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Titel: A Note on the Family F (a, q) of Summability Methods.

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A Note on the Family $F(a, q)$ of Summability Methods

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The family $F(a, q)$ of summability methods was introduced by Meir [7]. The summability matrix (c_{pk}) belongs to $F(a, q)$ if it satisfies the following conditions: p is a discrete or continuous parameter; a is a positive constant; $q=q(p)$ is a positive increasing function which tends to infinity as $p\to\infty$; for every fixed δ: $\frac{1}{2}<\delta<\frac{2}{3}$,

$$\begin{aligned} c_{pk} &= (a/\pi q)^{\frac{1}{2}}\, e^{-a(k-q)^2/q}\left\{1+O\left(\frac{|k-q|+1}{q}\right)+O\left(\frac{|k-q|^3}{q^2}\right)\right\} \\ &= (a/\pi q)^{\frac{1}{2}}\, e^{-a(k-q)^2/q}\, C(k, q) \quad \text{(say)} \end{aligned} \tag{1}$$

as $p\to\infty$ uniformly in k for $|k-q|\leqq q^{\delta}$,

$$\sum_{|k-q|>q^{\delta}} k c_{pk} = O(e^{-q^{\eta}}) \quad (p\to\infty), \tag{2}$$

where η is some positive number independent of p, and

$$c_{pk}\geqq 0. \tag{3}$$

As observed by Meir [7], the $F(a, q)$ family contains the summability methods Borel, Euler, S_α, Taylor and Valiron. The Tauberian constants for this family of methods have been obtained by Meir [7] and Ikeno [4]; the Gibbs' phenomenon and Lebesgue constants have been studied by Ikeno [5, 6].

It has been shown by Hardy ([2], p. 213, Theorem 149) that if the first Cesaro mean of the series $\sum a_n$ is of the form $s+o(n^{-\frac{1}{2}})$, then the series is summable by Borel's method to the sum s. That a similar result holds for the Valiron method under a more general hypothesis was shown by Hyslop [3]. In this note, we generalize Hardy's result for the family $F(a, q)$ of summability methods.

The following lemma, which is needed in the sequel, is due to Meir [7].

Lemma. If the summability matrix (c_{pk}) belongs to $F(a, q)$, then

$$\sum_{k=0}^{\infty} c_{pk} = 1+O(q^{-\frac{1}{2}}) \quad (p\to\infty).$$

Theorem [1]. *If the first Cesaro mean of the series* $\sum a_n$ *is such that*

$$C_n^{(1)} = s + o(n^{-\frac{1}{2}})$$

then the series is summable $F(a, q)$ *to the sum* s.

Proof. For a positive integer m, we have

$$\sum_{n=0}^{m} (s_n - s)\, c_{pn} = \sum_{n=0}^{m-1} \left[s_n^{(1)} - \binom{n+1}{n} s \right] \Delta c_{pn} + \left\{ s_m^{(1)} - \binom{m+1}{1} s \right\} c_{pm},$$

where $\Delta c_{pn} = c_{pn} - c_{p,n+1}$. Since by Lemma, $c_{pm} \to 0$ as $p \to \infty$, it follows that

$$F(p) = \sum_{n=0}^{\infty} (s_n - s)\, c_{pn} = \sum_{n=0}^{\infty} \left[s_n^{(1)} - \binom{n+1}{n} s \right] \Delta c_{pn}. \tag{4}$$

By hypothesis of the theorem,

$$s_n^{(1)} - \binom{n+1}{n} s = o(n^{\frac{1}{2}}) \quad (n \to \infty). \tag{5}$$

From (4) and (5), we get

$$F(p) = o\{G(p)\} + o(1) \quad (n \to \infty),$$

where

$$G(p) = \sum_{n=0}^{\infty} n^{\frac{1}{2}} |\Delta c_{pn}|.$$

We now assert that $G(p)$ is bounded for all values of p. For

$$\begin{aligned} G(p) &= \Big(\sum_{|n-q| \leq q^{\delta}} + \sum_{|n-q| > q^{\delta}} \Big) n^{\frac{1}{2}} |\Delta c_{pn}| \\ &= \sum{}' + \sum{}'', \quad \text{say.} \end{aligned} \tag{6}$$

If $|f'(x)|$ has a supremum at $x = n + \theta$ in the range $n \leqq x \leqq n+1$, then we have by an application of the Mean Value Theorem

$$|\Delta f(n)| \leqq A\, |f'(n+\theta)|,$$

where A is a positive constant. Accordingly,

$$\begin{aligned} \sum{}' &\leqq (a/\pi q)^{\frac{1}{2}} \sum_{|n-q| \leq q^{\delta}} n^{\frac{1}{2}} e^{-a(n-q)^2/q} [C(n, q) - 1] \\ &\quad + A(a/\pi q)^{\frac{1}{2}} \sum_{|n-q| \leq q^{\delta}} n^{\frac{1}{2}} e^{-a(n+\theta-q)^2/q} (2a/q)\, |n + \theta - q|\, C(n, q) \\ &= \sum{}_1' + \sum{}_2', \end{aligned} \tag{7}$$

[1] A similar result is true for the family $F(c; \mu)$ of summability methods given by Anjaneyulu [1].

say. We have

$$\sum_1' = O\left\{q^{-\frac{1}{2}} \int_0^{q^\delta} q^{\frac{1}{2}} e^{-au^2/q}(u/q + u^3/q^2)\, du\right\} = O(1) \quad (q \to \infty) \tag{8}$$

and

$$\sum_2' = O\left\{q^{-1} \int_0^{q^\delta} e^{-au^2/q}\, u(1 + u/q + u^3/q^2)\, du\right\}.$$

Here, the first term is $O(1)$, while the remaining two terms are $o(1)$ as $q \to \infty$, with the result that

$$\sum_2' = O(1) \quad (q \to \infty). \tag{9}$$

Using (7), (8), and (9), we obtain

$$\sum' = O(1) \quad (q \to \infty). \tag{10}$$

Next, we proceed to consider $\sum''$. We have from (6),

$$\begin{aligned} \sum'' &= \sum_{|n-q| > q^\delta} n^{\frac{1}{2}} |c_{pn} - c_{p,n+1}| \\ &\leqq \sum_{|n-q| > q^\delta} n^{\frac{1}{2}} c_{pn} + \sum_{|n-q| > q^\delta} n^{\frac{1}{2}} c_{p,n+1} \\ &= \sum_1'' + \sum_2'', \end{aligned} \tag{11}$$

say. Notice that

$$\sum_1'' < \sum_{|n-q| > q^\delta} n c_{pn} = O(e^{-q^\eta}) = o(1) \quad (q \to \infty) \tag{12}$$

by hypothesis. Again,

$$\begin{aligned} \sum_2'' &= \sum_{n > q + q^\delta} n^{\frac{1}{2}} c_{p,n+1} + \sum_{n < q - q^\delta} n^{\frac{1}{2}} c_{p,n+1} \\ &\leqq \sum_{n > q + q^\delta} (n-1)^{\frac{1}{2}} c_{pn} + \sum_{n < q - q^\delta} (n-1)^{\frac{1}{2}} c_{pn} + (N-1)^{\frac{1}{2}} c_{pN} \\ &= \sum_3 + \sum_4 + \sum_5, \end{aligned} \tag{13}$$

say, where N is a positive integer such that $q - q^\delta \leqq N < q - q^\delta + 1$. Here,

$$\sum_3 < \sum_{n > q + q^\delta} n c_{pn} = o(1) \quad (q \to \infty), \tag{14}$$

$$\sum_4 < \sum_{n < q - q^\delta} n c_{pn} = o(1) \quad (q \to \infty), \tag{15}$$

$$\sum_5 = (N-1)^{\frac{1}{2}} (a/\pi q)^{\frac{1}{2}} e^{-a(N-q)^2/q}\, C(N, q) = o(1) \quad (q \to \infty). \tag{16}$$

Thus from (13), (14), (15) and (16), we obtain

$$\sum_2'' = o(1) \quad (q \to \infty) \tag{17}$$

and from (11), (12), and (17), we have

$$\sum'' = o(1) \quad (q \to \infty). \tag{18}$$

The theorem now follows from (6), (10), and (18).

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