## Werk

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# Strongly Ovoidal Möbius Planes*

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The purpose of this paper is to give two generalizations of the following theorem of Krier:

**Theorem A** (Krier [7], Theorem 1). *If $\mathcal{M}$ is a Möbius plane of finite even order and of Hering class at least* II 2, *then $\mathcal{M}$ is miquelian.*

The proof of Theorem A is based on Dembowski's Theorem [3], which states that Möbius planes of finite even order are ovoidal, and that their automorphisms are induced by collineations of the associated 3-space. It is thus natural to ask whether Theorem A also applies to "strongly ovoidal" Möbius planes, i.e. to Möbius planes which are derived from an ovoid in projective 3-space over a commutative field $F$, and whose $(p, C)$ and $(p, q)$ automorphisms are induced by collineations of the 3-space. I am grateful to Professor H. Lüneburg for posing this question.

The strongly ovoidal Möbius planes of Hering class V–VII have been completely determined by Tits [9–11]. We shall be concerned here with certain strongly ovoidal Möbius planes of Hering class II–IV. § 1 contains definitions and known theorems which are required later. In § 2 we consider strongly ovoidal Möbius planes which are $(p, C)$ and $(p, q)$ transitive for some $p, q, C$ with $p \neq q$ and $p, q \in C$. We show that they are either miquelian or of Hering class III 2 with char $F = 2$, and give an example of the latter type. In § 3 we study strongly ovoidal Möbius planes which are $(p, C)$ transitive and $(p, q)$ half-transitive for some $p, q, C$ with $p \neq q$ and $p, q \in C$; here we assume char $F \neq 2$. We find a canonical form for the ovoids, and show that the corresponding Möbius planes are of class II 1 or IV 1. An example of the second type is the real "squashed sphere", described by Ewald [5], and Tits [11].

## § 1. Definitions and Known Theorems

For information about Möbius planes (there called "inversive planes"), see Dembowski [4], Chapter 6. A *Möbius plane*, $\mathcal{M} = (\mathfrak{P}, \mathfrak{C}, \in)$, is an incidence structure consisting of *points* $\in \mathfrak{P}$ and *circles* $\in \mathfrak{C}$ such that the circles are non-empty subsets of $\mathfrak{P}$ and 1) on 3 distinct points there is a unique circle, 2) if $p, q \in \mathfrak{P}$, if $C \in \mathfrak{C}$ and if $p \in C$, $q \notin C$, then there exists a unique $D \in \mathfrak{C}$ such that $q \in D$ and $C \cap D = \{p\}$, 3) there exist 4 points not all on the same circle. If $p \in \mathfrak{P}$ and if

$$\mathcal{M}_p = (\mathfrak{P} - \{p\}, \{C - \{p\} \mid p \in C \text{ and } C \in \mathfrak{C}\}, \in),$$

then $\mathcal{M}_p$ is an affine plane. A *tangent pencil* of circles at $p$ is a set of circles on $p$ which yields an equivalence class of parallel lines in $\mathcal{M}_p$. If $p \in C$, a $(p, C)$ *auto-*

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*morphism* is an automorphism of $\mathcal{M}$ which induces a translation in $\mathcal{M}_p$ and whose fixed circles are those of the tangent pencil containing $C$. $\mathcal{M}$ is $(p, C)$ *transitive* $\Leftrightarrow$ for each $x, y$ with $x, y \in C$ and $x, y \neq p$, there exists a $(p, C)$ automorphism which maps $x$ on $y$. If $p \neq q$, a $(p, q)$ *automorphism* is an automorphism of $\mathcal{M}$ which induces in $M_p$ a dilatation with centre $q$. $\mathcal{M}$ is $(p, q)$ *transitive* $\Leftrightarrow$ for each $x, y \neq p, q$ such that $x, y, p, q$ are concyclic, there exists a $(p, q)$ automorphism which maps $x$ on $y$. If $\mathcal{M}$ is $(p, C)$ transitive ($(p, q)$ transitive), and admits the automorphism $\alpha$, then $\mathcal{M}$ is $(p^\alpha, C^\alpha)$ transitive ($(p^\alpha, q^\alpha)$ transitive). If $\mathcal{M}$ is $(p, C)$ and $(q, C)$ transitive, with $p \neq q$, then $\mathcal{M}$ is $(x, C)$ transitive for all $x \in C$. If $\mathcal{M}$ is $(p, C)$ and $(p, D)$ transitive, with $C \neq D$, then $\mathcal{M}$ is $(p, X)$ transitive for all $X$ on $p$, and the group generated by the $(p, C)$ and $(p, D)$ automorphisms is transitive on $\mathfrak{P} - \{p\}$ ([4], p. 260).

Hering [6], has classified Möbius planes according to the configuration $\mathfrak{H}$ formed by their $(p, C)$ and $(p, q)$ transitivity pairs. In particular: $\mathcal{M}$ is *of class* II 1 $\Leftrightarrow$ for given $(p, C)$, $\mathfrak{H} = \{(p, C)\}$; $\mathcal{M}$ is *of class* II 2 $\Leftrightarrow$ for given $(p, C)$, $\mathfrak{H} = \{(p, C)\} \cup \{(p, x) | x \in C, x \neq p\}$; $\mathcal{M}$ is *of class* III 1 $\Leftrightarrow$ for given $p$, $\mathfrak{H} = \{(p, X) | p \in X\}$; $\mathcal{M}$ is *of class* III 2 $\Leftrightarrow$ for given $p$, $\mathfrak{H} = \{(p, X) | p \in X\} \cup \{(p, x) | x \neq p\}$; $\mathcal{M}$ is *of class* IV 1 $\Leftrightarrow$ for given $C$, $\mathfrak{H} = \{(x, C) | x \in C\}$; $\mathcal{M}$ is *of class* IV 2 $\Leftrightarrow$ for given $C$, $\mathfrak{H} = \{(x, C) | x \in C\} \cup \{(x, y) | x, y \in C \text{ and } x \neq y\}$; $\mathcal{M}$ is *of class* VII 2 $\Leftrightarrow$ $\mathfrak{H} = \{(x, X) | x \in X\} \cup \{(x, y) | x \neq y\}$, i.e. $\mathfrak{H}$ contains all possible pairs. A plane is *at least of Hering class h* $\Leftrightarrow$ $\mathfrak{H}$ contains at least the pairs of class $h$. Note that $\mathcal{M}$ is at least of class II 2 if $\mathcal{M}$ is $(p, C)$ and $(p, q)$ transitive for some $p, q, C$ with $p \neq q$ and $p, q \in C$, $\mathcal{M}$ is at least of class III 1 if $\mathcal{M}$ is $(p, C)$ and $(p, D)$ transitive for some $p, C, D$ with $p \in C, D$ and $C \neq D$, and that if $\mathcal{M}$ is at least of class III 2, then it is of class III 2 or VII 2.

Let $PG(3, F)$ denote the projective 3-space over the field $F$ (not necessarily commutative). An *ovoid*, $\Omega$, in $PG(3, F)$ is a non-empty set of points such that for every line $L$, i) $|L \cap \Omega| \leqq 2$, ii) $L \cap \Omega = \{p\} \Leftrightarrow L$ is contained in a well-defined plane $\pi(p)$, the *tangent plane* to $\Omega$ at $p$. If $\Omega$ is an ovoid in $PG(3, F)$, let $\mathcal{M}(\Omega, F) = (\mathfrak{P}, \mathfrak{C}, \in)$ where $\mathfrak{P} = \Omega$, $\mathfrak{C} = \{\pi \cap \Omega | \pi \text{ is a plane in } PG(3, F) \text{ and } |\pi \cap \Omega| \geqq 2\}$. Then $\mathcal{M}(\Omega, F)$ is a Möbius plane ([4], p. 254). The Möbius plane $\mathcal{M}$ is *ovoidal over* $F \Leftrightarrow \mathcal{M} \simeq \mathcal{M}(\Omega, F)$ for some ovoid $\Omega$ in $PG(3, F)$.

**Definition 1.** The Möbius plane $\mathcal{M}$ is *strongly ovoidal* (*over* $F$), $\Leftrightarrow$ (i) $\mathcal{M} \simeq \mathcal{M}(\Omega, F)$ and $F$ is commutative, (ii) all $(p, C)$ and $(p, q)$ automorphisms of $\mathcal{M}$ are induced by collineations of $PG(3, F)$ which preserve $\Omega$.

Note that if $\mathcal{M}$ is strongly ovoidal, then every collineation $\alpha$ of $PG(3, F)$ which induces a $(p, C)$ or $(p, q)$ automorphism in $\mathcal{M}$ is linear. For in the former case $\alpha$ fixes all planes on the line $\pi(p) \cap C$, and in the latter case $\alpha$ fixes all planes on the line $p \cup q$ (cf. [7], p. 4). The following definition is essentially that given by Tits [11].

**Definition 2.** Let $\mathcal{M} \simeq \mathcal{M}(\Omega, F)$ be ovoidal. The line $L$ of $PG(3, F)$ is a *transitive tangent* of $\Omega \Leftrightarrow L = \pi(p) \cap C$, where $\mathcal{M}$ is $(p, C)$ transitive and the $(p, C)$ automorphisms are induced by collineations of $PG(3, F)$.

**Definition 3.** Let $p, q$ be distinct points of the Möbius plane $\mathcal{M}$. Then $\mathcal{M}$ is $(p, q)$ *half-transitive* $\Leftrightarrow$ for some circle $C$ on $p$ and $q$ (and hence for every such $C$), the group of all $(p, q)$ automorphisms has exactly 2 orbits on $C - \{p, q\}$.

We shall require the followings results.

**Theorem I** (see [4], p. 254). *If $\mathcal{M}$ is ovoidal over $F$, then for each point $p \in \mathcal{M}$ the affine plane $\mathcal{M}_p$ is the desarguesian plane over $F$.*

**Theorem II** (Van der Waerden and Smid [12]). *$\mathcal{M}$ is miquelian $\Leftrightarrow \mathcal{M} \simeq \mathcal{M}(\Omega, F)$, where $F$ is commutative and $\Omega$ is a non-ruled quadric in $PG(3, F)$.*

**Theorem III** (Tits [9, 11]). *Let $\mathcal{M} \simeq \mathcal{M}(\Omega, F)$ be strongly ovoidal. Then* a) *if $\mathcal{M}$ is of Hering class* VII 2, *$\mathcal{M}$ is miquelian,* b) *if $\Omega$ admits 2 non-concurrent transitive tangents and if* char $F \neq 2$, *then $\mathcal{M}$ is miquelian.*

**Theorem IV** (Barlotti [2]; Panella [8]). *Let $F$ be a finite field of odd order. Then every ovoid in $PG(3, F)$ is a non-ruled quadric.*

It is not evident that every ovoidal Möbius plane over a commutative field must be strongly ovoidal, but it can be verified that this is in fact the case for the miquelian planes, for the planes derived from the translation ovoids and Suzuki-Tits ovoids [9, 10], and for the "squashed sphere". Finite ovoidal Möbius planes are always strongly ovoidal, by Dembowski's Theorem and Theorem IV.

In what follows we shall be concerned only with infinite planes, since finite planes of the type to be considered are necessarily miquelian – by Theorems II and IV if char $F \neq 2$, and by Theorem A if char $F = 2$.

## 2. Strongly Ovoidal Möbius Planes of Hering Class at Least II 2

Suppose $\mathcal{M} = \mathcal{M}(\Omega, F)$ is a strongly ovoidal Möbius plane. Let $p, q, C \in \mathcal{M}$, where $p \neq q$, $p \in C$ and $q \in C$. Choose homogeneous coordinates $\mathbf{x} = (x, y, z, w)$ in $PG(3, F)$, so that 1) $p = (0, 0, 1, 0)$ and the tangent plane at $p$ is $w = 0$, 2) $C = \Omega \cap (x = 0)$, 3) $q = (0, 0, 0, 1)$ and the tangent plane at $q$ is $z = 0$. Then

$$\Omega = \{p\} \cup \{(x, y, z, 1) \mid z = \phi(x, y)\},$$

where $\phi\colon F \times F \to F$ is a function of $x$ and $y$ with the property that $\phi(x, y) = 0 \Leftrightarrow x = y = 0$. We first derive conditions for the existence of $(p, C)$ and $(p, q)$ automorphisms. (Most of results (A)–(F) are given in [7] for the case char $F = 2$, see also [11].)

(A) *Suppose that $\mathcal{M}$ admits the $(p, C)$ automorphism $T_k$ which maps $q$ on $(0, k, \phi(0, k), 1)$. Then $T_k$ maps $\mathbf{x}$ on $\mathbf{x} T_k^*$, where*

$$T_k^* = \begin{pmatrix} 1 & 0 & a_1(k) & 0 \\ 0 & 1 & a_2(k) & 0 \\ 0 & 0 & a_3(k) & 0 \\ 0 & k & \phi(0, k) & 1 \end{pmatrix}.$$

*The $a_i(k)$, $i = 1, 2, 3$, are functions of $k$, and $a_3(k) \neq 0$.*

*Proof.* As noted in § 1, $T_k$ is linear. Let $T_k^* = (a_{ij})$, $i, j = 1, \ldots, 4$. Since $T_k$ fixes $p$, $a_{31} = a_{32} = a_{34} = 0$. Since $T_k$ fixes the planes $w = 0$ and $x = 0$, $a_{14} = a_{24} = 0$ and $a_{21} = a_{41} = 0$. Normalize $a_{44}$ to 1. (Clearly $a_{44} \neq 0$, since $T_k$ is non-singular.) Since $T_k$ maps $q$ on $(0, k, \phi(0, k), 1)$, $a_{42} = k$ and $a_{43} = \phi(0, k)$. In $\mathcal{M}$, $T_k$ fixes all circles in the tangent-pencil at $p$ which contains $C$. Hence in $PG(3, F)$, $T_k$ fixes all planes on the line $C \cap \pi(p)$, i.e. all planes $ax + bw = 0$, and it follows that $a_{11} = 1$. More-

over, in $\mathscr{M}$, $T_k$ fixes every other tangent-pencil at $p$ as a set, whence, in $PG(3, F)$, $T_k$ fixes every line $(ax+by=0)\cap(w=0)$. It follows that $a_{12}=0$ and $a_{22}=1$. Thus $T_k^*$ has the given form, where we write $a_{i3}=a_i(k)$, $i=1, 2, 3$. Since $T_k$ is non-singular, $a_3(k)\neq 0$ for any $k$.

(B) *If $\mathscr{M}$ is $(p, C)$ transitive, then for all $k, l\in F$,*

$$\phi(x, y+k)=a_1(k)\,x+a_2(k)\,y+a_3(k)\,\phi(x, y)+\phi(0, k), \tag{1}$$

$$a_1(k+l)=a_1(k)\,a_3(l)+a_1(l)=a_1(l)\,a_3(k)+a_1(k), \tag{2}$$

$$a_2(k+l)=a_2(k)\,a_3(l)+a_2(l)=a_2(l)\,a_3(k)+a_2(k), \tag{3}$$

$$a_3(k+l)=a_3(k)\,a_3(l), \tag{4}$$

$$\begin{aligned}\phi(0, k+l)&=k\,a_2(l)+a_3(l)\,\phi(0, k)+\phi(0, l)\\&=l\,a_2(k)+a_3(k)\,\phi(0, l)+\phi(0, k).\end{aligned} \tag{5}$$

*Proof.* (1) follows from the fact that $T_k$ preserves $\Omega$. By Theorem I, $\mathscr{M}_p$ is desarguesian. Hence the $(p, C)$ automorphisms form an abelian group. On evaluating $T_k^*\,T_l^*$, we find from the 4th row that $T_k\,T_l=T_{k+l}=T_l\,T_k$. Eqs. (2)–(5) are obtained by equating the corresponding matrices.

(C) *Suppose $\mathscr{M}$ is $(p, C)$ transitive. If* char $F=2$, *then $a_3(k)=1$ and $a_2(k)=0$ for all $k$. Conversely, if there exists $k\neq 0$ such that $a_3(k)=1$ and $a_2(k)=0$, then* char $F=2$.

*Proof.* If char $F=2$, then by (4), $[a_3(k)]^2=a_3(0)=1$, whence $a_3(k)=1$ for all $k$. Then by (5), $0=\phi(0, k+k)=k\,a_2(k)$, whence $a_2(k)=0$ for all $k\neq 0$; $a_2(0)=0$ because $T_0^*$ is the identity matrix. Conversely, if $a_3(k)=1$ and $a_2(k)=0$ for some $k\neq 0$, then $T_k$ induces a non-trivial elation $E_k$ in the plane $x=0$, with centre $(0, k, \phi(0, k), 0)$ (not on $\Omega$). Since $T_k$ preserves $\Omega$, it follows that $E_k$ is the identity, whence char $F=2$.

(D) *Suppose that $\mathscr{M}$ admits the $(p, q)$ automorphism $D_r$ which maps $(0, 1, \phi(0, 1), 1)$ on $(0, r, \phi(0, r), 1)$, $r\neq 0$. Then $D_r$ maps* $\mathbf{x}$ *on* $\mathbf{x}D_r^*$, *where*

$$D_r^*=\begin{pmatrix} r & 0 & 0 & 0\\ 0 & r & 0 & 0\\ 0 & 0 & \psi(r) & 0\\ 0 & 0 & 0 & 1\end{pmatrix},$$

*$\psi(r)\neq 0$ and $\psi(1)=1$.*

*Proof.* We can assume that $D_r$ is linear. Let $D_r^*=(a_{ij})$, $i, j=1, \ldots, 4$. Since $D_r$ fixes $p=(0, 0, 1, 0)$, $q=(0, 0, 0, 1)$ and the tangent planes $w=0$, $z=0$ at $p$ and $q$, $a_{31}=a_{32}=a_{34}=0$, $a_{41}=a_{42}=a_{43}=0$, $a_{14}=a_{24}=0$ and $a_{13}=a_{23}=0$. Since $D_r$ fixes all planes through the line $p\cup q$, in particular the lines $x=0$, $y=0$ and $x=y$, we also have $a_{21}=a_{12}=0$, $a_{11}=a_{22}$. Normalize $a_{44}$ to 1. Then $a_{22}=r$ because $(0, 1, \phi(0, 1), 1)$ is mapped to $(0, r, \phi(0, r), 1)$. Write $a_{33}=\psi(r)$; clearly $\psi(r)\neq 0$, since $D_r$ is non-singular, and $\psi(1)=1$, since $D_1$ is the identity.

(E) *If $\mathscr{M}$ is $(p, q)$ transitive, then*

$$\phi(xr, yr)=\psi(r)\,\phi(x, y) \qquad \textit{for all } x, y \textit{ and for all } r\neq 0, \tag{6}$$

$$\psi(r_1\,r_2)=\psi(r_1)\,\psi(r_2) \qquad \textit{for all } r_1, r_2\neq 0. \tag{7}$$

*Proof.* Eq. (6) expresses the fact that $D_r$ preserves $\Omega$, and (7) that the $(p, q)$ automorphisms form a group.

(F) *If* $\mathcal{M}$ *is* $(p, C)$ *and* $(p, q)$ *transitive, then for all* $k$ *and for all* $r \neq 0$,

$$a_1(kr) = a_1(k)\,\psi(r)\,r^{-1}, \tag{8}$$

$$a_2(kr) = a_2(k)\,\psi(r)\,r^{-1}, \tag{9}$$

$$a_3(kr) = a_3(k). \tag{10}$$

*Proof.* If $r \neq 0$, then $D_r^{-1}\,T_k D_r$ is a $(p, C)$ automorphism, and in fact is $T_{kr}$, as we find on computing the image of $q$. Eqs. (8)–(10) are obtained by equating the corresponding matrices.

**Theorem 1.** *If* $\mathcal{M} = \mathcal{M}(\Omega, F)$ *is strongly ovoidal and of Hering class at least* II 2, *then either* (a) $\mathcal{M}$ *is miquelian, or* (b) char $F = 2$ and $\mathcal{M}$ *is of class* III 2. *In case* (b), *the coordinate system can be chosen so that*

$$\Omega = \{(0, 0, 1, 0)\} \cup \{(x, y, z, 1) \mid z = a\psi(x) + b\psi(y)\},$$

*where* (i) $\psi\colon F \to F$ *is a field monomorphism which is not surjective,* (ii) *if* $k \in F$, $k \neq 0$, *then there exists a unique* $x \in F$ *with* $x \neq 0$ *such that* $\psi(x) = kx$, (iii) *there exists* $x \in F$ *such that* $\psi(x) \neq x^2$, (iv) $ab \neq 0$ *and* $ab^{-1} \notin \operatorname{Im}\psi$. *Conversely, if* char $F = 2$ *and if* $\psi$, $a$, $b$ *satisfy* (i)–(iv), *then* $\Omega$, *as defined in* (b), *is an ovoid in* $PG(3, F)$, *and* $\mathcal{M}(\Omega, F)$ *is strongly ovoidal of class* III 2.

*Proof.* Suppose that $\mathcal{M}$ is strongly ovoidal and of Hering class at least II 2. Then there exist $p, q, C \in \mathcal{M}$ with $p, q \in C$ and $p \neq q$ such that $\mathcal{M}$ is $(p, C)$ and $(p, q)$ transitive. If coordinates are chosen in $PG(3, F)$ as above, then the conclusions of (A)–(F) hold. If char $F = 2$, then $a_3(k) = 1$ for all $k$, by (C). If char $F \neq 2$, then $[a_3(k)]^2 = a_3(2k)$, by (4), $= a_3(k)$ by (10). Hence in this case also $a_3(k) = 1$ for all $k$, since $a_3(k) \neq 0$ by (A). It follows from (5) that $k\,a_2(l) = l\,a_2(k)$ for all $k$, $l \in F$, whence $a_2(k) = dk$ for all $k$, where $d$ is some constant independent of $k$; moreover $d = 0 \Leftrightarrow$ char $F = 2$, by (C).

*Case 1.* Suppose first that $d \neq 0$. By (9), $dkr = dk\,\psi(r)\,r^{-1}$ for all $k$ and all $r \neq 0$, whence $\psi(r) = r^2$ for all $r \neq 0$. By (8), $a_1(r) = cr$ for all $r \neq 0$, where $c = a_1(1)$. (Possibly $c = 0$.) By (6), $\phi(r, 0) = ar^2$ and $\phi(0, r) = br^2$ for all $r$, where $a = \phi(1, 0)$, $b = \phi(0, 1)$; note that this implies $ab \neq 0$. By (1) with $y = 0$, $\phi(x, k) = a_1(k)\,x + \phi(x, 0) + \phi(0, k)$ for all $x$, $k$, whence $\phi(x, k) = ckx + ax^2 + bk^2$ for all $x$, $k$. Hence $\Omega$ is a quadric (non-ruled because it is an ovoid), and $\mathcal{M}$ is miquelian by Theorem II.

*Case 2.* Suppose that $d = 0$ (so that char $F = 2$), but that $a_1(k_1) \neq 0$ for some $k_1$. By (8), $a_1(kr) = a_1(k)\,\psi(r)\,r^{-1}$ for all $k$ and all $r \neq 0$, whence $a_1(r) = c\,\psi(r)\,r^{-1}$ for all $r \neq 0$, where $c = a_1(1)$. Since $a_1(k_1) \neq 0$, $c \neq 0$. By (6), $\phi(r, 0) = a\psi(r)$ and $\phi(0, r) = b\psi(r)$ for all $r \neq 0$, where $a = \phi(1, 0)$ and $b = \phi(0, 1)$, so that $ab \neq 0$. By (5), $\phi(0, k + l) = \phi(0, k) + \phi(0, l)$ for all $k$, $l$. Define $\psi(0) = 0$. Then for all $k$, $l$, $\psi(k + l) = \psi(k) + \psi(l)$, since $b \neq 0$. Now if $k, l, k + l \neq 0$, $a_1(k + l) = c[\psi(k + l)](k + l)^{-1}$, $= c[\psi(k) + \psi(l)](k + l)^{-1}$, while $a_1(k) + a_1(l) = c\psi(k)k^{-1} + c\psi(l)l^{-1}$. But by (2), $a_1(k + l) = a_1(k) + a(l)$. Thus, since $c \neq 0$, $[\psi(k + l)](k + l)^{-1} = \psi(k)k^{-1} + \psi(l)l^{-1}$ for all $k$, $l$ with $k$, $l$, $k + l \neq 0$. It follows that for all such $k$, $l$, $\psi(k)l^2 + \psi(l)k^2 = 0$. In particular, with $l = 1$, $\psi(k) = k^2$ for all $k$. As in Case 1, this implies that $\mathcal{M}$ is miquelian.

*Case 3.* Suppose now that $a_2(k)=a_1(k)=0$ for all $k$. Since $a_3(k)=1$, char $F=2$ by (C). By (6), $\phi(r,0)=a\psi(r)$ and $\phi(0,r)=b\psi(r)$ for all $r\neq 0$, where $a=\phi(1,0)$ and $b=\phi(0,1)$, so that $ab\neq 0$. Define $\psi(0)=0$. Then by (5) and (7), $\psi\colon F\to F$ is a field homomorphism, and by (D), $\psi$ is injective. By (1) with $y=0$, $\phi(x,k)=\phi(x,0)+\phi(0,k)$ for all $x$, $k$, whence $\Omega=\{p\}\cup\{(x,y,z,1)\mid z=a\psi(x)+b\psi(y)\}$. Now, for all $x$, $y$, $k$, $\phi(x+k,y)=a\psi(x+k)+b\psi(y)=a\psi(x)+a\psi(k)+b\psi(y)=\phi(x,y)+\phi(k,0)$. Hence, for each $k$, the collineation $\hat{T}_k$ with matrix

$$\hat{T}_k^*=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\k&0&\phi(k,0)&1\end{pmatrix}$$

preserves $\Omega$. It is easily verified that $\hat{T}_k$ induces the $(p,D)$ automorphism which maps $q$ on $(k,0,\phi(k,0),1)$, where $D$ is the circle $\Omega\cap(y=0)$. (Exchange $x$ and $y$ in (A).) Thus $\mathcal{M}$ is $(p,D)$ transitive, whence, as noted in §1, $\mathcal{M}$ is of Hering class at least III 2, and therefore of class III 2 or VII 2. If $\mathcal{M}$ is of class VII 2 it is miquelian by Theorem III a), and $\psi(x)=x^2$ for all $x$.

Suppose that $\mathcal{M}$ is of class III 2; then Theorem 1 (iii) is satisfied. The group generated by the $(p,C)$ and $(p,D)$ automorphisms is transitive on $\mathfrak{P}-\{p\}$. Hence in finding the restrictions which ensure that $\Omega$ is an ovoid, it suffices to require that 1) a line through $q$ which lies in the "tangent plane" $z=0$ at $q$ meets $\Omega$ only at $q$, 2) a line through $q$ which is not in the plane $z=0$ meets $\Omega$ in exactly one other point. (The corresponding conditions at $p$ are already satisfied, by the choice of representation for $\Omega$.) Condition 1) is satisfied if and only if $\phi(x,y)=0\Rightarrow x=y=0$. Since $\psi$ is multiplicative, $\psi(0)=0$, $ab\neq 0$ and char $F=2$, this is equivalent to $ab^{-1}\notin\operatorname{Im}\psi$. Let $\mathbf{x}_1=(x_1,y_1,z_1,w_1)$ be a point $\neq q$ with $z_1\neq 0$, and let $L=q\cup\mathbf{x}_1$. Then any point $\mathbf{x}\neq q$ on $L$ has coordinates $(x_1,y_1,z_1,w_1+t)$ for some $t$. First note that $L=p\cup q\Leftrightarrow x_1=y_1=0$, and that in this case $L\cap\Omega=\{p,q\}$. Hence if $x_1$, $y_1$ are not both 0, then $\mathbf{x}\in\Omega\Leftrightarrow z_1/(w_1+t)=a\psi(x_1/w_1+t)+b\psi(y_1/w_1+t)$, i.e., since $\psi$ is multiplicative and $z_1\neq 0$, $\Leftrightarrow\psi(w_1+t)=(w_1+t)[\{a\psi(x_1)+b\psi(y_1)\}z_1^{-1}]$. Thus condition 2) is satisfied if and only if the equation $\psi(x)=kx$ has a unique solution $x\neq 0$ for each $k\neq 0$. Hence $\Omega$ is an ovoid but not a quadric $\Leftrightarrow$ (i)–(iv) of Theorem 1 are satisfied.

It remains to be shown that if (i)–(iv) are satisfied, then $\mathcal{M}(\Omega,F)$ is strongly ovoidal and of class III 2. As shown above, $\mathcal{M}$ is at least of class III 2, but conceivably, for some $x$, $y$, $X$ with $x\in X$ and $x,y\neq p$, $\mathcal{M}$ admits a non-trivial $(x,X)$ or $(x,y)$ automorphism which is not induced by a collineation of $PG(3,F)$. In this case, since $\mathcal{M}$ admits an automorphism which moves $p$, $\mathcal{M}$ is of Hering class VII 2, and in particular admits all $(q,Q)$ automorphisms with $q\in Q$. (Since $\mathcal{M}$ is not strongly ovoidal, we cannot apply Theorem III a).) Now $\mathcal{M}$ can also be represented by projecting $\Omega$ from $q$ onto the tangent plane $\pi(p)$ at $p$ ([4], p. 254). The affine plane $\alpha(q)$ obtained by omitting from $\pi(p)$ the line $\pi(p)\cap\pi(q)$ (i.e. the line $z=w=0$), is isomorphic to $\mathcal{M}_q$, and circles not through $q$ project onto ovals in $\alpha(q)$. The point $(x,y,\phi,1)$ projects onto $(x,y,\phi,0)$, and we can choose affine coordinates $(X,Y)$ in $\alpha(q)$, so that $X=x/\phi$, $Y=y/\phi$ for $\phi\neq 0$, and $(0,0)=p$. The point $(X,Y)$, $\neq(0,0)$, then corresponds to the point $(tX,tY,t,1)$ on $\Omega$, where $t=\phi(tX,tY)$. (It is clear

geometrically that $t$ is well-defined, and this follows algebraically from (ii).) A circle $S$ not through $q$ is the point-set $\{(x, y, \phi, 1) \mid Ax+By+C\phi+D=0\}$, for some constants $A, B, C, D$ with $D \neq 0$, and $S$ projects onto an oval $\mathfrak{O}$ with knot at $(-B, A, 0, 0)$, since the tangent plane to $\Omega$ at $(x, y, \phi, 1)$ is $z=\phi w$.

Suppose that $\mathscr{M}$ admits the $(q, Q)$ automorphism $\tau_h$ which induces the collineation $\hat{\tau}_h$ in $M_q$, where $\hat{\tau}_h(X, Y)=(X+h, Y)$, $h \neq 0$. Then $\tau_h(p)=(h, 0, a\psi(h), 1)$, $=p'$, $\tau_h(p')=p$, and for $(x, y, \phi, 1) \neq p', \ p$, $\tau_h(x, y, \phi, 1)=(t(x+h\phi)/\phi, ty/\phi, t, 1)$, where $t=\phi(t(x+h\phi)/\phi, ty/\phi)$. Since $\phi(t(x+h\phi)/\phi, ty/\phi)=a\psi[t(x+h\phi)/\phi]+b\psi[ty/\phi]$,

$$t/\psi(t)=[\phi/\psi(\phi)]+a\psi(h). \tag{11}$$

Let $\tau_h(S)=S'$, the section of $\Omega$ by the plane $A'x+B'y+C'z+D'w=0$, and let $\mathfrak{O}'$ be the projection of $S'$ on $\alpha(q)$. Then $\mathfrak{O}'$ has knot $(-B', A', 0, 0)$, and since $\hat{\tau}_h$ is a translation of $\alpha(q)$, $(-B', A', 0, 0)=(-B, A, 0, 0)$; without loss of generality, choose $A'=A$, $B'=B$. If $\mathbf{x}, \neq p, p'$, is on $S$, so that $Ax+By+C\phi+D=0$, then, since $\tau_h(\mathbf{x})$ is on $S'$, $A(x+h\phi)+By+C'\phi+D'(\phi/t)=0$. Thus for all such $\mathbf{x}$, $C\phi+D=(Ah+C')\phi+D'(\phi/t)$, i.e. $\phi/t=k_1\phi+k_2$, where $k_1=(Ah+C+C')/D'$, $k_2=D/D'$. Let $\mathbf{x}_i=(x_i, y_i, \phi_i, 1)$, $i=1, 2$, be distinct points on $\Omega$, $\neq q, p, p'$, and let $\pi$ be any plane on $\mathbf{x}_1$, $\mathbf{x}_2$. Then there exist $k_1$, $k_2$, depending only on $\pi$ and $h$, such that $\phi_i/t_i=k_1\phi_i+k_2$, $i=1, 2$. But $\mathbf{x}_1 \neq \mathbf{x}_2 \Rightarrow \phi_1 \neq \phi_2$, whence $k_1$ and $k_2$ are uniquely determined by $\mathbf{x}_1$ and $\mathbf{x}_2$. Thus the values of $k_1$ and $k_2$ are constant for every plane $\pi$ on $\mathbf{x}_1$ and $\mathbf{x}_2$. But every point of $\Omega$ is on some such plane $\pi$. Hence there exist constants $k_1$, $k_2$ such that $\phi/t=k_1\phi+k_2$ for all points $\mathbf{x} \neq p, p', q$ on $\Omega$. Since $\tau_h$ is involutory and $k_2=D/D'$, it follows that $D/D'=D'/D$, whence $D'=D$ and $k_2=1$. Now suppose $S$ is on $p'$, so that $S'$ is on $p$. Then, with $K=a\psi(h)=\phi(h, 0)$, $Ah+CK+D=0$, and $C'=0$. Thus

$$k=(Ah+C+C')/D'=(Ah+C)/D=(CK+C+D)/D.$$

In particular, with $C=D$, we have $k_1=K$. Thus $\phi/t=K\phi+1$ for all points $\mathbf{x} \neq p, p', q$ on $\Omega$. On solving for $t$ and substituting in (11) we obtain:

$$\psi(\phi)=K\phi^2/[(\psi(K)-K^2)\phi-K], \quad \text{for all } \phi \neq K. \tag{12}$$

But if $\phi_i=\phi(x_i, y_i)$, $i=1, 2$, then $\phi_1+\phi_2=\phi(x_1+x_2, y_1+y_2)$. Since $\psi(\phi_1+\phi_2)=\psi(\phi_1)+\psi(\phi_2)$, it follows from (12) that $\psi(K)=K^2$, and hence that $\psi(\phi)=\phi^2$ for all $\phi$. Possibly $\phi: F \times F \to F$ is not surjective. However, $\phi(x, 0)=a\psi(x)$ for all $x$, so that $\psi[a\psi(x)], \ =\psi(a)\psi^2(x), \ =a^2[\psi(x)]^2$ for all $x$. In particular, with $x=1$, $\psi(a)=a^2$, whence $\psi^2(x)=[\psi(x)]^2$ for all $x$. Let $k \in F$, $k \neq 0$, and let $x$ be the unique non-zero solution of $\psi(x)=kx$. Then $\psi^2(x)=\psi(k)\psi(x)$, whence $\psi(k)=\psi(x)$, and $x=k$, since $\psi$ is injective. Thus $\psi(k)=k^2$ for all $k \in F$, and $\mathscr{M}$ is miquelian, a contradiction. Hence $\mathscr{M}(\Omega, F)$ is of class III 2 and strongly ovoidal.

**Corollary.** *There is no strongly ovoidal Möbius plane of Hering class* II 2 *or* IV 2, *and a strongly ovoidal Möbius plane of class* III 2 *must be of the type described in Theorem* 1 b).

*Proof.* A strongly ovoidal Möbius plane of class II 2, III 2 or IV 2 satisfies the hypotheses of Theorem 1.

**Theorem 2.** *There exists a strongly ovoidal Möbius plane of class* III 2.

*Proof.* We construct a commutative field $F$, with char $F=2$, for which $\psi(x)=x^4$ satisfies (i)–(iii) of Theorem 1.

(a) *Suppose* char $F=2$, *and that* $x^2+x+1$ *is irreducible over* $F$. *If* $x^3-k$ *is irreducible over* $F$, *then* (i) $x^3=k$ *has a unique solution in* $F(\sqrt[3]{k})$, (ii) $a_0\in F$ *and* $a_0=x^2$ *with* $x\in F(\sqrt[3]{k}) \Rightarrow x\in F$.

For $F(\sqrt[3]{k})=\{a_0+a_1t+a_2t^2 \mid a_i\in F,\ i=1, 2, 3\}$, where $t^3=k$. If $x^3=k$ has more than one solution in $F(\sqrt[3]{k})$, then for some $a_0, a_1, a_2\in F$, $(a_0+a_1t+a_2t^2)^2+(a_0+a_1t+a_2t^2)+1=0$. But then $a_0^2+a_0+1=0$, contrary to hypothesis. If $a_0=(b_0+b_1t+b_2t^2)^2$, then $a_0=b_0^2$ and $b_1=b_2=0$.

(b) *Let* $F$ *be a countable field such that* char $F=2$ *and* $x^2+x+1$ *is irreducible over* $F$. *Then* $F$ *can be embedded in a countable field* $G(F)$ *such that* (i) *if* $k\in F$ *then* $x^3=k$ *has a unique solution* $x\in G(F)$, (ii) *if* $a_0\in F$ *and if* $a_0=x^2$ *with* $x\in G(F)$, *then* $x\in F$.

If $k\in F$, then $x^3=k$ has at most one solution in $F$. If $S=\{k\in F \mid x^3=k$ has no solution in $F\}$ is non-empty, then it can be indexed by $I$, where either $I$ is the set of all positive integers or $I=\{1, 2, \ldots, n\}$, for some $n$. Define $F_0=F$, and for each $i\in I$ define $F_i=F_{i-1}(\sqrt[3]{\lambda_i})$ if $x^3-k_i$ is irreducible over $F_{i-1}$, and define $F_i=F_{i-1}$ otherwise. Let $G(F)=\bigcup_{i\in I'} F_i$, where $I'=\{0\}\cup I$. Then $G(F)$ has the required properties (i) and (ii), by (a), induction on $i$ and the standard ascending chain argument. Each $F_i$ is countable, again by induction on $i$, whence $G(F)$ is countable.

(c) *Let* $F_0$ *be a countable field, with* char $F_0=2$, *such that* $x^2+x+1$ *is irreducible over* $F_0$. *Then* $F_0$ *can be embedded in a field* $F$ *such that* (i) *if* $k\in F$ *then* $x^3=k$ *has a unique solution in* $F$, (ii) *if* $a_0\in F_0$ *and if* $a_0=x^2$ *with* $x\in F$, *then* $x\in F_0$.

For each positive integer $i$, define $F_i=G(F_{i-1})$. Let $F=\bigcup_{i=0}^{\infty} F_i$. Then $F$ has the required properties, by (b) and induction on $i$.

Now let $F_0=Z_2(t)$ (the field of rational functions in one variable over the field of 2 elements). Then $F_0$ is countable, char $F_0=2$ and $x^2+x+1$ is irreducible over $F_0$. Let $F$ be the field described in (c), and let $\psi(x)=x^4$ for each $x\in F$. Then $\psi: F\to F$ is a monomorphism, and, by (c), if $k\neq 0$, $\psi(x)=kx$ has a unique solution $x\neq 0$. Since $t$ is not a square in $F_0$, $t$ is not a square in $F$, by (c) (ii), and a fortiori not a 4th power in $F$. Hence $\psi$ satisfies (i)–(iii) of Theorem 1, and we may choose $a=t$ and $b=1$ to satisfy (iv).

## 3. Strongly Ovoidal Möbius Planes which are $(p, C)$ Transitive and $(p, q)$ Half-Transitive

**Theorem 3.** *Let* $\mathcal{M}=\mathcal{M}(\Omega, F)$ *be a strongly ovoidal Möbius plane, where* char $F\neq 2$. *Suppose that* $\mathcal{M}$ *contains* $p$, $q$, $C$ *with* $p\neq q$ *and* $p, q\in C$ *such that* $\mathcal{M}$ *is* $(p, C)$ *transitive and* $(p, q)$ *half-transitive. Let* $F^*$ *denote the multiplicative group of* $F$. *Then* (a) $F^*$ *contains a subgroup* $H$ *of index* 2 *such that, with a suitable choice of coordinates in* $PG(3, F)$, $\Omega=\{(0, 0, 1, 0)\}\cup\{(x, y, z, 1) \mid z=\phi(x, y)\}$, *where* $\phi(x, y)=a_1x^2+y^2$ *if* $x\in H$, $\phi(x, y)=a_2x^2+y^2$ *if* $x\notin H$, *and* $a_1\neq a_2$, (b) $\mathcal{M}$ *is of Hering class* IV 1 $\Leftrightarrow$ $\phi(x, y)\in H$ *for all* $(x, y)\neq(0, 0)$, *and otherwise* $\mathcal{M}$ *is of class* II 1.

*Proof.* For the definition of "half-transitive", see §1, Definition 3. Assume the hypotheses of Theorem 3, and choose coordinates in $PG(3, F)$ as in §2. If $\mathcal{M}$ admits the $(p, q)$ automorphism $D_r$ which maps $(0, 1, \phi(0, 1), 1)$ on $(0, r, \phi(0, r), 1)$, then by (D), $D_r$ maps $(0, y, \phi(0, y), 1)$ on $(0, ry, \psi(r)\phi(0, y), 1)$; moreover, if $\mathcal{M}$ admits $D_r$, $D_t$, then $\mathcal{M}$ admits $D_{r^{-1}}$, $=(D_r)^{-1}$, and $D_{rt}$, $=D_rD_t$. Hence

$$H=\{h\in F^* \mid \mathcal{M} \text{ admits } D_h\}$$

is a subgroup of $F^*$. Let $s\notin H$, $s\neq 0$. Since the group of all $(p, q)$ automorphisms has exactly 2 orbits on $C-\{p, q\}$, every element $k\in F^*-H$ can be written as $k=hs$ for some $h\in H$. Thus $H$ has index 2 in $F^*$. Write $N=\{hs \mid h\in H\}$, $=F^*-H$.

Since $\mathcal{M}$ is $(p, C)$ transitive, the conclusions of (A)–(D) still hold, and since $\mathcal{M}$ is $(p, q)$ half-transitive, (E), (F) can be replaced by (E'), (F'), as follows:

(E') *If $\mathcal{M}$ is $(p, q)$ half-transitive, then*

$$\phi(xh, yh)=\psi(h)\,\phi(x, y) \quad \textit{for all } x, y\in F \textit{ and all } h\in H, \tag{6'}$$

$$\psi(h_1h_2)=\psi(h_1)\,\psi(h_2) \quad \textit{for all } h_1, h_2\in H. \tag{7'}$$

(F') *If $\mathcal{M}$ is $(p, C)$ transitive and $(p, q)$ half-transitive, then for all $k\in F$ and all $h\in H$,*

$$a_1(kh)=a_1(k)\,\psi(h)h^{-1}, \tag{8'}$$

$$a_2(kh)=a_2(k)\,\psi(h)h^{-1}, \tag{9'}$$

$$a_3(kh)=a_3(k). \tag{10'}$$

By (10'), for all $h\in H$, $a_3(h)=a_3(1)=c_1$, say, while $a_3(sh)=a_3(s)=c_2$. By (4), $a_3(k+l)=a_3(k)\,a_3(l)$ for all $k$, $l\in F$. Thus if $k\in H$ and $l\in N$, where we choose $k+l\neq 0$, then $c_1c_2=c_1$ or $c_2$ according as $k+l\in H$ or $\in N$, i.e. $c_2=1$ or $c_1=1$. Suppose $c_1=1$. Then with $k$, $l\in N$, $k+l\neq 0$, $c_2^2=1$ or $c_2$ according as $k+l\in H$ or $\in N$, whence $c_2=1$ or $-1$. Suppose $c_2=-1$. By (4), if $h_1$, $h_2\in H$ and $h_1+h_2\neq 0$, then $h_1+h_2\in H$, since $a_3(h_1+h_2)=1$, while $sh_1+sh_2\in H$, since $a_3(sh_1+sh_2)=(-1)^2=1$; this is impossible. Similarly, $c_2=1$ implies $c_1=1$. Hence $a_3(k)=1$ for all $k\in F$.

Since char $F\neq 2$, $a_2(k)=0$ only if $k=0$, by (C). By (5), $ka_2(l)=la_2(k)$ for all $k$, $l\in F^*$, whence $a_2(k)=dk$ for all $k$, where $d$ is a non-zero constant. By (9'), $a_2(h)=a_2(1)\psi(h)h^{-1}$ for all $h\in H$, i.e. $dh=d\psi(h)h^{-1}$, so that $\psi(h)=h^2$ for all $h\in H$. By (8'), $a_1(kh)=a_1(k)h$ for all $k\in F$ and all $h\in H$. Let $a_1(1)=c_1$, $a_1(s)=c_2s$. Then $a_1(h)=c_1h$ and $a_1(sh)=c_2sh$ for all $h\in H$. By (2), $a_1(k+l)=a_1(k)+a_1(l)$ for all $k$, $l\in F$. Hence $a_1(h+sh)$, $=c_1h+c_2sh$, $=c_1(h+sh)$, $c_2(h+sh)$ or 0, according as $h+sh\in H$. $\in N$ or $=0$. In each case we see that $c_1=c_2$. Thus $a_1(k)=ck$ for all $k\in F$, where $c$ is a constant, possibly 0.

By (6') and the fact that $\psi(h)=h^2$ for all $h\in H$, $\phi(h, 0)=ah^2$, where $a=\phi(1, 0)$, and $\phi(0, h)=bh^2$, where $b=\phi(0, 1)$. Also $\phi(sh, 0)=a'(sh)^2$, where $a'=\phi(s, 0)/s^2$ and $\phi(0, sh)=b'(sh)^2$, where $b'=\phi(0, s)/s^2$. By (5), $\phi(0, k+l)=dk+\phi(0, k)+\phi(0, l)$ for all $k$, $l\in F$. In particular, if $h_1$, $h_2\in H$ and if $h_1+sh_2\neq 0$, then $\phi(0, h_1+sh_2)=dsh_1h_2+bh_1^2+b's^2h_2^2$, $=b(h_1+sh_2)^2$ or $b'(h_1+sh_2)^2$ according as $h_1+sh_2\in H$ or $N$. In the first case, $2bsh_1h_2+bs^2h_2^2=dsh_1h_2+b's^2h_2^2$, and in the second case $b'h_1^2+2sb'h_1h_2=bh_1^2+dsh_1h_2$. By Theorem IV we can assume that $F$ is infinite, since otherwise $\mathcal{M}$ is miquelian, and hence $(p, q)$ transitive, not $(p, q)$ half-transitive.

Thus at least one of the above equations must hold for more than one value of the ratio $h_1/h_2$, and in either case $b=b'$ and $d=2b$. In particular $\phi(0, sh)=b(sh)^2$ for all $h\in H$. Now by (1), with $y=0$, $\phi(x, k)=ckx+ax^2+bk^2$ if $x\in H$, while $\phi(x, k)=ckx+a'x^2+bk^2$ if $x\in N$. (If $x=0$, $\phi(x, k)=bk^2$, and both formulae apply.)

Since char $F\neq 2$ and $b\neq 0$, the coordinate transformation $x'=x$, $y'=y+(c/2b)x$, $z'=z/b$, $w'=w$ is non-singular. Let $a_1=(4ab-c^2)/4b^2$, $a_2=(4a'b-c^2)/4b^2$. Then in the new coordinate-system (where we now suppress the superscripts), $\Omega=\{(0, 0, 1, 0)\}\cup\{(x, y, z, 1)|z=\phi(x, y)\}$, where $\phi(x, y)=a_1x^2+y^2$ if $x\in H$ and $\phi(x, y)=a_2x^2+y^2$ if $x\notin H$. Since $\mathcal{M}$ is not miquelian, $a_1\neq a_2$.

We now determine the Hering class of $\mathcal{M}$ (still under the assumption that $\mathcal{M}$ is strongly ovoidal). Write $T=F\times F-(0, 0)$. Suppose first that $\phi(x, y)\in H$ for all $(x, y)\in T$. Let $\alpha(x, y, z, w)=(x, y, w, z)$; then $\alpha$ is a collineation of $PG(3, F)$ which fixes $C$ and exchanges $p$ and $q$. If $(x, y)\in T$, $\alpha(x, y, \phi(x, y), 1)=(x/\phi, y/\phi, 1/\phi, 1)\in\Omega$, since $\phi\in H$. Thus $\alpha$ preserves $\Omega$. Since $\mathcal{M}$ is $(p, C)$ transitive and $\alpha$ maps $(p, C)$ on $(q, C)$, $\mathcal{M}$ is also $(q, C)$ transitive, and hence is at least of class IV 1. If $\mathcal{M}$ admits any additional transitivity pairs, there exists a collineation of $PG(3, F)$ which preserves $\Omega$ and does not fix $C$. But the transitive tangents which correspond to the transitivity pairs $(x, C)$ with $x\in C$ form the set of all tangents to $\Omega$ through $(0, 1, 0, 0)$ (since this point is the pole of $x=0$ with respect to each of the half-quadrics which together form $\Omega$). Hence $\Omega$ must admit 2 non-concurrent transitive tangents. But then $\mathcal{M}$ is miquelian by Theorem III b), a contradiction.

Suppose, conversely, that $\mathcal{M}$ is of class IV 1. Then $\mathcal{M}$ is $(q, C)$ transitive, and the group $\Gamma$ generated by all collineations which induce $(p, C)$ or $(q, C)$ automorphisms is 2-transitive on the points of $C$. Hence $\Gamma$ contains a linear collineation $\beta$ which fixes $C$, exchanges $p$ and $q$, and preserves $\Omega$. One calculates that $\beta(x, y, z, w)=(kx, y, lw, z/l)$ for some $k$, $l\in F^*$. Then for $(x, y)\in T$, $\beta(x, y, \phi, 1)=(k\,1\,x/\phi, l\,y/\phi, l^2/\phi, 1)$. Since $\beta$ preserves $\Omega$, if $k\,l/\phi\in H$ then $k^2=1$, while if $k\,l/\phi\in N$ then $k^2=a_1/a_2$ or $a_2/a_1$. Since $a_1\neq a_2$, it follows that either $\phi(x, y)\in H$ for all $(x, y)\in T$ or $\phi(x, y)\in N$ for all $(x, y)\in T$. In the latter case, by considering the pairs $(x, y)=(1, 0)$, $(1, a_1/a_2)$, $(1, a_2)$, we have $a_1$, $a_1+(a_1/a_2)^2$, $a_1+a_2^2\in N$. But since $a_1+(a_1/a_2)^2=(a_1/a_2^2)(a_1+a_2^2)$ and $H$ is of index 2 in $F^*$, this implies the contradiction $a_2^2\in N$. Hence $\phi(x, y)\in H$ for all $(x, y)\in T$.

Finally, if $\mathcal{M}$ is not of class IV 1 then it is of class II 1 or III 1, by Hering's Theorem and Theorem III b). If $\mathcal{M}$ is of class III 1, then it is $(p, D)$ transitive, where $D$ is the circle $\Omega\cap(y=0)$. Thus for each $k\in F$ there exists a collineation of $PG(3, F)$ which preserves $\Omega$ and maps

$$(x, y, z, 1) \quad \text{on} \quad (x+k, y, b_1x+b_2y+b_3z+\phi(k, 0), 1),$$

where $b_1$, $b_2$, $b_3$ depend only on $k$. (Exchange $x$ and $y$ in (A) of § 2.) It is easily verified that this contradicts the assumption $a_1\neq a_2$.

**Theorem 4.** *Let $F$ be a commutative field with* char $F\neq 2$. *Let $H$ be a subgroup of index 2 in $F^*$, and in $PG(3, F)$ let $\Omega=\{(0, 0, 1, 0)\}\cup\{(x, y, z, 1)|z=\phi(x, y)\}$, where $\phi(x, y)=a_1x^2+y^2$ if $x\in H$, $\phi(x, y)=a_2x^2+y^2$ if $x\notin H$, and $a_1\neq a_2$. Let $N=F^*-H$, $S=\{s\in F|s=t^2$ for some $t\in F\}$, $R=\{r\in F|r=(a_1+t^2)/(a_2+t^2)$ or $(a_2+t^2)/(a_1+t^2)$ for some $t\in F\}$. Then $\Omega$ is an ovoid with tangent plane $z=0$ at*

$(0, 0, 0, 1)$ *if and only if* (i) $0 \notin R$, (ii) $R \subseteq H$, (iii) *if* $v \in N$ *then* $(v+1)^2 - 4vr \in S$ *for all* $r \in R$, (iv) *if* $v \in H$, *then for each* $r \in R$, *the equation* $u^2 - (v+1)u + vr = 0$ *has no solution* $u \in N$.

*Proof.* It follows from the definition of $\Omega$ and the fact that $\phi: F \times F \to F$ is well-defined that the plane $w=0$ meets $\Omega$ only at $p=(0, 0, 1, 0)$, and that any line on $p$ which is not in this plane meets $\Omega$ in exactly 2 points. Let $\Omega_1, \Omega_2$ be the subsets of $\Omega$ for which $x \in H$, $x \in N$ respectively, and let $C = \Omega \cap (x=0)$. Let $q=(0, 0, 0, 1)$. Then $\Omega \cap (z=0) = \{q\}$ if and only if $\phi(x, y) = 0 \Rightarrow x = y = 0$. This is equivalent to condition (i). Let $L$ be a line on $q$ which does not lie in the plane $z=0$. Then $L = q \cup \mathbf{x}_1$, where $\mathbf{x}_1 = (x_1, y_1, z_1, 0)$ for some $x_1, y_1, z_1$ with $z_1 \neq 0$. We can assume $\mathbf{x}_1 \neq p$, since $\Omega \cap (q \cup p) = \{p, q\}$; note that $\mathbf{x}_1 \neq p \Rightarrow \mathbf{x}_1 \notin \Omega$. Any point $\mathbf{x} \neq q, \mathbf{x}_1$ on $L$ has coordinates $(x_1/u, y_1/u, z_1/u, 1)$ for some $u \in F^*$. Now $\mathbf{x} \in \Omega, \Leftrightarrow u = (a_1 x_1^2 + y_1^2)/z_1$ and $x_1/u \in H$, while $\mathbf{x} \in \Omega_2 \Leftrightarrow u = (a_2 x_1^2 + y_1^2)/z_1$ and $x_1/u \in N$, and $\mathbf{x} \in C \Leftrightarrow u = y_1^2/z_1$ and $x_1 = 0$. If $x_1 = 0$, $L$ meets $\Omega$ in exactly one point $\neq q$. The same is true for $x_1 \neq 0$ if and only if $(a_1 x_1^2 + y_1^2)/(a_2 x_1^2 + y_1^2) \in H$; since $y_1^2/x_1^2$ is an arbitrary square, this is equivalent to condition (ii). Since Aut $PG(3, F)$ contains a subgroup which fixes $\Omega$ and is transitive on $C - \{p\}$, it is unnecessary to consider the ovoid conditions at other points on $C$.

Let $\mathbf{x}_1 = (x_1, y_1, z_1, 1) \in \Omega_1$, so that $\mathbf{x}_1 \in H$ and $z_1 = a_1 x_1^2 + y_1^2$. Let $\mathbf{x}_2 = (0, y_2, z_2, 1)$, $\neq p$. Let $L = \mathbf{x}_1 \cup \mathbf{x}_2$; then $L \cap (w=0) \notin \Omega$. If $\mathbf{x} \in L$, with $\mathbf{x} \neq \mathbf{x}_2$, and if $\mathbf{x}$ is not in the plane $w=0$, then $\mathbf{x} = (x_1/u, [y_1 + y_2(u-1)]/u, [z_1 + z_2(u-1)]/u, 1)$ for some $u \in F^*$; the value $u=1$ corresponds to $\mathbf{x}_1$. Now $\mathbf{x} \in \Omega_1 \Leftrightarrow u \in H$ and $[z_1 + z_2(u-1)]\, u = a_1 x_1^2 + [y_1 + y_2(u-1)]^2$. On simplifying, and using the fact that $\mathbf{x}_1 \in \Omega_1$, we find:

$$\begin{aligned} &\mathbf{x} \in \Omega_1 \Leftrightarrow u = 1, \quad \text{or } u \in H \\ \text{and} \qquad &u(y_2^2 - z_2) = y_2^2 + z_1 - 2y_1 y_2 = a_1 x_1^2 + (y_1 - y_2)^2. \end{aligned} \tag{13}$$

The point $\mathbf{x}$ satisfies the equation of

$$\begin{aligned} \Omega_2 &\Leftrightarrow u[z_1 + z_2(u-1)] = a_2 x_1^2 + [y_1 + y_2(u-1)]^2, \\ &\Leftrightarrow u^2(y_2^2 - z_2) + u[2y_1 y_2 - 2y_2^2 + z_2 - z_1] + [a_2 x_1^2 + y_1^2 - 2y_1 y_2 + y_2^2] = 0, \\ &\Leftrightarrow u^2(y_2^2 - z_2) - u[(y_2^2 + z_1 - 2y_1 y_2) + (y_2^2 - z_2)] + [a_2 x_1^2 + (y_1 - y_2)^2] = 0. \end{aligned}$$

Thus:

$$\begin{aligned} &x \in \Omega_2 \Leftrightarrow u \in N \\ \text{and} \qquad &u^2(y_2^2 - z_2) - u[\{a_1 x_1^2 + (y_1 - y_2)^2\} + (y_2^2 - z_2)] + [a_2 x_1^2 + (y_1 - y_2)^2] = 0. \end{aligned} \tag{14}$$

If $y_2^2 - z_2 = 0$, then $\mathbf{x}_2 \in C$, and in this case $\Omega \cap L = \{\mathbf{x}_1, \mathbf{x}_2\}$, as we have seen above. So assume $y_2^2 - z_2 \neq 0$. Let $u = v$ be the solution of (13), and let $r_2 = [a_2 x_1^2 + (y_1 - y_2)^2]/[a_1 x_1^2 + (y_1 - y_2)^2]$, so that $r_2 \in R$. Then (14) can be written:

$$\text{If } y_2^2 - z_2 \neq 0, \text{ then } \mathbf{x} \in \Omega_2 \Leftrightarrow u \in N \quad \text{and} \quad u^2 - u(v+1) + v r_2 = 0. \tag{15}$$

If $z_2 + z_1 = 2y_1 y_2$, then $v=1$ in (13). If (15) with $v=1$ has repeated roots, then $r_2 = 1$, in contradiction to $a_1 \neq a_2$, and since $r_2 \in H$, the roots of (15) with $v \in H$ are either both in $H$ or both in $N$. Hence the "ordinary" tangent plane to $\Omega_1$ at $\mathbf{x}_1$, i.e. $z + z_1 = 2a_1 x_1 x + 2y_1 y$, which intersects $x=0$ in the line $x=0$, $z + z_1 = 2y_1 y$, must in fact be the tangent plane to $\Omega$ at $\mathbf{x}_1$, if $\Omega$ is indeed an ovoid. This is the case provided that (15) has no roots in $N$ for $v=1$, a special case of (iv). If $v \in N$, then (15)

must have exactly one root in $N$; since $vr_2 \in N$, this is so if and only if $(v+1)^2 - 4vr_2 \in S$. If $v \in H$, then (15) must have no root in $N$.

Similarly, choosing $\mathbf{x}_1 \in \Omega_2$, we find analogues of (13)–(15) with $a_1$ and $a_2$ interchanged throughout; in particular $r_2$ is replaced in (15) by

$$r_1 = [a_1 x_1^2 + (y_1 - y_2)^2]/[a_2 x_1^2 + (y_1 - y_2)^2].$$

Each $r \in R$ will appear in (15) or its analogue for some $\mathbf{x}_1 \in \Omega_1 \cup \Omega_2$. Hence $\Omega$ is an ovoid if and only if $F$ satisfies (i)–(iv). This completes the proof of Theorem 4.

The conditions of Theorem 4 are certainly satisfied if $F$ is ordered, with $H$ as the set of positive elements, and if every element in $H$ is a square. In this case $a_1, a_2 \in H$ by Theorem 4(i), whence $\phi(x, y) \in H$ for all $(x, y) \neq (0, 0)$. The projective coordinate transformation $x = x'$, $y = y'$, $z = z' + w'$, $w = -z' + w'$, and the affine specialization $X = x'/w'$, $Y = y'/w'$, $Z = z'/w'$ then allow us to write $\Omega = \{(X, Y, Z) | X \geqq 0$ and $a_1 X^2 + Y^2 + Z^2 = 1\} \cup \{(X, Y, Z) | X < 0$ and $a_2 X^2 + Y^2 + Z^2 = 1\}$, where $a_1, a_2 > 0$ and $a_1 \neq a_2$. In this form, for $F = \mathbb{R}$, $a_1 = 1$, $\Omega$ is recognizable as the "squashed sphere" which appears as an example in [5] and [11]. Ewald [5], has shown that $\mathcal{M}(\Omega, \mathbb{R})$ is of class IV 1, without assuming it is strongly ovoidal; his proof is topological.

The Möbius planes discussed in § 3, particularly those of class IV 1, have a good deal in common with the projective "generalized Moulton planes" which were characterized geometrically by André [1], and Yaqub [13], and one might hope to obtain an analogous characterization. I know no example of a Möbius plane which is $(p, C)$ transitive, $(p, q)$ half-transitive and of Hering class II 1.

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