Math. Z. 138, 273–294 (1974)

On the Conjugacy Problem for Knot Groups

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1. Introduction

In this paper a technique is developed for solving the conjugacy problem for a large class of knot groups. In particular it provides the first known proof of the solvability of the conjugacy problem for the group of a non-alternating knot. The technique is applied to solve the conjugacy problem for the infinite class of (non-alternating) cable knots of type (2,1) on knotted tori. It is also applied to solve the conjugacy problem for two more complicated knots, and to provide a different proof for the theorem of Weinbaum [12] on prime alternating knots.

The approach is based on the small cancellation arguments of Lyndon and Schupp. We employ the Wirtinger presentation of the knot group (rather than the Dehn presentation employed in [12] and [2]). Since this presentation satisfies almost none of the small cancellation conditions on relator regions, we are forced to use somewhat differently defined faces in the graph theoretic part of the approach. The principal tool employed is the dual of a (small cancellation type) conjugacy diagram, and the resulting natural faces which we call sections. In this paper we consider only prime knots. With considerable additional machinery the approach was extended to all alternating knots in the work announced in [1], but it seems reasonable that the method in [2] will eventually show that the conjugacy problem for a composite knot is solvable if those for its prime parts are solvable.

While the basic techniques are developed in a uniform manner, their application makes use of specific properties of the knots examined. There is as yet no uniform procedure for applying the technique to all knots, but there are no known cases where it does not apply.

This research was supported by the National Science Foundation. It profited from many conversations with P. E. Schupp, C. F. Miller, R. Finney, G. Francis, R. C. Lyndon, and especially W. Haken.

2. Background

a) *Knot Theory* (see [4, 8, 9])

A *knot* is a subset of 3-space which is homeomorphic to a circle. We consider only tame knots (see references for precise definition). Such a knot may be deformed into a polygon (i.e., a union of a finite number of line segments). This polygon may be projected into a plane disjoint from the knot (and intuitively below it) in such a way that the following conditions are satisfied: i) the only multiple points of the projection are double points; ii) no double point is the image of any vertex of the polygon; iii) there are finitely many double points. When the term *projection* is used we shall always mean such a projection.

We will use the term *crossing* for the preimage of a double point of the projection. The "higher" of the two preimage points will be called the *overcrossing point* and the lower will be called the *undercrossing point.* The images of small arcs of the knot including these points are called *overcrossing* and *undercrossing curves*, respectively.

Let K be a knot and let S be a 2-sphere which intersects K in precisely two points. Define K_1 and K_2 to be components if K_1 is the union of the part of K lying outside of S, the two points of intersection and an arc on the sphere joining the points and K_2 is similarly defined for the part of K lying inside S. K is called *prime* if no such sphere S splits K into two components neither of which has complement homeomorphic to that of the circle. A projection is called *minimal* if no other projection of the knot has fewer crossings. A knot is called *alternating* if it has some projection for which undercrossing and overcrossing points alternate on the knot.

We will use Fig. 1, which gives a minimal projection of the prime non-alternating knot with eleven crossings constructed by Little [5] to illustrate some further definitions.

The *group* of a knot K is defined to be the fundamental group of the complement of K. Given a projection π' of a knot K, we obtain a *Wirtinger presentation* for the group of K as follows: With each arc of the projection joining consecutive undercrossing points, we associate a generator a_i numbered so that successive generators in the orientation of the projection have successive subscripts. (In practice we will often use $a, b, c, \dots$ for $a_1, a_2, a_3, \dots$.) Associate a relator with each crossing as follows: draw a circle about the crossing, with radius small compared to the minimal distance between crossings. Read off, in counterclockwise order, the generators crossed by the circle, assigning exponent 1 if the generator is directed into the circle, and -1 if it is directed out of the circle. For example, in Fig. 1, the relator R_1 associated with the crossing R_1^* is $agb^{-1}g^{-1}$, while the relator R_3, associated with R_3^*, is $ca^{-1}d^{-1}a$. In general, relators for a Wirtinger presentation of the group of an n-crossing knot will have the form $a_i a_{j(i)}^{\varepsilon_i} a_{i+1}^{-1} a_{j(i)}^{-\varepsilon_i}$ (where subscripts are taken modulo n and where $j(i)$ is the subscript of the overcrossing generator at the crossing). Thus, a Wirtinger presentation has n generators and n relators. It is well known (and easy to prove using techniques developed below) that any one of these relators is a product of conjugates of the other $n-1$, and hence one relator may be omitted. On the other hand, it is known that the abelianization of any knot group is infinite cyclic, so that at most one relator may be omitted. We will always eliminate one of the n such relators from our group presentation. The crossing corresponding to the omitted relator will be called the special crossing of π' and denoted by X. The resulting presentation of the knot group will be called a *special Wirtinger presentation.*

b) Small Cancellation Theory and Graph Theory (see [7, 10])

Let G be a group with presentation $\langle a_1, \dots, a_n; R_1, \dots, R_m \rangle$. Let W_1 and W_2 be words on the a_i which represent conjugate elements of G. We may construct

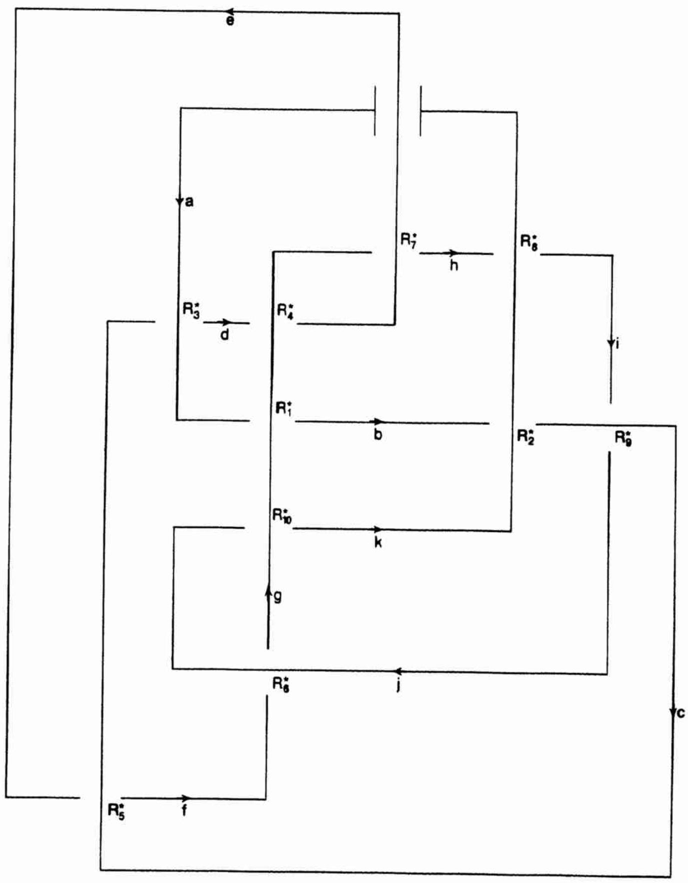


Fig. 1. Little's 11-crossing non-alternating knot

a connected diagram with the following properties: i) the diagram is an annular graph whose directed edges correspond to occurrences of generators a_i and their inverses; ii) the word obtained from each face by reading counterclockwise from any starting point the sequence of generators and inverses corresponding to the edges on that face is a cyclic conjugate of a relator or its inverse; iii) the word corresponding to the inner boundary is W_1; and iv) the word corresponding to the outer boundary is W_2. (Here we think of the boundaries as bounding faces in the complement and use the term boundary word for the word obtained from such a "face".)

A critical tool in the work of Lyndon and Schupp is an elegant counting formula derived from Euler's formula (see Schupp [10, Lemma 2.1]). We shall use a special case of this formula. Let M be a connected planar graph, such that each face has connected boundary, and each edge and vertex lies on a face. Let $d(v)$ (the *degree* of vertex v) be the number of edges lying on v, and let $d(D)$ (the *degree* of face D) be the number of edges on D. Let $\sum'$ mean a sum taken over boundary vertices, and $\sum^0$ a sum taken over interior vertices. Let h be the number of holes in M. Let $E^{\cdot}$ and $V^{\cdot}$ be the number of boundary edges and faces respectively. Then the formula states

$$4(1-h)=\sum'(3-d(v))+\sum^{0}(4-d(v))+\sum(4-d(D))+V^{\cdot}+E^{\cdot}. \qquad (1)$$

3. On Conjugacy Annuli

We remark that in solving the conjugacy problem we may assume that no conjugacy annulus contains a pair of points whose deletion would disconnect the annulus. For if such a pair exists, then the deletion of one of them reduces the question of the conjugacy of a pair of words to the question of whether a single word (the boundary word of the new figure) equals the identity of G. But this is just the word problem which was solved for all knot groups by Waldhausen [11]. One might note that the arguments in this paper may be applied to disks to yield a separate proof of Waldhausen's result for the groups we consider.

We observe that the conjugacy problem for a group is solvable if we have a recursive function f of two variables such that whenever W_1 is conjugate to W_2 in the group then there is a word V such that if W_i has length w_i, $i=1,2$, as a word on the generators of the group then V has length at most $f(w_1,w_2)$, and $V^{-1}W_1V=W_2$. For, since we are dealing with finitely generated groups, we could exhaust all words of length at most $f(w_1,w_2)$ and know that if no one of them conjugated W_1 into W_2 then the words were not conjugate.

A *conjugating word* V for W_1 and W_2 in a conjugacy annulus they determine is a word obtained by reading the generators on a path in the annulus from the beginning of W_1 to the beginning of W_2. Thus we will be able to use the following procedure to show the conjugacy problem solvable for certain knot groups. First, we will show that there exists a recursive function $g(n,k)$ such that if W_1 and W_2 are words on the generators of a group G with n generators with $w_1+w_2=k$ then if W_1 and W_2 are conjugate in G then there is a conjugacy annulus attesting to this fact which has fewer than $g(n,k)$ faces. Then we will simply observe that any conjugating word must have length less than four times the number of faces.

In Section 4 we shall describe the construction of the *dual* of a given conjugacy annulus A. We first describe here a construction which produces a copy of A which we hereafter identify with A itself. The construction is as follows:

Stage 0. Define A_0 to be any face of A.

Stage $k+1$. *Case 1.* Suppose that A_k has two adjacent boundary edges, e_1 and e_2 corresponding to the same interior edge of A. Suppose further that these edges bear inverse labels a_g and a_g^{-1}. Then A_{k+1} is formed from A_k by identifying e_1 and e_2. (We will describe this operation as cancelling e_1 against e_2.)

Case 2. Suppose that Case 1 does not apply and there is at least one face R of A which is not a face of A_k. Furthermore, suppose that some edge e_1 of R is identified (in A) with a boundary edge e_2 of A_k. In this case, form A_{k+1} by adding face R to A_k and identifying e_1 with e_2.

Case 3. Suppose that neither Case 1 nor Case 2 applies, but A is not simply connected and there are two non-adjacent boundary edges e_1 and e_2 in A_k which are identified in A. In this case, form A_{k+1} by cancelling e_1 against e_2.

The construction is finished when none of Cases 1, 2, 3 occurs.

We note that all regions of A appear in A_m, where $m+1$ is the number of the stage at which Case 3 first occurs. Clearly, A_m is itself simply connected. Hence, if A is simply connected the construction will apply with no instances of Case 3, while if A is a conjugacy annulus then one instance of Case 3, possibly followed by further instances of Case 2, will suffice. The number of stages in the construction of A is precisely the number of interior edges of A, since each stage forms an interior edge. We note that the construction only requires that A have no finite set of points by whose removal A becomes disconnected.

4. On the Dual of an Annulus

First, we shall introduce some further terminology. A projection π' of a knot K may be thought of as a graph whose vertices are the crossings. The edges of this graph will be called *strings of the projection.* The preimage of a string of the projection, namely an arc of the knot joining crossing points, will be called a *string of the knot.* Thus a generator may be viewed as a union of consecutive strings and overcrossing points.

Let π' be a minimal projection of the prime knot K. We will use the term *component of* π' to mean a maximal connected component of the complement of π' in the plane. Two components will be said to *lie on a string of* π' if that string is part of the boundary of each of the components. We note that since K is prime two components of π' cannot lie on more than one common string. Choose $\varepsilon>0$ but small relative to the minimal distance between crossings of π'. We delete from π' those points in the image of each undercrossing string of the knot lying within distance ε of the crossing, and call the deleted projection π. We will refer to components of π' as *components of* π. We note that the maximal arcs of π are just the generators of the presentation. If C and D are components of π' lying on a string of π' which is crossed by a generator g then we note that the definition of π requires that a terminal segment of the string be deleted. Thus in the complement of π there is a *passage from C to D* connecting these components.

Using this terminology, we will describe the construction of the dual $\delta(A)$ of annulus A, imagining that the construction of $\delta(A)$ takes place simultaneously with that of A. In forming the dual, we shall give measurements for certain radii and distances. Although not necessary for the theory, this will enable the reader to construct a dual which has only finitely many self intersections, all of which are double points. Now we may define $\delta(A_k)$ for each k as follows.

Stage 0. A_0 is a face of A which is labelled by some relator, say R_i. Let R_i^* be the crossing of the projection corresponding to A_0. $\delta(A_0)$ is defined to be an oriented circle of radius $3\varepsilon/2$ with center at R_i^*. $\delta(A_0)$ is oriented in the counterclockwise direction if the face is determined by an instance of R_i. It is oriented in the clockwise direction if the face is determined by an instance of R_i^{-1}.

We call the circle $\delta(A_0)$ the dual of face A_0 for the following reasons. First, starting at any point on the circle and proceeding in the direction of the orientation, the circle is intersected by four generators in their order of occurrence in a cyclic conjugate of R_i if the circle is oriented in the counterclockwise direction while the order is that of R_i^{-1} if the circle is oriented in the clockwise direction. We interpret an intersection of the circle with the generator as a positive occurrence if the (oriented) generator crosses the circle from right to left (from the point of view of an observer proceeding in the direction of the oriented circle) and as a negative occurrence if the generator crosses the circle from left to right. With this convention, the signed generator occurrences agree with their appearances in the relator (or relator inverse) corresponding to the circle. Having associated the generator occurrences, which determine edges on the face, with points of the dual, we associate vertices of the face with arcs of the dual, namely the arcs joining the associated pairs of edge duals.

At each stage of the construction the dual will be related to A_k in a manner similar to that described above. In particular, we shall construct $\delta(A_k)$ so that the following conditions are satisfied.

1) If $k \leqq m$ (where stage $m+1$ was the first stage at which Case 3 was used) then $\delta(A_k)$ is a set of distinct closed curves, exactly one of which intersects π. If $k > m$ then $\delta(A_k)$ is a set of distinct curves more than one of which intersects π.

2) A curve in $\delta(A_k)$ which intersects π is called a *dual boundary curve of* A_k. It intersects π in a sequence of edges corresponding to a boundary word of A_k in the manner described above.

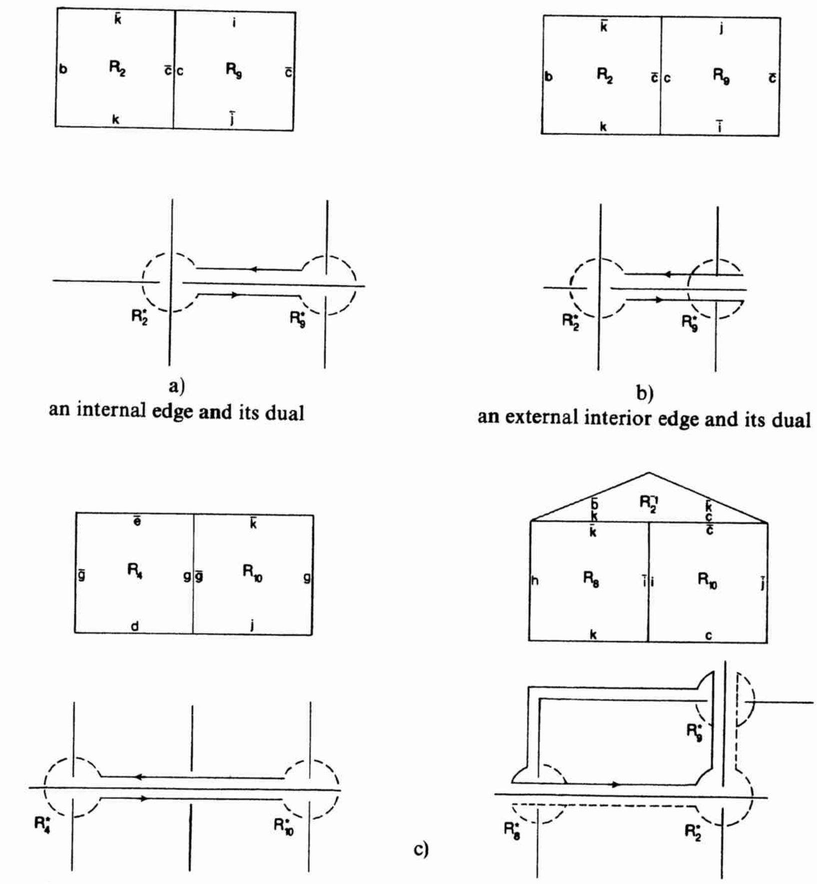
3) The dual of an interior edge of A_k will be a pair of arcs in $\delta(A_k)$ which do not intersect π. These arcs are parallel to the generator corresponding to the label on the edge.

4) The dual of a vertex of A_k is a curve containing intersection points with generators corresponding to the boundary edges on the vertex and arcs which belong to the duals of interior edges on the vertex. If v is an interior vertex, then its dual $\delta(v)$ is a closed curve not intersecting π.

It may be verified from the construction that these conditions hold at Stage 0. It will be clear that the conditions are preserved by the inductive step.

Stage $k+1$. We proceed in two steps. First we define a set δ^* of curves and then obtain $\delta(A_{k+1})$ by a cancellation. In Cases 1 or 3 of the construction of A_{k+1}, δ^* is $\delta(A_k)$. In Case 2, δ^* is the union of $\delta(A_k)$ and a circle of radius $(2^{k+4}-2)\varepsilon/2^{k-3}$, with center at the crossing R^* determined by the relator associated with the added face. The circle is oriented counterclockwise or clockwise according as the face corresponds to R or R^{-1}. The edges which are cancelled at stage $k+1$ of the construction of A are each dual to an intersection of π with a generator. In order for cancellation to take place the same generator must be involved in both intersections and the arcs containing these points must be oriented so that they cross the generator in opposite directions. Thus we may assume that e_1 corresponds to some a_j^γ and e_2 corresponds to $a_j^{-\gamma}$.

Fig. 2 illustrates certain situations which arise in the cancellation of a pair of edges in this construction. In each of the four pairs of figures, the dual of the entire top diagram is shown as the union of the solid and dotted curves. The solid part of the dual curve for a), b), and c), is the dual of the interior edge which is cancelled. The names listed for the interior edges and vertices will be defined and explained in Section 5. The projection used is that of Fig. 1.



a) an internal edge and its dual

b) an external interior edge and its dual

c) another external interior edge and its dual; an external interior vertex and its dual

Fig. 2

The second step of stage $k+1$ is to construct $\delta(A_{k+1})$ from δ^* as follows. First, add to δ^* two line segments parallel to generator a_j and at distance $(2^{k+3}-1)\,\varepsilon/2^{k+3}$ from a_j on either side beginning and ending at δ^* on arcs containing the points dual to the cancelled edges. Next, delete the small arcs containing the cancelled edges which join the new segments. Last, choose an orientation for each new segment consistent with the curve to which it is joined.

Note that the new segments do not intersect π since the distances have been chosen so that they "fit through" passages. The segments added are defined to be the dual of the new interior edge of A_{k+1}. The duals of the two vertices on this edge are the curves of $\delta(A_{k+1})$ to which the dual of the edge belongs.

In Cases 1 and 3, the replacement of $\delta(A_k)$ by $\delta(A_{k+1})$ increases the number of curves of the dual by one. In Case 2, the number of curves is unchanged. We now verify some further properties of the dual.

Lemma. *Of the curves in $\delta(A_k)$, $k \leq m$, exactly one intersects π.*

Proof. This may be seen by induction as follows. It is true by definition for $k=0$. Consider stage $k+1$. If Case 1 is used, the unique curve in $\delta(A_k)$ is replaced by two curves. Since the identified edges are adjacent, their common vertex v becomes an interior vertex. One of the two curves obtained is the dual of this vertex, and since the vertex lies on no boundary edges, the dual curve cannot intersect π. The other curve must contain the duals of all boundary edges and thus must intersect π. It is the only curve in $\delta(A_{k+1})$ which intersects π and is called the *dual boundary curve of* A_{k+1}. In Case 2, no new interior vertex is created but the dual boundary curve of A_k is replaced by a curve with three more intersections with π. Since $k<m$, Case 3 does not apply.

At stage $m+1$, the disk A_m is replaced by an annulus A_{m+1}. In this situation the dual boundary curve of A_m is replaced by two curves both of which intersect π. Since A is an annulus, no further applications of Case 3 are made and an argument similar to the lemma above shows that A itself has exactly two curves in its dual which intersect π.

5. Sections of A

As previously indicated, the graph whose faces correspond to individual relators does not satisfy the small cancellation conditions of Lyndon and Schupp. We will now define a new graph whose faces are unions of the faces of our previous graph. We will henceforth call the faces of our original graph *relator regions.*

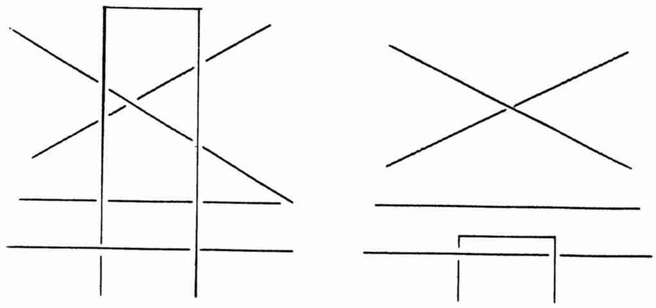
If e is an interior edge of A, its dual consists of two parallel curves in $\delta(A)$ (see Fig. 2). If each of these curves lies in a single component of π then the edge e is called an *internal edge*. (It will be internal to a face of our new graph.) Note that if e is an internal edge then the relators it separates must have dual circles of the same orientation. If e is not an internal edge but is an interior edge of A it is called an *external interior edge.*

Suppose v is an interior vertex of A which lies on an external interior edge. Its dual $\delta(v)$ contains one of the curves in the dual of e so it cannot lie in a single component. The number of edge duals in $\delta(v)$ which lie in more than one component is just the number of external edges on v.

Lemma. *No interior vertex lies on precisely one external interior edge.*

Proof. If v was an external vertex which lay on just one external edge then $\delta(v)$ would contain precisely one edge dual which lay in more than one component of π'. Hence this edge dual would begin and end in the same component. Thus the generator along which it overcrosses out of and into this component overcrosses every string of π' it intersects between these crossings. But this means π could not be minimal, for the part lying between the exit from the component and the reentry could be "pulled back" (as in Fig. 3) to reduce the number of crossings of K, contradicting the minimality of the projection.

Thus we note that every external vertex lies on a path of external edges which either joins two points of the boundary of A or is a closed path. This means that we may use the external edges to form a coarser partition of A than was obtained by using all edges. The graph that we obtain will be called the *section graph* of A. Its faces will be called *sections.* We will show that this graph is much more amenable to small cancellation type arguments than is the original graph whose faces were the relator regions.



non-minimal situation after pullback

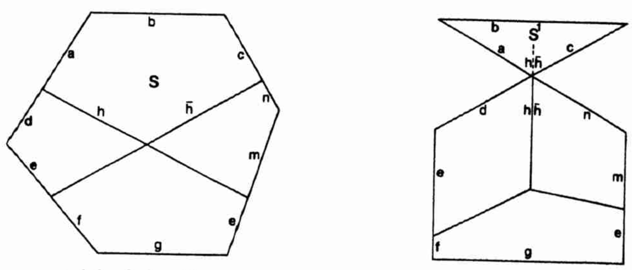
Fig. 3

An edge of A is interior to a face of the section graph (except possibly for its end vertices) if and only if it is an internal edge of A. This definition permits us to refine our minimality condition for A. A *is a minimal annulus* if among all annuli with the same boundaries (i.e. boundary edge sequences determined by the same pair of conjugate words), A has the smallest number of regions, and among those annuli with the smallest number of regions A has the smallest number of external edges. It is important to note that this definition forbids adjacent regions to be inverse and in position to cancel totally against one another. An immediate consequence of the minimality condition is the following lemma.

Lemma A. *Let v be an external interior vertex and let $e_0, \ldots, e_{m-1}$ be the consecutive external edges on v. Let e_i and e_{i+1} (subscripts modulo m) correspond to the same generator h. Then the duals of e_i and e_{i+1} must be oriented in the same direction and hence a face with e_i and e_{i+1} as consecutive boundary edges will not have consecutive labels h^{ε} and $h^{-\varepsilon}$ at v.*

Proof. Suppose not. Consider the section S on which the edges lie (see Fig. 4). If S is cancelled along these edges we would obtain a new section S' with perimeter smaller than that of S by the removal of the two cancelled edges. Thus we obtain an annulus with the same number of regions but fewer external edges, contradicting minimality.

The construction of a dual given in Section 4 may be applied to define the dual of a section. In most cases of interest we will deal with simply connected sections and will seek information about the sections from their duals.



original situation reduced situation

Fig. 4

We will use some elementary properties of winding numbers. We take the viewpoint of [3] where the *winding number* of a closed curve C about a point P is shown to be the number of positive crossings minus the number of negative crossings of the curve over any ray from P to infinity. The concept of negative and positive crossing corresponds with our definition of negative and positive generator occurrences on vertex and edge duals so the theory is immediately applicable.

If a relator R determines the labels of m faces of a section S we say that R *occurs m times in S*.

Lemma. *If relator R occurs m times in section S then the absolute value of the sum of the winding numbers of the curves in $\delta(S)$ about R^* is m.*

Proof. First, we note that since every interior edge of a section is internal, all regions must be determined by relator occurrences with the same sign. Since the lemma is stated in terms of absolute value we lose no generality in assuming that every relator occurs positively in S. In the construction of the dual, an occurrence of Case 2 which adds a circle about R^* increases the winding number of $\delta(A_k)$ about R^* by one. Other occurrences of Case 2 do not affect this winding number. In Cases 1 and 3 the number of curves in the dual is increased but the winding numbers about the crossings are not changed. Thus the lemma is true by induction.

An interior vertex of a section lies on internal edges only. Hence, its dual lies entirely in one component of π. Such a curve must have winding number zero about each crossing. Thus, $W_r(S)$, the number of occurrences of relator R in section S, which shall be called the *weight of R in S*, is the absolute value of the sum of the winding numbers of the dual boundary curves of S about R^*. We define the *weight of S*, $W(S)$ to be the maximum (over all relators) of $W_R(S)$.

Let C be an oriented closed curve with no multiple points except for isolated double points. At each double point we will call the two arcs oriented towards the point *incoming curves* and the two arcs *outgoing curves*. Each incoming curve is associated with an outgoing curve by the parametrization of C. By a *switch* at double point p we mean a reparametrization which changes the association of incoming curves with outgoing curves at p. We define *the simplification of C* to be the reparametrization which effects a switch at each double point of C. Simplification replaces C by a union of simple closed curves which intersect only in the double points of C. Clearly the sum of the winding numbers about any point of the curves in the simplification of C is the same as the winding number of C about that point. Thus, if C has winding number m about a point q there must be at least m curves in the simplification of C with q in their interiors. (Note that a simple closed curve has winding number -1, 0, or 1 about any point.)

Let S be a section and let $\Delta(S)$ be the collection of dual boundary curves of S. (Usually this will be a single curve.) Let R be a relator such that $W(S) = W_R(S)$. By the remarks above there must be at least $W(S)$ closed curves in the simplification of $\Delta(S)$ which contain R^* as an interior point. But, it is easily observed from the construction of the dual that X, the special crossing, must lie exterior to every such curve. Hence each of these curves must separate R^* from X.

Since a knot projection is a graph all vertices of which have degree four, a standard lemma of graph theory (see [6; p. 244] or [8]) to note that complementary components may be "properly two colored". This means that they may be colored with two colors in such a way that whenever two components both lie on the same string their colors are different.

Since S is a section, the duals of interior edges do not contain segments which lie in more than one component. Thus, the dual boundary curves (and hence any curve in their simplification) can only change components at the duals of boundary edges. This means that each curve in the simplification of $\Delta(S)$ must intersect π at an even number of points. If K is prime, by definition any curve which contains a crossing in its interior and another in its exterior must intersect π in at least four points. We may state this in terms of degrees and weights as follows.

Lemma B. *If S is a section of a conjugacy diagram for the group of a knot K then $d(S) \geqq 2\,W(S)$. Moreover, if K is prime then $d(S) \geqq 4\,W(S)$.*

Recall that we only consider conjugacy annuli without any pair of points whose removal would disconnect the annuli. Even with this restriction it is possible to obtain a partition into sections without connected boundaries. (For example, a section can itself be an annulus.) It is easy to show that by addition of edges and vertices on such boundaries we may insure that every section has a connected boundary. We will henceforth assume that this has been done.

6. Sound Assignments

For an annulus with our restrictions, in the terminology of Formula 1, $V^{\cdot} = E^{\cdot}$ and $h = 1$ so that we obtain the somewhat simpler formula:

$$\sum{}'(3 - d(v)) + \sum{}^{\circ}(4 - d(v)) + \sum(4 - d(D)) = 0. \tag{1'}$$

Observe that Formula 1′ tells us that the sum of the degrees of all vertices and faces is precisely four times the number of faces plus four times the number of interior vertices plus three times the number of boundary vertices.

We wish to interpret Formula 1′ in a somewhat different manner. Note that every edge contributes to the degrees of the vertices and faces on which it lies. Each interior edge "contributes one unit" to the degrees of the two faces and the two vertices on which it lies, a total contribution of four. Similarly, boundary edges make total contributions of three. We wish to generalize this procedure slightly. Each edge will contribute the same number of units to the degree sum, but the contributions will not necessarily be made to the faces and vertices on which it lies in the same uniform manner.

More formally, we shall define a *generalized degree* assignment for a graph as an assignment of integers to the faces and vertices based on the following rules.

1) The assignment to each face and vertex is a sum of integer units "contributed" by the edges on the face or vertex.

2) The total contribution of an interior edge is four units. (These must be contributed to vertices or faces on the edge.)

3) The total contribution of a boundary edge is three units.

We will write $p(v)$ or $p(D)$ for the generalized degree of a vertex or face. Define $A \dot{-} B$ to be the maximum of $A - B$ and 0. Then, by the above reasoning we obtain the following analogue of Formula 1′ for our conjugacy annulus.

$$\sum{}'(3 \dot{-} p(v)) = \sum{}^{0}(p(v)-4) + \sum(p(D)-4) + \sum{}'(p(v) \dot{-} 3). \tag{2}$$

(Note that no boundary vertex contributes non-trivially to both sides since at most one of $(p(v)-3)$ and $(3-p(v))$ is positive.)

Certainly the most important generalized degree assignment is the ordinary degree assignment. The properties of the degree assignment are sufficient for our argument for alternating knots. We add further restrictions to the generalized degree to bring it closer to the ordinary degree. A generalized degree assignment is called *sound* if all of the following hold:

a) $p(v) \geqq 4$ for each interior vertex v;

b) $p(D) \geqq W(D)+3$ for each face D;

c) $p(v) \geqq 2$ for each boundary vertex v;

d) there is a bound B depending only on the knot such that $p(v) > d(v) - B$ for every vertex v.

If p is a sound generalized degree assignment on a conjugacy annulus then the left side of the equation in Formula 2 is at most the number of boundary vertices, i.e., $w_1 + w_2$, the sum of the lengths of the boundary words. Each summand of each term on the right must be non-negative (by the soundness condition). We shall use these conditions to bound the number of relator regions in the conjugacy annulus.

Let A be a conjugacy annulus which satisfies our conditions. We will define a collection of bands and layers of A by induction. The *zero-th band* of A is the inner boundary of A. The *zero-th layer* of A consists of all faces (sections) incident on the inner boundary. For $k>0$, the *k-th band* of A is the outer boundary of the subannulus of A consisting of the union of the zero-th through $(k-1)$-st layers of A. The *k-th layer* consists of those faces incident on the k-th band but not belonging to the union of the previous layers. A is said to have a *proper k-th band* if its k-th layer is non-empty.

With each band of A we associate a set of words in a natural way. The words are obtained by reading off the (signed) generator occurrences on the edges in the band, reading in a counterclockwise direction from any starting point. All of the associated words of the band are cyclically conjugate. Since A is a minimal annulus, no distinct pair of proper bands can have an associated word in common. For if they did we could delete the layers separating the bands and form a smaller annulus with the same boundary words (W_1 and W_2).

Let K be a prime knot with n crossings. Let A be a minimal conjugacy annulus which attests to the conjugacy of words W_1 and W_2 in the group of K. We list some conditions on the boundedness of certain numbers which we shall use below.

i) If S is a section of weight m in A then the *perimeter* of S (i.e., the number of boundary edges of S) is greater than $4m/n$ and less than $3nm$. The lower bound is a consequence of Lemma B while the upper bound results from the fact that S cannot have more than $(n-1)m$ regions.

ii) The degrees of the vertices of A are all bounded by $(w_1+w_2)+B$. This is a consequence of the definition of soundness and the fact that each term on the right side of Formula 2 must be non-negative.

iii) A layer of A is called a *full layer* if the two bands which form its boundary are disjoint. Let L be a full layer of A and let B_1 and B_2 be its inner and outer bands respectively. Let b_i be the length of B_i, $i=1,2$. The ratio b_2/b_1 is bounded both above and below by recursive functions of n and k (where $k=w_1+w_2$). This can be seen from i and ii.

iv) If L is a non-full layer of A then its inner boundary band must contain some boundary edge of A. Thus the number of non-full layers of A is at most w_2.

v) Since each term on the right of Formula 2 is non-negative no more than $2k$ disjoint bands can lie on either vertices or faces of generalized degree greater than 4.

From conditions i–v it is clear that one can obtain a recursive function $h(n, k, r)$ such that if A is an annulus for a knot with n crossings and the length of the perimeter of A is k, then if A has r relator regions it must have at least $h(n, k, r)$ consecutive disjoint bands no one of which lies on a vertex or face with generalized degree greater than four under any sound assignment of generalized degrees.

Consider a subannulus of A consisting of the layers bounded by a pair of bands in such a consecutive sequence. By soundness, every face and every interior vertex of such a subannulus would have generalized degree four. If the generalized degree assignment chosen were the degree assignment we would think of this as part of a tiling of the plane by rectangles, hence we extend the usage and call such an annulus a *tiled annulus*.

If we were able to bound the number of bands which could occur in such a tiled annulus then by the above reasoning we could recursively bound the number of relator regions in A. We will show that if A is a minimal conjugacy annulus we can usually show the existence of such a bound. In general we will use two kinds of arguments. We will show that some sections of generalized degree four cannot lie in a tiled annulus with arbitrarily many faces because the properties of the dual of the knot require that it lie within some fixed distance of either a boundary vertex or a vertex or face of generalized degree greater than four. Last we will show that if a tiled subannulus has no such face it must have two bands with the same associated word, contradicting minimality.

We shall not give the recursive functions required in any of the arguments which follow. We shall just show that the number of consecutive bands in a tiled subannulus is effectively bounded. It is clear that in any particular instance it is a routine, if tedious, exercise to provide the recursive function required by an analysis of the argument given.

7. On Prime Alternating Knots

We provide a proof of the theorem of Weinbaum [12] using our methods.

Theorem (Weinbaum). *The conjugacy problem of any prime alternating knot is solvable.*

Proof. We choose our sound generalized degree assignment to be the ordinary assignment of degrees to the faces and vertices. In the case of an alternating knot it is immediate by definition that every generator is the union of two strings.

Thus the primeness of the knot and the two coloring property of the complementary components require that every interior vertex have degree four. By *Lemma B* every face of degree four has weight one so the number of possible faces in a tiled subannulus is bounded by a recursive function of n. Because all vertices and faces have degree four, each layer of the annulus must have the same number of faces, hence each band in a tiled subannulus has the same number of edges. But the number of words of any fixed length is a recursive function of n and that length, and the number of bands is easily seen to be recursively bounded using the bound on the ratio of the lengths of successive bands. Thus the conjugacy problem is solvable.

8. Non-Alternating Knots

The principal reason that the proof of the theorem of the previous section does not apply to non-alternating knots is that generators may be unions of more than two strings. In this case the argument which shows that interior vertices have degree four is neither applicable nor true. We try to find a generalized degree assignment which is as close as possible to the degree assignment but does not permit vertices of generalized degree less than four. It is convenient to define the generalized degree assignment by a diagram.

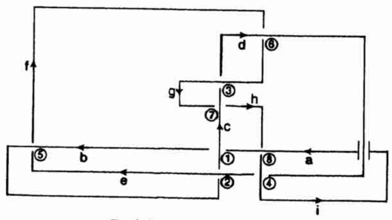
To illustrate our procedure we will consider the knot numbered 9–48 in the table of [8] (see Fig. 5 – top). Henceforth in diagrams we will use subscripts instead of full designations of relators. Thus "5" stands for R_5 while encircled numerals represent the associated crossings in the projection. Inverses are barred (e.g., "$\bar{5}$").

Below the knot projection are six pairs of diagrams labelled a–f. In each case the lower member shows the dual of the upper. Vertex duals are partially dashed for reasons which will be given below. Each of the diagrams illustrates a vertex of (section) degree less than four. The vertices in a, b, c, d, and f have degree two, that in e has degree three. The duals of a–e are simple closed curves, that in f is not. Note that the dual of the vertex in f contains a simple closed curve which traces the entire boundary of the component in which it lies. We say that this dual curve *loops* in a component. Note that a vertex of degree four lies on a face of weight greater than one if and only if its dual curve loops in a component.

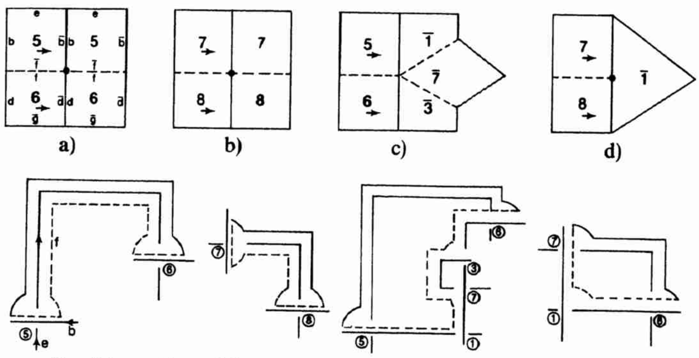
One can see by inspection that a–e are the only vertices of degree less than four which only lie on faces of weight one. (Note that the dual of such a vertex is a simple closed curve which uses parts of relator dual circles in fewer than four components and does not use a circle on the special crossing.)

In our generalized degree assignment, soundness requires that an interior vertex be assigned generalized degree at least four. Since the generalized degree of a vertex is determined by the way in which the unit contributions of the edges on the vertex are distributed, we must make use of two external interior edges on each vertex of degree two and one on each vertex of degree three. In Fig. 5 it is clear that diagrams d and f are related by the loop in diagram f. In what follows it will be evident that if we define our assignment for vertices on sections of weight one the assignment may be extended naturally to all vertices of low degree.

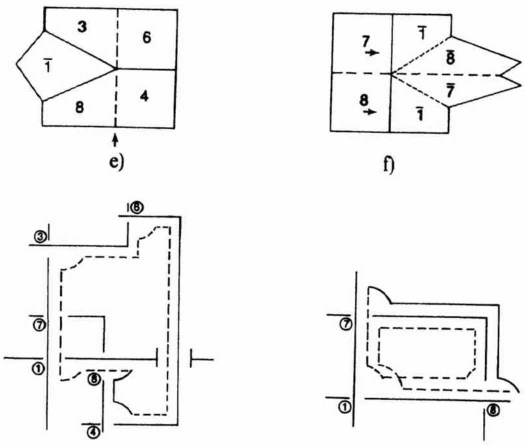
In diagrams a–d two arrows point to edges lying on the center vertex. These arrows indicate that the contribution which would have been made to the section



Reidemeister Knot 9–48



Possible vertices of degree 2 on sections of weight one and duals



Vertex of degree 2 not on section of weight 1

Fig. 5

in which the arrow lies by the edge at the head of the arrow is made instead to the central vertex. In diagram e the arrow on the bottom vertex indicates that the contribution which would have been made to that vertex by the edge joining it to the central vertex is instead made to the central vertex. Thus, in each case the central vertex has generalized degree four. The faces shown in a–d all have degree six so the reduction gives them generalized degree four. The bottom vertex of diagram e has a dual which cannot (as can be seen by tedious inspection) be completed to the dual of an interior vertex of degree less than five, so this reduction is also safe. However, we must show that any face must have generalized degree consistent with the soundness definition under this assignment.

Since simplification (see Lemma B) gives considerable leeway and many extra boundary edges, it is not difficult to see that we need only treat in detail faces of weight one. What we must show is that the edges bearing arrows are, in effect, surplus to the sections to which they belong. By this we mean that the dual boundary curve of the section must have four intersections with generators in addition to those intersections dual to edges bearing arrows. In effect this means that if a face has a dual boundary curve containing the solid portion of one of the vertex duals in diagrams a–d then the boundary vertices on that portion cannot be counted towards the generalized degree of the face. Since we need only consider simple closed curves, inspection shows that under these restrictions there is no face of generalized degree less than four.

Once we have chosen a sound generalized degree assignment we must examine those sections of weight one which have generalized degree exactly four under the assignment. These are the candidates to be faces in a tiled subannulus. Since no internal edge may be a boundary edge of an interior face of A, we restrict attention to sections all of whose boundary edges are external edges.

Suppose that a generator g consists of a single string of the projection. If an interior edge has its dual curves parallel to g then the curves must each lie in a single component and the edge must be internal. Thus, if two relator regions cancel along such a generator, they must be paired in any interior face to which either of them belongs. For example, there could be no interior face containing the region corresponding to R_5 without the region corresponding to R_6. Those sections of generalized degree four which satisfy the minimality condition of Lemma A and have no boundary edges whose duals intersect generators consisting of a single string are called tiles. One can enumerate the tiles of a projection quite easily; we have done it for our example in Fig. 6.

We shall distinguish a class of tiles called *standard tiles.* A tile is called standard if it satisfies one of the following conditions:

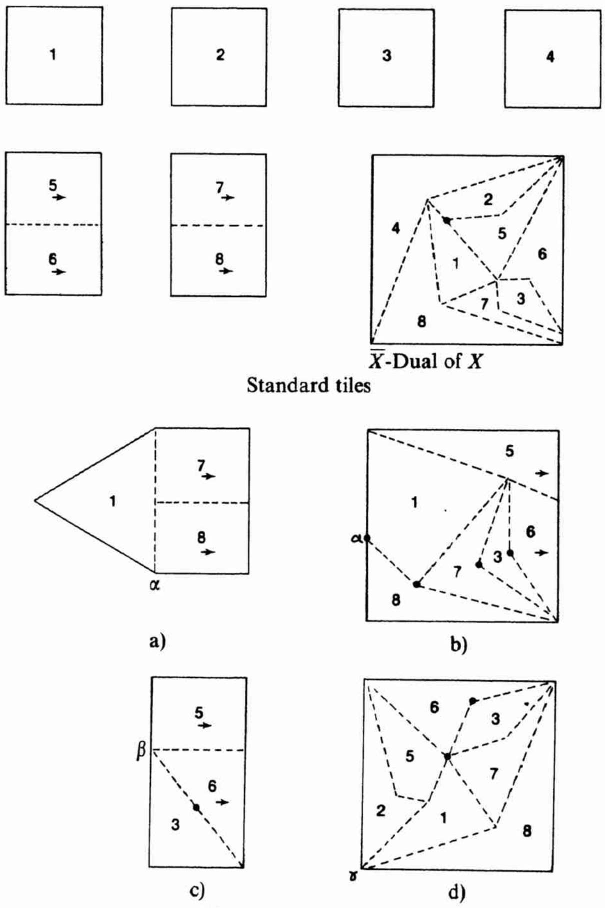
i) It is a relator region.

ii) It is a union of regions corresponding to consecutively numbered relators with its only interior edges having duals which are parallel to generators consisting of single strings.

iii) It contains one region corresponding to each relator of the special Wirtinger presentation. (The boundary word of such a tile is the inverse of the relator of the full Wirtinger presentation which would be dual to the special crossing.)

The standard tiles of our example are shown at the top of Fig. 6. A consequence of the condition on single string generators is the fact that every tile must be a union of standard tiles of the first two types.

A tile cannot be interior to a tiled annulus if it has a vertex whose dual cannot be completed to a closed vertex dual which enters at most four components in addition to any loss due to vertices losing their edge contributions due to the generalized degree assignment. In particular this completion cannot permit



Non-standard tiles

Fig. 6. Tiles for the knot of Fig. 5

the boundary edges on the vertex to have duals which lie in a single component. This property may be checked by careful inspection for any vertex.

In our example, the non-standard tiles satisfy:

Lemma. *No non-standard tile for our example can occur as an interior face of a tiled annulus.*

Proof. We examine the four possibilities. In diagrams a and b of Fig. 6 the same vertex, marked α occurs. By examination of its dual we can easily check that it cannot be part of an interior vertex of degree four, hence cannot be an interior vertex of a tiled annulus. In diagram c the same remarks may be applied to β. After elimination of the first three we see that vertex γ of diagram d cannot occur as a vertex on four tiles of remaining types.

In a tiling consisting of standard tiles it is easily seen that the faces may be arranged in only one way to obtain vertices of degree four. Each tile must be

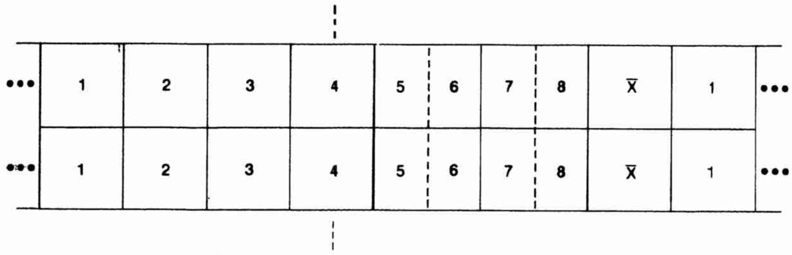


Fig. 7. The standard tiling of our example

bordered on two sides by copies of itself and on the other two sides by its numerical predecessor and successor – as shown in Fig. 7.

The tiling shown in Fig. 7 is called the *standard tiling* corresponding to the projection. It is immediate that if any subannulus is tiled by this tiling all of its bands have the same length. Moreover, depending on the manner in which the tiles are "laid" either every band has the same associated word or each band has an associated word which is the same power of some generator. Thus the tiled subannulus can have no more than seven bands. The above analysis proves the following theorem.

Theorem. *Knot 9–48 of Reidemeister's table is a non-alternating knot with solvable conjugacy problem.*

The technique used in this section has been applied to other knots with similar results. In the next section we will show how the procedure may be applied to an infinite class of knots. We conclude this section by one further example.

Theorem. *Little's eleven-crossing non-alternating knot* (Fig. 1) *has solvable conjugacy problem.*

Proof. We will give diagrams for the generalized degree assignment and the elimination of non-standard tiles. The procedure will be essentially the same as in the previous theorem. The generalized degree assignment is described by Fig. 8.

The one new feature arises in the elimination of three of the non-standard tiles. In these cases we cannot guarantee that any particular vertex has degree greater than four but we note that if both vertex γ and vertex δ are interior vertices then at least one must have degree greater than four.

In Little's knot there is only one non-standard tile (see Fig. 9) and in this case the vertex α cannot have degree four. Thus the theorem is proved.

9. Double Cable Knots

A *cable knot of type* (p, q) is a curve on the surface of a non-trivially knotted torus which cuts each meridian in p points and each longtitude in q points. We shall call a knot a *double cable knot* if it is a cable knot of type (2, 1) on a torus which is itself knotted by a prime alternating knot. A minimal presentation may be obtained by taking a pair of parallel copies of a minimal projection of the alternating knot of the torus and connecting them by a single crossing (see Fig. 10). It is convenient to take this "joining" crossing as the special crossing to make our treatment uniform.

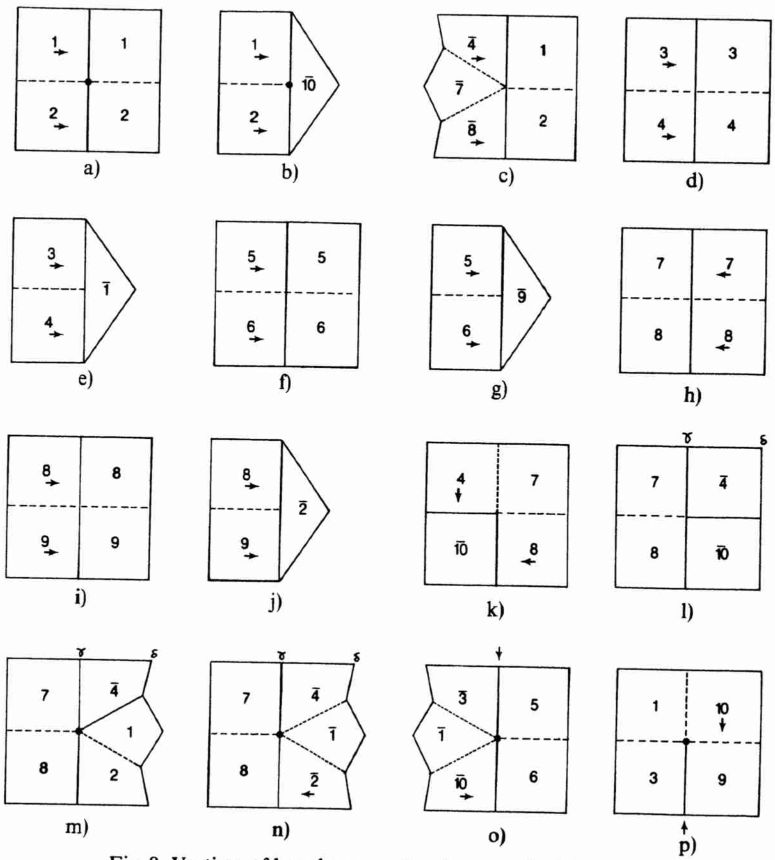


Fig. 8. Vertices of low degree and assignment for Little's knot

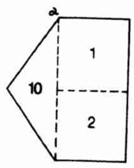


Fig. 9. The non-standard tile for Little's knot

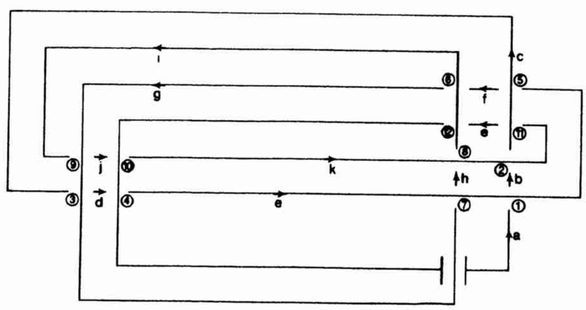


Fig. 10. Double cable trefoil

Theorem. *The conjugacy problem of any double cable knot is solvable.*

Proof. We observe that every crossing of a double cable knot lies on a *single string generator* (i.e., a generator consisting of a single string). Let a be a single string generator on the special crossing. By definition of dual, a cannot be the label on an interior edge of our conjugacy annulus; hence any face with an edge labelled by a must be a boundary face. As before, the simplification procedure of Lemma 2 permits us to restrict attention to sections of weight one in defining and checking our generalized degree assignment. In particular there can be at most $w_1 + w_2$ sections with a as a boundary edge. Hence, if we ignore these sections in the requirement that a face must have generalized degree at least four, the recursive bound obtained from Formula 2 will still be applicable.

We will call two crossings *paired* if they are joined by a single string generator. In our example the crossings $(2i-1, 2i)$, $i = 1, 6$, are paired. Two crossings are called *cross paired* if they are adjacent crossings on one generator and also lie on a pair of parallel generators of the two parallel copies of the knot of the torus used to obtain the projection. In our example the crossings $(i, i+6)$, $i = 1, 6$, are cross paired. Neither of the two definitions is to be applied to the special crossing.

We define the *collapse* of a double cable projection to be the identification of the two parallel copies of the knot of the torus used to define it and the consequent disappearance of the special crossing and the identification of paired and cross-paired crossings.

Let C be the dual boundary curve of a section of weight one. Then C is a simple closed curve in the plane of the projection π. The effect of the collapse of π on C is as follows. If any crossing of a set of four crossings linked by the pairing and cross pairing bears a relator circle then the image crossing bears a circle. The interior edge duals between circles not so linked in the preimage parallel the image of the generator on which they lay in the preimage. This is illustrated by Fig. 11.

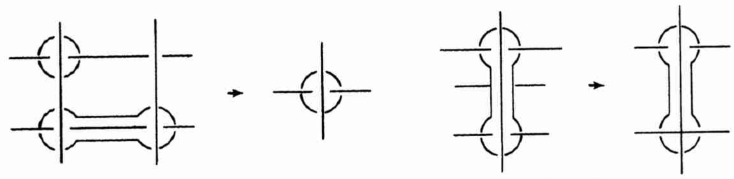


Fig. 11. The effect on boundary curves of a collapse

It is clear from examining the projection of a double cable knot that every vertex of degree less than four has degree two, and involves paired or cross-paired crossings. Fig. 12 gives some examples from our double trefoil.

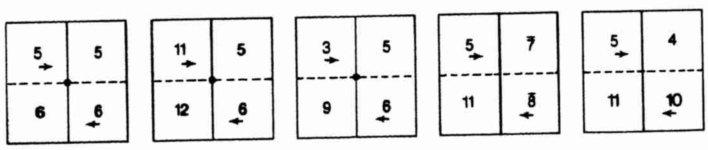


Fig. 12

The assignment we choose just assigns the contribution of one edge of each face to the vertex. This means that one of each pair of paired or cross-paired edges changes its contribution from the face to the vertex. We will use the definition of the collapse to show that this type of assignment is sound.

Call a set of four crossings closed under pairing and cross pairing a *complex*. If S is a section of weight one such that there is a complex H with no relator dual to a crossing of H corresponding to a region of S then the collapse of the dual boundary curve of S must have one crossing in its interior and the collapse of H in its exterior. Thus it must intersect the collapse of the projection in at least four points. But at least one preimage of each such intersection point is an edge dual which has not changed its contribution from the face to a vertex, since such a contribution occurs only if both members of a paired or cross-paired set of edges occurs in the boundary of the face.

Suppose, on the other hand, that at least one member of every complex is dual to a relator region of the section. From our previous remarks we know that relator R_1 which lies on generator a cannot supply a relator region to the section. Thus not every complex is represented by four relator regions in S. Now we redefine the effect of a collapse on a dual boundary curve so that a crossing of the collapse is interior to the image only if the entire complex is interior ro the preimage. In this way we show that the section has degree four under the generalized degree assignment if any entire complex is contained in the interior of its dual curve. If neither of the cases above holds, the number of intersections is large and a simple counting argument suffices.

One can easily see that the only possible tiles are the standard tiles, sections whose relator regions are dual to all of the crossings of a complex, and sections of perimeter eight whose relator regions are the duals of a union of complexes. The collapse argument shows that no other union of sections could be an internal face of generalized degree four. But in any of these faces of degree six or eight the requirement of generalized degree four for an interior tile of a tiled annulus forces vertices of degree two in such a way that one can easily see that all bands must have the same length. But this suffices to prove the theorem.

It appears that the approach of this paper is applicable to more general cable knots and to torus knots but the proofs seem more complicated. One would like to find a uniform manner to apply the argument without as great a reliance on special arguments but so far attempts at finding such a uniform method have proved unsuccessful.

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(Received April 4, 1974)