overline{\dot{V}_k(e^{i\phi})}\, \ddot{V}_{\lambda-k}(e^{i\phi}) = \sum_{\mu=0}^{\lambda-2} A_{\mu}(\lambda, n) \cos^{2\mu}\frac{\phi}{2}$$

where

$$A_{\mu}(\lambda, n) = \begin{cases} \displaystyle\sum_{k=1}^{\mu+1} \binom{n}{k} \binom{n}{\lambda-k} a_{\mu}(k, \lambda-k) - 1/2\, a_{\mu}(\mu+1, \mu+1), & \text{if } \lambda = 2\mu+2 \\ \displaystyle\sum_{k=1}^{q} \binom{n}{k} \binom{n}{\lambda-k} a_{\mu}(k, \lambda-k), & \text{otherwise,} \end{cases}$$

and $q = \min\{\mu+1, \lambda-\mu-1\}$.

Our theorem is proved once the following Lemma has been established.

Lemma 7.2. *For* $0 \leqq \mu \leqq \lambda - 2$, $2 \leqq \lambda \leqq 2n$, $n \in \mathbb{N}$, *we have* $A_\mu(\lambda, n) > 0$.

Proof. i) $0 \leqq \mu \leqq \lambda - \mu - 2$. Lemma 7.1 shows that our assertion is equivalent to

$$\binom{n}{\mu+1}\binom{n}{\lambda-\mu-1}(\mu+1)(2\mu+1) > 3\sum_{k=1}^{\mu}\binom{n}{k}\binom{n}{\lambda-k}\frac{k(\lambda-k)(\lambda-2k)}{(\lambda-\mu)(\lambda-\mu-1)}. \tag{7.6}$$

But since

$$(n-\mu-1)!\,(n-\lambda+\mu+1)! < (n-k)!\,(n-\lambda+k)!, \qquad k \leqq \mu < \lambda-\mu-1,$$

it remains to prove

$$(\mu+1)(2\mu+1) > 3\sum_{k=1}^{\mu}\frac{(\mu+1)!\,(\lambda-\mu-1)!}{k!\,(\lambda-k)!}\,\frac{k(\lambda-k)(\lambda-2k)}{(\lambda-\mu)(\lambda-\mu-1)},$$

which is easily reduced to $(\lambda-\mu)(2\mu+1) > 3\mu$, an obviously valid inequality in the range under consideration.

ii) $1 \leqq \lambda - \mu - 1 < \mu + 1$. In this case we have to show

$$\binom{n}{\mu+1}\binom{n}{\lambda-\mu-1}\frac{(\mu+1)(\mu+2)(2\lambda-2\mu-3)}{\lambda-\mu} > 3\sum_{k=1}^{\lambda-\mu-2}\binom{n}{\lambda-k}\frac{k(\lambda-k)(\lambda-2k)}{(\lambda-\mu)(\lambda-\mu-1)}.$$

Replacing μ by $\lambda - \tilde{\mu} - 2$ this can be reduced to case i).

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