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Titel: Transitive Collineation Groups on Affine Planes.

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Transitive Collineation Groups on Affine Planes

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1. Introduction

One of the first problems that arose in the theory of affine planes was the following question: If the finite affine plane π has a collineation group G transitive on the points or lines of π, is π a translation plane? Wagner [14] gave an affirmative answer when G is transitive on the affine lines. However, if G is transitive on the points the answer is, in general, negative. Ostrom [10] showed that the dual Lüneburg planes and the planes derived from them are counterexamples.

However, the problem has been solved in the affirmative if there is one or more additional assumptions. Ostrom and Wagner [12] give an affirmative answer if G is doubly transitive on the points. An affirmative answer holds if G is primitive on the points and π does not have order m^2 with $m \equiv 0 \pmod 4$ (Keiser [8]). Kallaher [4] and Liebler [9] prove π is a translation plane when G has rank 3 on the affine points. For this last result, it was shown in [4] that G not flag transitive implies G_ℓ, $\mathcal{O}$ an affine point, is transitive on the points of $\ell - \{\mathcal{O}\}$ for all lines ℓ through $\mathcal{O}$. Also, if $\bar{\pi}$ is a translation plane derived from a Desarguesian plane π by a series of homology type replacements then, for $\mathcal{O}$ an affine point of $\bar{\pi}$ and m a line of $\pi \cap \bar{\pi}$, $G_{\ell,m}$ is transitive on the points of $m - \{\ell\}$.

In view of the above, it is natural to ask the following question: If a collineation group G is transitive on the points of an affine plane π *and* for some line ℓ of π $G_{\ell,\ell}$ is transitive on the points of $\ell - \{\ell\}$, is π a translation plane? The transitive groups of the dual Lüneburg planes and those derived from them do not satisfy the hypothesis.

In Section 2, we formalize the above condition by introducing "block orbits" (see (2.1)). If an affine plane π of order n has a transitive collineation group G such that G_ℓ, $\mathcal{O}$ an affine point of π, possesses a block orbit Γ of degree $\gamma \geqq 2$ (see (2.1)) then π is a translation plane provided either n is even or n is a non-square. If G_ℓ has one block orbit of even degree or two block orbits of degree greater than or equal to 2, then π is a translation plane without any restriction on order. We feel that the results of this paper tend to indicate that the existence of block orbits for transitive collineation groups imply the plane is a translation plane.

In Section 3 we introduce two invariants of a transitive collineation group G on a finite affine plane. G has rank (r, s), and π is of rank (r, s) with respect to G, if G has rank r on the affine points of π and G has s point orbits on the line at infinity of π. Every transitive collineation group has rank (r, s) for some $r \geqq 2$ and some $s \geqq 1$. The main result of Section 3 is that an affine plane of order n and rank (r, s) with respect to a collineation group G is a translation plane if $s > \frac{r}{2}$ and n is either even or nonsquare. Also, without any restriction on n, if $s = r - 1$, π is a translation plane. Section 4 applies the results of Section 3 to planes where r is small – in particular, when $r = 4$.

Section 5 is concerned with transitive collineation groups G with a block orbit Γ for G_ℓ such that the permutation group induced on the set of lines of Γ is a regular group (see (5.1)). If the degree of the block orbit is at least two, the plane is a translation plane. Finally, in Section 6, we consider affine planes of odd order with a $\frac{3}{2}$-transitive collineation group G and show the existence of a block orbit for G_ℓ implies the plane is a translation plane.

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Notation. $p^r \| m$ will denote $p^r | m$ and $p^{r+1} \nmid m$. We will use the notation and nomenclature of the field of projective planes as found in [2] and [11]. Also we assume the reader is familiar with the theory of permutation groups as given in [13]. Finally, by *transitive collineation group* we will always mean a collineation group transitive on the points.

2. Block Orbits

(2.1) **Definition.** Let π be an affine plane, G a group of collineations of π and let Γ be a point orbit of G_P where P is a point of π. Γ is a *block orbit* of G_P if and only if $\Gamma \cup \{P\}$ is a set theoretic union of lines of π incident with P. The *degree* of the block orbit Γ shall be the number of lines involved in the union.

Remark. If Γ is a block orbit of G_ℓ and $\alpha \in G$ with $\ell\alpha = P$, then $\Gamma\alpha$ is a block orbit of G_P.

Notation. Let H be a group of collineations of an affine plane π and Δ be a point orbit of H_P, P a point of π. We shall denote by Δ_∞ the point orbit on ℓ_∞ of H_P corresponding to Δ. That is, let $\alpha \in H_P$ and Q a point of π. α induces the mapping $\tilde{\alpha}$ on ℓ_∞ where $PQ \cap \ell_\infty \xrightarrow{\tilde{\alpha}} P(Q\alpha) \cap \ell_\infty$. Note that it is possible that $\Delta_\infty = \Gamma_\infty$ and $\Delta \neq \Gamma$ where Δ, Γ are point orbits of H_P. Conversely, a point orbit of H_P on ℓ_∞ gives rise to a set of H_P-point orbits.

(2.2) **Lemma.** *Let G be a group of collineations of an affine plane π such that for an affine point $\mathcal{O}$ of π, $G_{\mathcal{O}}$ is transitive on the points not equal to $\mathcal{O}$ of a line ℓ incident with $\mathcal{O}$. If G possesses a non-identity translation τ with center $D=\ell\cap\ell_\infty$, then π is (D,ℓ_∞)-transitive.*

Proof. If $R\in\ell-\{\mathcal{O}\}$ there is a $\sigma\in G_{\mathcal{O}}$ such that $(\mathcal{O}\tau)\sigma=R$. Then $\mathcal{O}\sigma^{-1}\tau\sigma=\mathcal{O}\tau\sigma=R$. Clearly, $\sigma\in G_\ell$ so that σ fixes D. Thus, $\sigma^{-1}\tau\sigma$ is a translation with center D which maps $\mathcal{O}$ onto R. Since R is arbitrary, π is (D,ℓ_∞)-transitive.

(2.3) **Lemma.** *Let G be a collineation group of a finite affine plane π of odd order n. Assume that for an affine point $\mathcal{O}$ of π, $G_{\mathcal{O}}$ is transitive on the points not equal to $\mathcal{O}$ of a line ℓ incident with $\mathcal{O}$. Then G_ℓ contains an involutory homology fixing $\mathcal{O}$.*

Proof. $|G_{\mathcal{O},\ell}|=(n-1)\,|G_{\mathcal{O},Q}|$ where Q is a point of $\ell-\{\mathcal{O}\}$. $n-1$ is even by assumption. Let $2^a\|(n-1)$ and $2^b\|\,|G_{\mathcal{O},Q}|$. Let S be a Sylow 2-subgroup of $G_{\mathcal{O},\ell}$; then $|S|=2^{a+b}$. Let $\sigma\in Z(S)$, the center of S, with $|\sigma|=2$. If σ is a planar involution then σ fixes exactly $\sqrt{n}$ points of ℓ (since σ fixes ℓ) and $\mathcal{O}$ must be one of the fixed points. Clearly, S permutes the points fixed by σ and fixes $\mathcal{O}$. If $Q\neq\mathcal{O}$ is a point fixed by σ, then $\Lambda_Q=\{Q\tau\,|\,\tau\in S\}$ is a subset of the set of fixed points of σ. Hence

$$2^{a+b}=|S|=|\Lambda_Q|\,|S_Q| \quad\text{and}\quad |S_Q|\,|\,2^b$$

since S_Q is a subgroup of G_Q. Therefore $2^a\,|\,|\Lambda_Q|$. Now

$$\Big|\bigcup_{Q\in\ell-\{\mathcal{O}\}}\Lambda_Q\Big|=\sqrt{n}-1,$$

so $2^a\,|\,\sqrt{n}-1$ and hence $2^{a+1}\,|\,(\sqrt{n}-1)(\sqrt{n}+1)=n-1$. But $2^a\|n-1$, so we have a contradiction. Therefore σ is a central collineation and, since n is odd, σ is a homology.

(2.4) **Lemma.** *Let G be a collineation group on a finite affine plane π of even order n. If G_ℓ is transitive on the points of a line ℓ of π then G_ℓ contains an involutory elation.*

Proof. Clearly, $|G_\ell|=n\,|G_Q|$ where Q is a point of ℓ. Let $2^a\|n$ and $2^b\|\,|G_Q|$. Let S be a Sylow 2-subgroup of G_Q and $\sigma\in Z(S)$ such that $|\sigma|=2$. Assume σ is a planar involution. Thus σ fixes exactly $\sqrt{n}$ points of ℓ. If Q is a fixed point of σ on ℓ, let $\Lambda_Q=\{Q\alpha\,|\,\alpha\in S\}$. As in the proof of (2.3), Λ_Q consists of points fixed by σ and $2^{a+b}=|\Lambda_Q|\,|S_Q|$ so that $2^a\,|\,|\Lambda_Q|$. Thus $2^a\,\big|\,\big|\bigcup_{Q\in\ell}\Lambda_Q\big|=\sqrt{n}$ so that $2^{2a}|n$, which contradicts $2^a\|n$. Therefore, σ is an involutory elation since n is even.

If G is a transitive group on an affine plane π the point orbits of G on ℓ_∞ and those of $G_{\mathcal{O}}$ may not be identical. However, if $G_{\mathcal{O}}$ admits two

block orbits Γ, Δ then (2.5) shows that Γ_∞ and Δ_α cannot belong to the same point orbit of G on ℓ_∞.

(2.5) **Lemma.** *Let G be a transitive collineation group of an affine plane π and $\mathcal{O}$ an affine point of π. If there is a point orbit of G on ℓ_∞ which is the union of at least two point orbits of $G_\mathcal{O}$ then these point orbits of $G_\mathcal{O}$ on ℓ_α can give rise to at most one block orbit of $G_\mathcal{O}$.*

Proof. Let Δ and Γ be block orbits of $G_\mathcal{O}$ such that $\Delta_\infty \cup \Gamma_\infty$ is contained in a point orbit of G. Let $\alpha \in G$ and $P \in \Delta_\infty$, $Q \in \Gamma_\infty$ be such that $P\alpha = Q$. There is an element $\sigma \in G_Q$ such that $((\mathcal{O}\alpha)Q)\sigma = \mathcal{O}Q$ (note that G is point transitive so that G_Q is transitive on the lines of π incident with Q; see [12]). $\mathcal{O}P$ is a line of the block orbit Δ so $(\mathcal{O}\alpha)Q$ is a line of a block orbit of $G_{\mathcal{O}\alpha}$ and $((\mathcal{O}\alpha)\sigma)Q = \mathcal{O}Q$ is a line of a block orbit of $G_{\mathcal{O}\alpha\sigma}$. But $\mathcal{O}Q$ is also a line of the block orbit Γ of $G_\mathcal{O}$.

If $\mathcal{O}\alpha\sigma = \mathcal{O}$ then $\alpha\sigma \in G_\mathcal{O}$ and $P\alpha\sigma = Q$, contrary to our assumptions. Assume $\mathcal{O}\alpha\sigma \neq \mathcal{O}$. $G_{\mathcal{O},Q}$ is transitive on the points of $\mathcal{O}Q - \{\mathcal{O}\}$ since $\mathcal{O}Q$ is a line of Γ. Hence $\mathcal{O}Q$ is a line of a block orbit of G_R for every point R of $\mathcal{O}Q$. Choose $R \neq \mathcal{O}\alpha\sigma$, $\mathcal{O}$; there is an element $\kappa \in G_{R,Q}$ such that $\mathcal{O}\alpha\sigma\kappa = \mathcal{O}$. Then $\alpha\sigma\kappa \in G_\mathcal{O}$ and $P\alpha\sigma\kappa = Q$, which is again contrary to our assumptions. This proves (2.5).

The question of when the point orbit Γ_∞ corresponding to the block orbit Γ of $G_\mathcal{O}$ is an orbit of G is directly related to asking whether G_m, m a line of a block orbit of $G_\mathcal{O}$, is doubly-transitive on the points of m. For we have:

(2.6) **Lemma.** *Let G be a transitive collineation group of an affine plane π and let Γ be a block orbit of $G_\mathcal{O}$. Γ_∞ is an orbit of G if and only if G_m, m a line of Γ, is doubly-transitive on the points of m.*

Proof. Let $P \in m - \{\mathcal{O}\}$. There exists $\alpha \in G$ such that $P\alpha = \mathcal{O}$. So $\mathcal{O} \in m\alpha$. If Γ_∞ is an orbit of G there is an element $\beta \in G_\mathcal{O}$ such that $m\alpha\beta = m$, so $\alpha\beta \in G_m$. Since $P\alpha\beta = \mathcal{O}$ and $G_{\mathcal{O},m}$ is transitive on $m - \{\mathcal{O}\}$, we have that G_m is doubly transitive on the points of m.

Conversely, let Δ_∞ be the point orbit of G on ℓ_∞ containing Γ_∞. Assume there exists $\alpha \in G$ such that $\Gamma_\infty \alpha \not\subseteq \Gamma_\infty$, and let $P \in \Gamma_\infty$ with $P\alpha \in \Delta_\infty - \Gamma_\infty$. If m is a line of Γ incident with $\mathcal{O}$, there is an element $\beta \in G_{P\alpha}$ such that $\alpha\beta$ contains $\mathcal{O}$. If $\mathcal{O}\alpha\beta = \mathcal{O}$ then $\alpha\beta \in G_\mathcal{O}$; but $P\alpha\beta = P\alpha \in \Delta_\infty - \Gamma_\infty$. Hence $\mathcal{O}\alpha\beta \neq \mathcal{O}$. G_m doubly transitive on the points of m implies $G_{m\alpha\beta}$ is doubly transitive on the points of $m\alpha\beta$. Hence there is an element $\gamma \in G_{m\alpha\beta}$ such that $\mathcal{O}\alpha\beta\gamma = \mathcal{O}$. Thus $\alpha\beta\gamma \in G_\mathcal{O}$ and $P\alpha\beta\gamma \notin \Gamma_\infty$, which is a contradiction.

(2.7) **Theorem.** *Let G be a transitive collineation group of a finite affine plane π of order n and let Γ be a block orbit of $G_\mathcal{O}$, $\mathcal{O}$ an affine point. For a line m of Γ, G_m is doubly transitive (and hence Γ_∞ is an orbit of G) if one of the following conditions hold:*

(i) *The degree of* Γ *is greater than or equal to* 2.

(ii) *The number of orbits of* G *on* ℓ_∞ *is greater than* 2.

(iii) G_ℓ *has at least three block orbits.*

Proof. Suppose G_m is not transitive on the points of m. Then $G_m \subseteq G_\ell$. Since G_P, $P = \ell_\infty \cap m$, is transitive on the affine lines of π incident with P, we have that $[G: G_m] = n\,|\Delta_\alpha|$ where Δ_α is the orbit of G containing P.

Since $n^2 = [G: G_\ell] \leqq [G: G_m] = n\,|\Delta_\alpha|$ we have that $n \leqq |\Delta_\alpha| \leqq n+1$. If $|\Delta_\alpha| = n+1$ then G is flag-transitive (Wagner [14]) and G_m is transitive on the points of m—a contradiction. Hence $|\Delta_\alpha| = n$, $|G_\ell| = |G_m|$, and thus $G_\ell = G_m$ for every line m of the block orbit Γ. Therefore G_ℓ fixes all points of Γ_α and this implies $|\Gamma_\alpha| = 1$. Thus if (i) holds, we have a contradiction. Since $|\Delta_\alpha| = n$, there is a point P on ℓ_α such that $G = G_P$. This implies G has exactly two orbits on ℓ_α; therefore (ii) does not hold. If there are at least three block orbits we have a contradiction by (2.5), so (iii) cannot hold.

(2.8) **Definition.** Let G be a point transitive collineation group of an affine plane π and let Γ be a block orbit of G_ℓ, ℓ an affine point. Γ is a *strong block orbit* if Γ_α is an orbit of G on ℓ_α.

(2.9) *Remarks.* It seems probable that every block orbit Γ of G_ℓ is strong. In fact, the situation becomes extremely pathological by assuming the existence of a block orbit which is not strong. That is, as seen in (2.7), in this case there can be at most two block orbits. If two block orbits exist, one of the block orbits is strong and both have degree one. In any case G has two orbits on ℓ_α, one of length one and one of length n, the order of π.

If $\{P\}$ is the orbit of length 1 and n is odd then it can be shown that for every affine line ℓ incident with P there is a unique point Q of $\ell_\alpha - \{P\}$ such that G contains a (Q, ℓ)-involutory homology and $G_\ell = G_Q$. Also, G is doubly transitive on the affine lines incident with P and doubly transitive on the points of $\ell_\alpha - \{P\}$. Furthermore, G can contain no central collineation with center P. If n is even then it can be shown that π is (P, ℓ_∞)-transitive but G contains no other translations and no other elations with center P.

(2.10) **Lemma.** *Let* π *be an affine plane with a transitive collineation group* G. *Assume* G_ℓ, ℓ *an affine point, has a strong block orbit* Γ. *Then for every line* m *of* Γ *there is a non-planar involution in* G_m.

Proof. Follows from (2.3), (2.4), and (2.6).

(2.11) **Theorem.** *Let* π *be a finite affine plane of order* n *with a transitive collineation group* G. *Assume that* G_ℓ, ℓ *an affine point of* π, *has a block orbit* Γ *of degree* $\gamma \geqq 2$. *Then* π *is a translation plane and* G *contains the translation group of* π *if one of the following conditions hold:*

(i) *n is even;*

(ii) *n is a nonsquare;*

(iii) *γ is even.*

Proof. Assume (i) holds. Let m be a line of Γ and let $P=m\cap\ell_\infty$. By (2.10) and (2.6) G_m contains an involutory elation σ. σ has center P and its axis is either ℓ_∞ or m. If the axis is m then G_m has a nontrivial translation with center P. For G_P is transitive on the (affine) lines of π incident with P. Thus there are nontrivial (P,ℓ)-elations for all affine lines ℓ incident with P and by a result of Hughes (see [2], p. 188) P is the center of a nontrivial translation. By (2.2), π is (P,ℓ_∞)-transitive. Since $\gamma\geqq 2$, π is (P,ℓ_∞)- and (Q,ℓ_∞)-transitive for distinct points P and Q of ℓ_∞ and π is a translation plane. Clearly, G contains the full translation group.

For conditions (ii) and (iii) we may assume that n is odd. Assume (ii) holds. Let m be a line of Γ and let $P=m\cap\ell_\infty$. By (2.4) and (2.10), G_m contains an involutory homology σ. If σ has axis ℓ_∞ then by André [1] π is a translation plane and G contains the translation group. Assume that σ has affine axis so that either m is the axis of σ or P is the center of σ. By (2.10) and (2.16), there is an element $\alpha\in G_m$ such that $\mathcal{O}\alpha=Q$, $Q\in m$, $Q\neq\mathcal{O}$.

Assume P is the center of σ. If h is the axis of σ then $h\alpha\neq h$. $\alpha^{-1}\sigma\alpha$ is an involutory homology in $G_{m\alpha}$ with axis $h\alpha$ and center P. By André [1] G_m contains an elation with center P. We can use the argument above to show that π is (P,ℓ_∞)-transitive. Since $\gamma\geqq 2$, π is a translation plane and G contains the translation group.

Now assume that m is the axis of σ and let $Q\in\ell_\infty$ be the center of σ. If $\mathcal{O}Q$ is a line of Γ then we can apply the previous argument. Therefore, we can assume that $Q\notin\Gamma_\infty$. Since σ fixes P and no other point of Γ_∞ we have $|\Gamma_\infty|=\gamma$ is odd. If $G_{\mathcal{O},P}$ contains a collineation α moving Q then G_m contains a homology $\alpha^{-1}\sigma\alpha$ with axis m and center $Q\alpha\neq Q$. We have by André [1] that G_m contains an elation with center P. Arguing as above, π is a translation plane and G contains the translation group.

Thus, we may assume that $G_{\mathcal{O},P}$ fixes Q. $|G_{\mathcal{O},P}|=(n-1)|G_{\mathcal{O},R}|$, R a point of $m-\{\mathcal{O}\}$. Let $\Lambda_1,\ldots,\Lambda_t$ be the point orbits on $\mathcal{O}Q$ of $G_{\mathcal{O},Q}$. Then for each i, $|G_{\mathcal{O},Q}|=|\Lambda_i|\,|G_{\mathcal{O},S}|$, S a point of Λ_i. Since the axis of σ is m, we have $2\big||G_{\mathcal{O},R}|$. Let $2^a\|(n-1)$ so that $2^{a+1}\big||G_{\mathcal{O},P}|$ and since $G_{\mathcal{O},P}=G_{\mathcal{O},Q}$, $2^{a+1}\big||G_{\mathcal{O},Q}|$. If $2^{a+1}\big||\Lambda_i|$ for all $i=1,\ldots,t$ then $2^{a+1}\Big|\sum_{i=1}^{t}|\Lambda_i|=n-1$, which is a contradiction. Hence, there is a Λ_i such that $2^{a+1}\nmid|\Lambda_i|$ so that $2\big||G_{\mathcal{O},S}|$, S a point of Λ_i. Since n is an odd nonsquare there is an involutory homology τ in $G_{\mathcal{O},S}$ so that the axis of τ is $\mathcal{O}S=\mathcal{O}Q$. Since $|\Gamma_\infty|$ is odd, the center of τ must be in Γ_∞ and we can apply the previous argument.

Assume (iii) holds. In the proof of (i) and (ii), we showed that either π is a translation plane or for every line m of Γ, m is the axis of an homology σ of order 2. And, we have seen in this situation that the dimension of Γ must be odd. Thus (2.11) is proved.

Note that the hypothesis "n is a nonsquare" or "the degree of Γ is even" was required only in the situation where for each line m of Γ, m is the axis of an involutory homology. We can get around this difficulty if two block orbits each of degree at least 2 exist.

(2.12) **Theorem.** *Let π be a finite affine plane with a transitive collineation group G. If G_ℓ, $\mathcal{O}$ an affine point, admits two block orbits Γ, Δ such that the degree of each is at least 2, then π is a translation plane and G contains all of the translations.*

Proof. By (2.11) we may assume that $|\Gamma_\infty|$ and $|\Delta_\infty|$ are odd, and the order of π is odd. Let m be a line of Γ. By (2.3) G_m contains an involutory homology σ fixing $\mathcal{O}$. By an argument used in the previous proof, if σ has center $m \cap \ell_\infty$ the theorem holds. Otherwise σ has axis m and its center is in Δ_∞ since $|\Delta_\infty|$ is odd. But then the same argument shows that the theorem holds.

(2.13) **Theorem.** *Let π be a finite affine plane with a transitive collineation group G. If G_ℓ, $\mathcal{O}$ an affine point, admits three distinct block orbits or two strong block orbits, then π is a translation plane and G contains the translation group if one of the following conditions hold:*

(i) *The order n of π is even;*

(ii) *n is nonsquare.*

Proof. By (2.7) and (2.12) we can assume that we have at least two strong block orbits Γ and Δ both of degree 1, say $\Gamma_\infty = \{P\}$ and $\Delta_\infty = \{Q\}$. First assume that n is even. By (2.10) there is an involutory elation in G_m and in G_ℓ, where $m = \mathcal{O}P$ and $\ell = \mathcal{O}Q$, and by Ostrom-Wagner [12] and Hughes [2; p. 188] there is a translation in G_m with center $m \cap \ell_\infty = P$. By (2.2), π is (P, ℓ_∞)-transitive. Similarly, π is (Q, ℓ_∞)-transitive and hence π is a translation plane.

Now assume n is an odd nonsquare. By (2.3), there is an involutory homology σ in G_m fixing $\mathcal{O}$. Thus σ has axis $\mathcal{O}P$ and center Q or it has axis $\mathcal{O}Q$ and center P. Without loss of generality we may assume that we have an involutory homology σ with center P and axis $\mathcal{O}Q$. $|G_\ell| = (n-1)\,|G_{\ell,R}|$, R a point of Γ and $|G_\ell| = (n-1)\,|G_{\ell,S}|$, S a point of Δ. By assumption, $2 \mid |G_{\ell,S}|$ since σ has axis $\mathcal{O}Q$ and thus $2 \mid |G_{\ell,R}|$. But, n is nonsquare so there is an involutory homology τ in $G_{\ell,R}$ with axis $\mathcal{O}R = \mathcal{O}P$. The center of τ is clearly Q. Therefore (see [2], p. 120), $\sigma\tau$ is an involutory homology with center $\mathcal{O}P \cap \mathcal{O}Q = \mathcal{O}$ and axis $PQ = \ell_\infty$. Hence π is a translation plane.

3. Rank (r, s) Planes

(3.1) **Definition.** Let π be a finite affine plane. A transitive collineation group G of π has *rank* (r, s) if G has s orbits on ℓ_∞ and has rank r on the affine points of π (i.e., $G_\mathscr{O}$ has r orbits in π including $\{\mathscr{O}\}$). We shall also say in this case that π is a rank (r, s) plane with respect to the group G.

If we wish to distinguish between the orbits of G on ℓ_∞ and those of $G_\mathscr{O}$ on ℓ_∞ we will use the following convention. G has *rank* (r, s, t) on π if and only if G has rank (r, s) on π and $G_\mathscr{O}$ has t orbits on ℓ_∞. We define *rank* (r, s, t)-planes similarly as above.

(3.2) *Remarks.* (i) To say that a transitive group G has rank (r, s) or (r, s, t) is no restriction. That is, every transitive collineation group on a finite affine plane has rank (r, s, t) where $r \geqq 2$, $s \geqq 1$, $t \geqq 1$. Also $1 \leqq s \leqq t \leqq r-1$.

(ii) Note that Wagner's theorem [14] on line transitive affine planes can be rephrased as follows: "If a finite affine plane π is of rank $(r, 1)$, $r \geqq 2$, with respect to a group G then π is a translation plane and G contains the translations." A similar rephrasing of the Ostrom-Wagner theorem on affine planes with doubly-transitive groups may also be given.

(3.3) **Lemma.** *Let π be a rank (r, s, t) affine plane with respect to the collineation group G, and let $\mathscr{O}$ be an affine point of π.*

(i) *If $\dfrac{r-1}{2} < t$ then there is at least one block orbit of $G_\mathscr{O}$*

(ii) *If $\dfrac{r}{2} < t$ then either there are two block orbits of $G_\mathscr{O}$ or there is a block orbit of $G_\mathscr{O}$ of degree at least 2.*

(iii) *If $\dfrac{r+1}{2} < t$ then either there are three block orbits of $G_\mathscr{O}$ or there is a block orbit of $G_\mathscr{O}$ of degree at least 2.*

Proof. Assume $G_\mathscr{O}$ has no block orbits. Then every line ℓ incident with $\mathscr{O}$ intersects at least two nontrivial point orbits of $G_\mathscr{O}$ and thus $r \geqq 2t+1$. Thus (i) holds.

Assume $\dfrac{r}{2} < t$. By (i), $G_\mathscr{O}$ admits at least one block orbit Γ. Assume the degree of Γ is 1 and $G_\mathscr{O}$ admits exactly one block orbit. Thus $\Gamma = \ell - \{\mathscr{O}\}$ where ℓ is some line of π incident with $\mathscr{O}$. $G_\mathscr{O}$ has $t-1$ orbits besides Γ_∞ on ℓ_∞. Each line $m \neq \ell$ incident with $\mathscr{O}$ must intersect at least two nontrivial (i.e., $\neq \{\mathscr{O}\}$) point orbits of $G_\mathscr{O}$. Thus $2(t-1)+1 \leqq r-1$ so $2t \leqq r$. This proves (ii). The proof of (iii) is virtually identical to the proof of (ii) and is omitted.

(3.4) **Theorem.** *Let π be a finite affine plane of order n and of rank (r, s) with respect to the collineation group G. If n is even or nonsquare and $\frac{r}{2} < s$ then π is a translation plane and G contains all the translations.*

Proof. If $r=2$ then a rank $(2, s)$ group G is doubly transitive on the affine points. The result then follows from Ostrom-Wagner [12]. If $r=3$ the result follows from Kallaher [4] and Liebler [9]. Thus we may assume that $r \geqq 4$ and $2 < s$. Since π is also of rank (r, s, t) with $t \geqq s$ we may apply (3.3). G_ℓ has either two block orbits of degree one or a block orbit of degree at least 2. By (2.7) these block orbits are strong (see (2.8) and (2.6)). (3.4) then follows from (2.11) and (2.13).

(3.5) **Theorem.** *Let π be a finite affine plane of order n and of rank (r, s, t) with respect to the collineation group G. If n is even or nonsquare and $\frac{r+1}{2} < t$ then π is a translation plane and G contains all the translations.*

Proof. Follow from (3.3) (iii), (2.1) (i), (ii), and (2.13).

As maintained at the beginning of this section, Wagner's result on line-transitive affine planes can be stated in terms of affine planes of rank $(r, 1)$. Since $1 \leqq s \leqq r-1$, on the opposite end of the spectrum we have affine planes of rank $(r, r-1)$. Note that in an affine plane of rank $(r, r-1)$ with respect to a collineation group G, every nontrivial point orbit of G_ℓ is a block orbit. Clearly, a rank $(r, r-1)$ affine plane is a rank $(r, r-1, r-1)$ affine plane and by (2.6), (2.7) the orbits of G and G_ℓ on ℓ_∞ are identical. The following result is, in some sense, a companion to Wagner's theorem on line-transitive planes.

(3.6) **Theorem.** *If π is a finite affine plane of rank $(r, r-1)$ with respect to a collineation group G, then π is a translation plane.*

Proof. Let $\mathcal{O}$ be an affine point of π. By the above remark, every point orbit of G_ℓ is a block orbit. As in the proof of (3.4) we may assume $r \geqq 4$ and by (2.13) we may assume the order of π is an odd square. By (2.3) for a line m of π incident with $\mathcal{O}$, G_m contains an involutory homology σ fixing $\mathcal{O}$. Hence the axis h of σ is incident with $\mathcal{O}$. Let Q be the center of σ. By (2.6), $G_{\ell Q}$ is transitive on the points of $\mathcal{O}Q$ so that if $R \in \sigma Q$ there is an element $\alpha \in G_{\ell Q}$ such that $\mathcal{O}\alpha = R$. Then $h\alpha \neq h$ is the axis and Q the center of $\alpha^{-1}\sigma\alpha$. By André [1] there is an elation with center Q and it follows as in the previous arguments (see (2.3) and (2.4)) that π is (Q, ℓ_∞)-transitive.

Therefore, either π is a translation plane or G fixes Q. In the latter case, by the previous paragraph every line m not going through Q is the axis of an involutory homology with center Q. We can now directly use Liebler's argument [9; p. 92–93] to obtain a contradiction.

We close this section with a result about affine planes of rank $(r, s, 2)$.

(3.7) **Theorem.** *Let π be a finite affine plane of order n and of rank $(r, s, 2)$ with respect to the collineation group G. If $G_\ell, \mathcal{O}$ an affine point, admits a block orbit Γ of degree one and n is odd then π is either a translation plane or a dual translation plane.*

Proof. Since $1 \leqq s \leqq 2$ we can assume that $s=2$ by Wagner [14]. Thus the orbits of G and $G_\mathcal{O}$ on ℓ_∞ are identical, so that we can assume Γ is strong. By (2.3), there is an involutory homology σ in G_m where $\Gamma = m - \{\mathcal{O}\}$ and σ fixes $\mathcal{O}$. Either the center of σ is P, where $\Gamma_\infty = \{P\}$, or the axis is m. In both cases, $G_\ell = G_{\ell,P}$ and $G_\mathcal{O}$ is transitive on the points of $\ell_\infty - \{P\}$. Let Q and $\mathcal{O}R$, where $R \in \ell_\infty$, be the center and axis of σ. We have either $(Q, \mathcal{O}R)$-involutory homologies for all $\bar{Q} \in \ell_\infty - \{P\}$ or $(Q, \mathcal{O}\bar{R})$-involutory homologies for all $\bar{R} \in \ell_\infty - \{P\}$. By André [1] in both of these cases the group of elations with axis m and center P is transitive on $\ell_\infty - \{P\}$. Since G_P is transitive on the affine lines of π which are incident with P, π is a dual translation plane.

4. Affine Planes of Small Rank

If π is an affine plane of rank 2 or 3 then π is a translation plane. We consider first the rank 4 situation.

(4.1) **Lemma.** *Every rank 4 affine plane π with respect to a group G is of rank $(4, s, s)$ for $s=1, 2$ or 3 and hence G and $G_\ell, \mathcal{O}$ an affine point, have the same orbits on ℓ_∞.*

Proof. If π is of rank (4, 1) or (4, 3), the lemma follows from previous discussion. Assume π is of rank (4, 2). Since π is of rank $(4, 2, t)$ for $t \geqq 2$, we can assume that $t=2$ or we are in one of the cases considered above. Clearly, in this case G and $G_\mathcal{O}$ have the same orbits on ℓ_∞.

(4.2) **Theorem.** *Let π be a finite affine plane of rank 4 with respect to the collineation group G. If the order n of π is nonsquare then π is a translation plane or its dual. In particular:*

(i) *Without restriction on the order of π, if π is of rank (4, 1) or (4, 3) with respect to the collineation group G then π is a translation plane and G contains the translations.*

(ii) *If π is of rank (4, 2) with respect to G and of even nonsquare order, then π is a translation plane and G contains the translations.*

(iii) *If π is of rank (4, 2) with respect to G and of odd nonsquare order, then π is a translation plane or a dual translation plane and G contains the translations or dual translations, respectively.*

Proof. (i) follows from Wagner [14] and (3.6); (iii) follows from (2.11) and (3.6). We prove (ii). By (4.1), (2.10) and (3.3) $G_\ell, \mathcal{O}$ an affine point admits a block orbit Γ. By (2.11) and (2.13) (note any block orbit Γ is

strong by (4.1)), we may assume that the degree of Γ is one and that Γ is the only block orbit of $G_{\mathscr{O}}$. $G_{\mathscr{O}}$ has 2 point orbits besides Γ and $\{\mathscr{O}\}$, and $G_{\mathscr{O}}$ has one orbit on ℓ_∞ besides Γ_∞.

Thus, every line ℓ incident with $\mathscr{O}$ and not in Γ intersects exactly two nontrivial (point) orbits of $G_{\mathscr{O}}$. Let the order of π be n. If ℓ intersects each nontrivial orbit of $G_{\mathscr{O}}$ in an odd number of points, then $n-1$ is even which contradicts our assumption. Therefore, let Λ be a point orbit of $G_{\mathscr{O}}$ which intersects ℓ in an even number of points.

Since $|\Lambda \cap \ell| \big| |G_{\mathscr{O},\ell}|, 2 \big| |G_{\mathscr{O},\ell}|$. n a non-square implies $G_{\mathscr{O},\ell}$ has an involutory homology σ with axis ℓ. Since $|\Gamma_x| = 1$, G fixes a point P on ℓ_x (see (4.1)). But $\ell \cap \ell_\infty \neq P$ by assumption, so σ must move P. Thus we have a contradiction and (iii) is proved.

(4.3) *Remarks.* Every nearfield plane has a rank (4, 3) collineation group in which two of the three ℓ_x-point orbits are single points. There are also generalized André planes having (4, 3)-collineation groups with this property (see [7]). It seems likely that almost all such planes are generalized André. In fact, the following is an easy consequence of the arguments given in [5] and [7].

(4.4) **Theorem.** *Let π be a rank* (4, 3) *affine plane in which two of the three orbits on ℓ_∞ are single points. If π has order n and $n \neq 5^2, 11^2, 19^2, 23^2, 29^2, 59^2$ or a Mersenne prime, then π is a generalized André plane.*

We remark that the restrictions on n are necessary since the irregular nearfield planes are rank (4, 3) but not generalized André planes.

(4.5) **Lemma.** *Any rank 5 affine plane π with respect to a group G of even nonsquare order is a rank* (5, s, s) *plane and so G and $G_{\mathscr{O}}$, $\mathscr{O}$ an affine point of π, have the same orbits on ℓ_∞.*

Proof. Let π be of rank (5, s, t). If $t = 1$ or 4, (4.5) clearly follows. Let $t = 2$. Clearly, we must have $s = 2$ and (4.5) is obvious. Finally, let $t = 3$. By (3.1) $G_{\mathscr{O}}$ has either two block orbits or a block orbit of degree $\gamma \geqq 2$. In the latter case the statement follows from (2.11). Thus, we may assume that $s = 2$ and that we have two block orbits of degree 1. Since the order of π is even, every line ℓ incident with $\mathscr{O}$ and not belonging to either of the block orbits must intersect one of the remaining two nontrivial point orbits of $G_{\mathscr{O}}$ in an even number of points. π having non-square order implies $G_{\mathscr{O},\ell}$ has an involutory elation τ. But then neither of the two block orbits can have degree 1. This proves (4.5).

The proof of the following corollary is straightforward and will be left to the reader.

(4.6) **Corollary.** *Let π be a finite affine plane of even nonsquare order and of rank 5 with respect to a collineation group G. If G does not have exactly two point orbits on ℓ_∞, then π is a translation plane and G contains the translations.*

5. Regular Block Orbits

If G is a transitive collineation group on an affine plane and Γ is a block orbit of $G_\mathcal{O}$, the representation of G on the lines of Γ is of special interest. The Desarguesian planes, Hall planes, nearfield planes, generalized André planes, as well as a class of non-generalized André planes constructed by one of the authors (see [3] and (5.3)) admit groups G having block orbits Γ such that the representation on Γ_∞ is regular.

(5.1) **Definition.** Let π be a rank (r, s) affine plane with associated group G. Let Γ be a block orbit of $G_\mathcal{O}$, $\mathcal{O}$ an affine point. Γ is a *regular block orbit* if and only if $G_\mathcal{O}$ induces a regular group on the lines of the block orbit; i.e., $G_\mathcal{O}$ induces a regular group on its orbit Γ_∞.

(5.2) **Theorem.** *Let π be a finite affine plane of rank (r, s) with respect to the collineation group G. If $G_\mathcal{O}$, $\mathcal{O}$ an affine point, admits a regular block orbit Γ of degree $\gamma \geqq 2$ then π is a translation plane.*

Proof. If the order n of π is even or if $\gamma = 2$ the theorem follows from (2.11). Thus we may assume n is odd and $\gamma \geqq 3$. Let m be a line through $\mathcal{O}$ in Γ. By (2.3) G_m contains an involutory homology σ fixing $\mathcal{O}$. σ also fixes $m \cap \ell_\infty \in \Gamma_\infty$. $G_\mathcal{O}$ regular on Γ_∞ implies σ fixes Γ_∞ pointwise. Since $\gamma \geqq 3$, this implies the axis of σ is ℓ_∞. By André [1], π is a translation plane.

(5.3) *Remarks.* (i) Let π be a finite affine plane of rank $(r, 2)$ with respect to the collineation group G such that $G_\mathcal{O}$ admits a regulr block orbit of degree $\gamma \geqq 3$. If the order of π is nonsquare then π is Desarguesian. For Kallaher's proof [6; p. 9–10] can be directly applied in this situation.

(ii) We cannot conclude that rank (r, s) planes with a regular block orbit of degree $\gamma \geqq 3$ are Desarguesian. For the planes π of [3] of order p^2 are affine planes of rank $\left(\frac{p^2-1}{3}+3, \frac{p^2-1}{3}+2-p\right)$. The collineation group is regular on all but one of its orbits on ℓ_∞. There are

$$\frac{p^2-1}{3}-p+1 \quad .$$

regular block orbits two of which have degree 1 and the others have degree 3. The remaining orbit on ℓ_∞ does not correspond to a block orbit. (See [3] for further details.)

6. $\frac{3}{2}$-Transitive Groups on Affine Planes

(6.1) **Lemma.** *Let π be an affine plane of order n, n odd, with collineation group G acting as a Frobenius group on π. If for a point $\mathcal{O} \in \pi$ $|G_\mathcal{O}|$ is even, then π is a translation plane and the Frobenius kernel of G is the group of translations of π.*

Proof. $|G_{\mathcal{O}}|$ even implies $G_{\mathcal{O}}$ has an element σ of order 2. Since $|G_{\mathcal{O},P}|=1$ for every point $P\in\pi$, $P\neq\mathcal{O}$, σ is a homology with axis ℓ_∞ and center $\mathcal{O}$. But then G transitive on π implies for every affine point P we have a nontrivial homology with center P and axis ℓ_∞. By André [1], π is a translation plane and the Frobenius kernel of G is the group of translations.

(6.2) **Theorem.** *Let π be an affine plane of odd order n with a collineation group G acting as a Frobenius group. π is a translation plane and the group of translations of π is the Frobenius kernel of G if one of the following holds:*

(i) *G has rank r, r even;*

(ii) *For a point $\mathcal{O}$ of π, $G_{\mathcal{O}}$ has a block orbit Γ.*

Proof. Assume (i) holds. Then for a point $\mathcal{O}\in\pi$, $G_{\mathcal{O}}$ has r orbits (including $\{\mathcal{O}\}$). Let $\Delta_1,\Delta_2,\ldots,\Delta_r$ be the orbits with $\Delta_1=\{\mathcal{O}\}$. G Frobenius implies $|\Delta_i|=|G_{\mathcal{O}}|$ for all $i=2,\ldots,r$. Therefore,

$$n^2=1+\sum_{i=2}^{r}|\Delta_i|=1+(r-1)\,|G_{\mathcal{O}}|.$$

Hence $n^2-1=(r-1)\,|G_{\mathcal{O}}|$. Since r is even we must have $2\,|\,|G_{\mathcal{O}}|$. Thus the Lemma applies.

Assume (ii) holds. Then $|\Gamma|=\gamma(n-1)$ where γ is the degree of Γ. Thus $\gamma(n-1)\,|\,|G_{\mathcal{O}}|$, and since $n-1$ is even, $2\,|\,|G_{\mathcal{O}}|$.

(6.3) **Theorem.** *Let π be a finite affine plane with a (point) primitive collineation group G. If for a point $\mathcal{O}$ of π $G_{\mathcal{O}}$ has a block orbit Γ, then π is a translation plane and G contains the translations of π.*

Proof. Clearly, G can fix no point on ℓ_∞ if G is primitive. Thus Γ is a strong block orbit by the proof of (2.7). By (2.10) G has an involutory central collineation. Keiser [8] proves that if a primitive collineation group has an involutory central collineation, then the affine plane is a translation plane and the group contains all the translations. This proves (6.3).

(6.4) **Theorem.** *Let π be a finite affine plane of order n with a collineation group G acting as a $\frac{3}{2}$-transitive group on the points of π. Assume $G_{\mathcal{O}}$, $\mathcal{O}$ an affine point, admits a block orbit Γ. π is a translation plane and G contains the translations if one of the following conditions hold:*

(i) *n is odd;*

(ii) *G is primitive;*

(iii) *G and $G_{\mathcal{O}}$ have the same orbits on ℓ_∞.*

Proof. G $\frac{3}{2}$-transitive implies either G is a Frobenius group or G is primitive on π (Passman [13]). If G is Frobenius and n is odd, then the theorem follows from (6.2). If G is primitive, the theorem follows from (6.3).

Thus assume n is even, G is a Frobenius group, and G and G_ℓ have the same orbits on ℓ_∞. Because of (2.11) we may also assume Γ has degree 1.

Let $\Gamma_\alpha = \{Q\}$. We have

$$|G| = n^2 |G_\ell| = n^2(n-1);$$

for G fixes Q and hence $G_\ell = G_{\ell,Q}$, and $G_{\ell,Q}$ is transitive on the points other than $\mathcal{O}$ of $\mathcal{O}Q$. Also $|G_{\ell,P}| = 1$, where P is an affine point of $\mathcal{O}Q$ with $P \neq \mathcal{O}$, since G is Frobenius.

(2.6) says that G_h, $h = \mathcal{O}Q$, is doubly transitive on h and (2.10) says that G_h has an involutory elation σ. Since $2 \nmid |G_\ell|$, σ has axis ℓ_α and center $h \cap \ell_\alpha = Q$. By (2.2), π is (Q, ℓ_∞)-transitive and G contains the group T of (Q, ℓ_∞)-elations. G_h is a Frobenius group on h and $|G_{\ell,P}| = 1$ for any two affine points $\mathcal{O}$, P on h. Hence, since $|G_h| = n(n-1)$, $n = 2^r$ by Passman ([13], p. 62). Also if K is the Frobenius kernel of G, then $G = KG_\ell$ and $|K| = n^2$.

Let R be a point of ℓ_α, $R \neq Q$, and assume K contains an element τ such that $R\tau = S \neq R$. Let Δ_α be the orbit of G containing R and let m be an affine line of π through R. Since G_R is transitive on the affine lines through R (Wagner [14]), the length of the orbit containing m is $n\,t$, where $t = |\Delta_\alpha|$. Hence

$$n^2(n-1) = |G| = n\,t\,|G_m|.$$

If $\mathcal{O}$ is an affine point on m and if $u = |G_{\ell,R}|$, then

$$n - 1 = |G_\ell| = u\,|G_{\ell,R}| = t\,u$$

since Δ_α is also an orbit of G_ℓ. Hence $|G_m| = n\,u$. Thus $n = 2^r \,||\, |G_m|$ and $(2, u) = 1$ since u is odd.

If Σ is the orbit of K containing m, then $n^2 = |K| = |\Sigma|\,|K_m|$; if $|K_m| = 1$, then $|\Sigma| = n^2$ and this implies K is transitive on the points of $\ell_\alpha - \{Q\}$. But then $t = |\Delta_\alpha| = n$ and since Δ_α is also an orbit of G_ℓ, $n \,||\, |G_\ell| - a$ contradiction. Therefore $|K_m| > 1$.

Since K transitive on the affine points implies K is transitive on the affine lines through R and S, respectively, the orbit Σ of m under K has length at least $2n$. Hence,

$$2n\,|K_m| \leqq |\Sigma|\,|K_m| = |K| = n^2$$

$$|K_m| \leqq \frac{n}{2} = 2^{r-1}.$$

On the other hand, $K_m = K \cap G_m$ and G_m is a Frobenius group with Frobenius kernel K_m (Passman [13], p. 183). Hence $G_m = K_m H$ and the orders of H and K_m are relatively prime. Since $|K_m| \,||\, |K| = n^2 = 2^{2r}$, this

implies $|H|$ is odd. Then

$$2^r u = n u = |G_m| = |K_m|\,|H|$$

implies $2^r \mid |K_m|$ – a contradiction.

Thus every element of K must fix R. Since R was an arbitrary point of ℓ_∞, K consists of elations with axis ℓ_∞. Therefore π is a translation plane and K is the group of translations.

(6.5) **Corollary.** *Let π be a finite affine plane of odd order and of rank (r, s, t) with respect to a collineation group G. If G is $\frac{3}{2}$-transitive and $r-1<2t$ then π is a translation plane and G contains the translations.*

(6.6) *Examples.* (i) Let π be a Desarguesian plane and $\bar{\pi}$ a translation plane constructed from π by homology type replacement. Let H denote the $(\mathcal{O}, \ell_\infty)$-homology group of π and T the translation group. The full collineation group of $\bar{\pi}$ admits the $(\mathcal{O}, \ell_\infty)$-homologies of π, but this latter group acts as a group of homologies only on the common net of $\pi \cap \bar{\pi}$. If m is any line of $\pi \cap \bar{\pi}$ which is incident with $\mathcal{O}$, then m is a block orbit of $(HT)_\mathcal{O} = G_\mathcal{O}$. Moreover, G is a Frobenius group of *both* π and $\bar{\pi}$ and each $G_\mathcal{O}$ block orbit has degree one.

(ii) Let π be a non-Desarguesian translation plane of order p^2 with kernel K and translation group T. Then $G = KT$ is a Frobenius group but does not admit block orbits.

(iii) Let π be a dual Lüneburg plane of order $n = 2^{(2m+1)^2}$ (see Ostrom [10]). Let Q be a left quasifield coordinatizing π. Q is a right 2-dimensional vector space over $F = GF(2^{2m+1})$. Let $\{t, 1\}$ be a basis for Q over F. There is a collineation group G of π which is regular on the affine points of π and induces permutations $\rho(\alpha, \beta)$ on ℓ_∞ with $\alpha, \beta \in F$ such that

$$\rho(\alpha, \beta)\colon (t\lambda_1 + \lambda_2) \to (t\lambda_1 + (\lambda_2 + \lambda_1 \alpha^{2m})),\ (\infty) \to (\infty).$$

Note that $\rho(\alpha, \beta)$ leaves (∞), (α), $\alpha \in F$, fixed and permutes the remaining points of ℓ_∞ in orbits of length 2^{2m+1}. Thus if $n = q^2$, π is of rank $(q^4, 2q, q^2+1)$ with respect to G and obviously $G_\mathcal{O}$ does not admit block orbits.

(iv) Let π be the affine Desarguesian plane of prime order p, φ a nontrivial $((\infty), \ell_\infty)$-translation, τ a nontrivial $((0), \ell_\infty)$-translation, and κ a nontrivial $((\infty), x=0)$-elation. If $G = \langle \tau\kappa \rangle \cdot \langle \varphi \rangle$ then G is regular on the points of π. Also G and $\langle \kappa \rangle$ induce the same permutation group on ℓ_∞. Hence π has rank $(p^2, 2, p+1)$ with respect to G.

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