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Sylow Subgroups of Transitive Permutation Groups

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Let G be a finite group of permutations of a set Ω, which is transitive on Ω, and let P be a Sylow p-subgroup of G for some prime p dividing $|G|$. Jordan (c. 1873) and Manning (c. 1918) obtained some strong bounds on the number of points of Ω fixed by P in the case where G is primitive but not alternating or symmetric, and P permutes only a small number of points (see [2], 13.10, and [1]; under these conditions $|G|$ is not divisible by p^2). Their results are useful, but we often require some limit on the size of the fixed point set of P (which we shall denote as $\mathrm{fix}_\Omega P$), when we have hardly any information about P or G. In this note we shall prove the following result.

Theorem. *If G is a transitive permutation group on a set Ω of degree n, and if P is a Sylow p-subgroup of G for some prime p dividing $|G|$, then the number of points of Ω fixed by P is less than $\frac{1}{2}n$.*

This is the best result possible if the only restriction on G is that it is transitive, because in the alternating group of degree $2p-1$, a Sylow p-subgroup has degree p and fixes $p-1$ points. We may construct many similar examples which are imprimitive groups; if K is a transitive group of degree k, such that $|K|$ is prime to p, then the wreath products $K \,\mathrm{wr}\, A_{2p-1}$ and $A_{2p-1} \,\mathrm{wr}\, K$ are both transitive imprimitive groups of degree $(2p-1)k$ in which a Sylow p-subgroup fixes $(p-1)k$ points.

I know of no primitive groups other than the alternating or symmetric groups, for which $|\mathrm{fix}\, P|$ is greater than $n/4$. However, if we consider A_{2p-1} acting (primitively) on its set of $(2p-1)(p-1)$ unordered pairs of distinct points, a Sylow p-subgroup fixes $\frac{1}{2}(p-1)(p-2)$ pairs, and this is almost a quarter of them. The only other example I know for which $|\mathrm{fix}\, P|$ is as large as this is $PSL(2, 7)$, a transitive group of degree 8 in which the Sylow 3-subgroups fix 2 points. This suggests the question: can a better bound be found for $|\mathrm{fix}\, P|$ for primitive non-alternating groups, and in particular, is $|\mathrm{fix}\, P| \leqq n/4$?

Proof. The proof is by induction on the degree n of G. Clearly the theorem is true if n is 2 or 3, and we assume that for transitive groups of degree r, $r < n$, any Sylow subgroup fixes less than $r/2$ points.

Case 1. Suppose that G is imprimitive on Ω, and let Σ be a set of r nontrivial blocks of imprimitivity for G. Then G is transitive on Σ of degree r, $r<n$. If P acts nontrivially on Σ, then by induction, $|\mathrm{fix}_\Sigma P|<r/2$. Also, if P fixes a point in a block B, then P fixes B setwise, and so $|\mathrm{fix}_\Omega P| \leqq |B|\,|\mathrm{fix}_\Sigma P|<|B|r/2=n/2$. If P fixes all the blocks setwise, then P is a Sylow p-subgroup of the kernel H of G acting on Σ. We claim that P fixes the same number of points in each block, and in particular that P acts nontrivially on each block. Let B, C be distinct blocks in Σ and let $g\in G$ be such that $B^g=C$. Then P^g is a Sylow p-subgroup of H and so $P=P^{gh}$ for some h in H. Thus we have $\mathrm{fix}_B P=\mathrm{fix}_C P$. Hence by induction, $|\mathrm{fix}_\Omega P|=r\,|\mathrm{fix}_B P|<r\,|B|/2=n/2$.

Case 2. Now suppose that G is primitive. Let α be a point of fix P, and let Γ denote any orbit of G_α in $\Omega-\{\alpha\}$. Then by a theorem of Jordan ([2], 18.4), P acts nontrivially on Γ. Hence by induction, $|\mathrm{fix}_\Omega P|=1+\Sigma\,|\mathrm{fix}_\Gamma P|<1+\Sigma\,|\Gamma|/2=(n+1)/2$, and since $|\mathrm{fix}_\Omega P|$ is an integer, $|\mathrm{fix}_\Omega P| \leqq n/2$. Thus the theorem is proved unless $|\mathrm{fix}_\Omega P|=n/2$. However in this case, the number of points permuted by P (which is divisible by p) is $n/2$. Hence $|G:G_\alpha|=n$ is divisible by p, and so P cannot be contained in G_α, a contradiction.

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