Werk

Titel: A Drived Homotopy Product.

Autor: Rutter, John W.

Jahr: 1973

PURL: https://resolver.sub.uni-goettingen.de/purl?266833020_0133|log35

Kontakt/Contact

Digizeitschriften e.V.
SUB Göttingen
Platz der Göttinger Sieben 1
37073 Göttingen

info@digizeitschriften.de

Math. Z. 133, 343 – 356 (1973)
© by Springer-Verlag 1973

A Derived Homotopy Product

John W. Rutter

1.

In this article I introduce and investigate the properties of a homotopy product

$$[SA, X] \times [B, X] \to [A \barwedge B, X]$$

which is defined if B is an h-coloop and which is the Whitehead product if B is a suspension: the construction gives two new methods of defining the Whitehead product in the latter case.

Each simply connected CW complex having a comultiplication is an h-coloop. There are therefore spaces $S^3 \cup e^{2n+1}$ which are h-coloops but are not homotopy associative ([1]), whilst $S^3 \vee S^{15}$, for example, has h-cogroup structures which are not suspensions (4.3 of [2]): the new homotopy product is defined when B is either of these spaces.

Let A and B have nondegenerate base points and let B be an h-coloop, then the sequence

$$B \underset{p}{\overset{i}{\rightleftarrows}} A \times B/A \times * \xrightarrow{q} A \barwedge B$$

is a split short exact sequence of h-coloops. Let

$$\psi\colon\ A \times B/A \times * \to SA \vee A \times B/A \times *$$

be the coaction in the cofibre sequence $A \to A \times B \to A \times B/A \times * \to \cdots$, then there is a unique class of maps $\omega\colon A \barwedge B \to SA \vee B$ satisfying $\omega q + p \sim (1 \vee p)\psi\colon A \times B/A \times * \to SA \vee B$. The class of ω is the universal example for the cooperator product

$$[SA, X] \times [B, X] \to [A \barwedge B, X].$$

Functoriality, deviation from bilinearity, and the relation of the product with the operation $\pi_1(X) \times [B, X] \to [B, X]$ of the fundamental group are investigated in § 3. In the case where B is an h-cogroup I define a semi-direct sum h-cogroup structure on $SA \vee A \times B/A \times *$ and hence obtain a short exact sequence of h-cogroups

$$SA \vee B \to SA \vee A \times B/A \times * \xrightarrow{(*, q)} A \barwedge B.$$

In case $B=SK$ is a suspension this latter h-cogroup extension sequence is equivalent to the sequence

$$SA \vee SK \to S(A \times K) \to S(A \barwedge K).$$

Moreover in the general case where B is an h-cogroup, the class $\omega\colon A \barwedge B \to SA \vee B$ is alternatively obtained (Theorem 4.2) by lifting back the class of the commutator

$$1_{SA} + p - 1_{SA} - p\colon\ SA \vee (A \times B/A \times *) \to SA \vee B.$$

This construction extends the commutator definition for the Whitehead product map, and coincides with it (Theorem 4.4) in case $B=SK$ is a suspension. An identity for the iterated product, related to the Jacobi-Witt identity, is given in § 5.

In [5] I show that the cooperator product $A \barwedge B \to SA \vee B$ coclassifies the inclusion $SA \vee B \subset SA \times B$ if either B is an h-cogroup or SA and B are simply connected CW complexes.

I am grateful to the referee for pointing out a way of shortening some of the proofs given in an earlier version of this article.

2. Preliminaries

I consider hausdorff spaces, X, with base point usually denoted $*$, and base point preserving maps and homotopies. The (reduced) cone and suspension are $CX = X \times I/X \times \{0\} \cup * \times I$ and $SX = CX/X \times \{1\}$. Let $X^+ = X \cup \{e\}$ be the (disjoint) union of X with a one point space $\{e\}$: e is the basepoint of X^+. Let $S^0 = \{0, 1\}$ be the 0-sphere with basepoint 0 and let $i\colon S^0 \to X^+$ be given by $i(0)=e$ and $i(1)=* \in X$. Consider the mapping cone cofibre sequence

$$S^0 \xrightarrow{i} X^+ \to X^+ \cup CS^0 \doteq X \vee I \to S^1 \to \cdots.$$

The base point $*$ of X is said to be *nondegenerate* when the projection $(X \vee I, e) \to (X, *)$ is a (based) homotopy equivalence [1].

Let $f\colon P \to Q$ induce the cofibre sequence

$$P \xrightarrow{f} Q \xrightarrow{r} C_f = Q \cup CP \xrightarrow{s} SP = C_f/Q \xrightarrow{Sf} SQ \to \cdots$$

then there is a coaction of the h-cogroup SP on C_f given by $\phi\colon C_f \to SP \vee C_f$ where

$$\begin{aligned} \phi(x,t) &= (x, 2t) && \in SP && t \leqq \tfrac{1}{2} \\ \phi(x,t) &= (x, 2t-1) && \in C_f && t \geqq \tfrac{1}{2} \\ \phi(y) &= y && \in C_f. \end{aligned}$$

[1] This is the definition of Puppe in Theorem 14 of [3]. Other authors have used the same term for an inequivalent concept.

In particular the composite function $C_f \xrightarrow{\phi} SP \vee C_f \xrightarrow{(*,1)} C_f$ is homotopic to the identity map of C_f and the composite $C_f \xrightarrow{\phi} SP \vee C_f \xrightarrow{(1,*)} SP$ is homotopic to $s\colon C_f \to SP$.

A *loop* is a set having a multiplication with two sided identity and unique solutions $x = b/a$ and $y = a \backslash b$ for the equations $x + a = b$ and $a + y = b$. An *M'-space* is a space, X, with a comultiplication $m\colon X \to X \vee X$ having homotopy identity, an *h-coloop* is an *M'*-space whose comultiplication induces a loop structure on $[X, Z]$ for each space Z,[2] and an *h-cogroup* is a homotopy associative *h*-coloop. The Hilton-Eckmann dual objects are called *M-space*, *h-loop* and *h-group*. A *homomorphism* is a map between two *M'*-spaces or two *M*-spaces which preserves the comultiplications or multiplications up to homotopy. A comultiplication on a space X induces, for each space V, a comultiplication on the *collapsed product* $V \between X = V \times X / V \vee X$.

3. The Cooperator Product

Given that A has nondegenerate base point consider the cofibre sequence [3]

$$S^0 \underset{p}{\overset{i}{\rightleftarrows}} A^+ \xrightarrow{q} A \to S^1 \to \cdots$$

where $i(0) = e$, $i(1) = * \in A$, $p(e) = 0$, $p(A) = 1$ and q identifies e with $*$. On taking the collapsed product of this sequence with B the following Lemma is obtained.

Lemma 3.1. *Let B be an h-coloop, then there is a split short exact sequence of h-coloops:*

$$B \underset{p}{\overset{i}{\rightleftarrows}} A^+ \between B \underset{r}{\overset{q}{\rightleftarrows}} A \between B.$$

The map $r\colon A \between B \to A^+ \between B$ satisfying $rq \sim 1/ip$ is unique up to homotopy. Here 1 is the identity map and $1/ip$ is a solution of $x + ip \sim 1$.

Given that B also has nondegenerate base point then the coaction in the cofibre sequence $S^0 \to B^+ \to B \to S^1 \to \cdots$ induces coactions

$$\psi\colon A^+ \between B \to SA^+ \vee A^+ \between B \to SA \vee A^+ \between B$$

and

$$\phi\colon A \between B \to SA \vee A \between B.$$

Define

$$\omega = (1 \vee p)\psi r\colon A \between B \to SA \vee B.$$

Thus, up to homotopy, ω is the unique map satisfying $\omega q \sim (1 \vee p)\psi / p$.

[2] Equivalently $[X, X \vee X]$ is a loop.

[3] This is the mapping cone sequence $S^0 \to A^+ \to A^+ \cup CS^0 \to S^1 \to \cdots$ modified by the homotopy equivalence $A^+ \cup CS^0 \to A$.

Definition 3.2. *A cooperator product map*[4] is ω: $A \divideontimes B \to SA \vee B$ defined as above. The *cooperator product* of y in $[SA, X]$ and z in $[B, X]$ is $[y, z] = \omega^*(y, z)$ in $[A \divideontimes B, X]$.

Some of the properties satisfied by the cooperator product $[SA, X] \times [B, X] \to [A \divideontimes B, X]$ are now described. Their proofs are given at the end of this section.

Theorem 3.3. (i) *Let* f: $X \to X_1$, *then* $f_*[y, z] = [f_* y, f_* z]$.

(ii) *Let* g: $A_1 \to A$, *then* $(g \divideontimes 1)^*[y, z] = [(Sg)^* y, z]$.

(iii) *Let* h: $B_1 \to B$ *be a homomorphism of h-coloops, then*

$$(1 \divideontimes h)^*[y, z] = [y, h^* z].$$

Theorem 3.4. *Let* X *be an* M*-space, then the cooperator product* $[SA, X] \times [B, X] \to [A \divideontimes B, X]$ *is zero.*

Corollary 3.5. *Let* S_*: $[A \divideontimes B, X] \to [S(A \divideontimes B), SX]$ *be the suspension; then, for any space* X, $S_*[y, z] = 0$.

Let $(\cdot)^z$ be the automorphism of $[A \divideontimes B, X]$ induced by the inner automorphism $w \to z\,p + w - z\,p$ of $[A^+ \divideontimes B, X]$. Thus $(t^z)\,q = z\,p + t\,q - z\,p$.

Theorem 3.6. *When* B *is an h-cogroup, the cooperator product* $[SA, X] \times [B, X] \to [A \divideontimes B, X]$ *satisfies*

(i) $[y_1 + y_2, z] = \phi^*(y_1, [y_2, z]) + [y_1, z]$,

(ii) $[y, z_1 + z_2] = [y, z_1] + [y, z_2]^{z_1}$.

In particular it is linear in the first variable if A *is an* M'*-space*[5] *and in the second variable if* B *is homotopy abelian.*

Let B be an h-coloop with nondegenerate base point, then the coaction ϕ: $B \to S^1 \vee B$ in the cofibre sequence $S^0 \to B^+ \to B \to S^1 \to \cdots$ induces, for any X, the usual (left) action of the fundamental group $\pi_1(X)$ on $[B, X]$ by $l_\sharp\{f\} = \phi^*(\{l\}, \{f\})$. The inclusion map $B \to S^1 \vee B$ is denoted ι_2.

Theorem 3.7. *Let* B *be an h-coloop with nondegenerate base point, then* $\omega \sim \phi/\iota_2$: $B \to S^1 \vee B$ *and hence the cooperator product* $[S^1, X] \times [B, X] \to [B, X]$ *satisfies*[6] $[l, z] = l_\sharp(z)/z$.

---

[4] This map is defined more generally when B has an M'-structure with right homotopy inverse provided that $[A \divideontimes B, SA \vee B] \to [A^+ \divideontimes B, SA \vee B]$ is injective. In the category of 1-connected CW complexes all M'-spaces are h-coloops.

[5] If B is a suspension, the cooperator product is the Whitehead product (Theorem 4.4) which is linear in its first variable provided only that SA is abelian (by Lemma 6.2(ii) for example). If SA is abelian the cooperator product satisfies $[y_1 + y_2, z] = \phi^*(y_2, [y_1, z]) + [y_2, z]$ but the author is not able to show that it is linear in y without restriction on B.

[6] This formula is well known for the Whitehead product in case B is a sphere.

The remainder of this section is devoted to proving the above Theorems. First several Lemmas are required.

Lemma 3.8. *Let B be an h-coloop, then there is a homotopy equivalence*

$$q+p\colon\ A^{+}\ltimes B \rightleftarrows A\ltimes B\vee B\colon\ (r, i)$$

and

$$(q+p)^{*}\colon\ [A\ltimes B, X]\times[B, X]\to[A^{+}\ltimes B, X]$$

is an isomorphism with respect to the semi-direct product structure, on $[A\ltimes B, X]\times[B, X]$, given by

$$(t_1, z_1)+(t_2, z_2)=(t_1+t_2^{z_1}, z_1+z_2).$$

Let

$$\bar{\psi}\colon\ A\ltimes B\vee B\to SA\vee A\ltimes B\vee B$$

be the coaction induced by ψ and the homotopy equivalence $q+p$.

Lemma 3.9.

$$\bar{\psi}\sim(\phi+\omega, 1_B).$$

Proof.

$$\bar{\psi}|B=(1\vee(q+p))\,\psi\, i\sim(1\vee(q+p))\, i\sim 1_B$$

$$\begin{aligned}(\psi|A\ltimes B)\, q&=(1\vee(q+p))\,\psi(1/i\, p)\\ &\sim(1\vee(q+p))\,\psi/p\\ &\sim((1\vee q)\,\psi+(1\vee p)\,\psi)/p \quad \text{by Lemma 6.3}\\ &\sim(\phi\, q+(1\vee p)\,\psi/p) \quad \text{since } B\to S^1 \text{ is trivial.}\end{aligned}$$

Proof of Theorem 3.4. By Lemma 6.2(i)

$$\bar{\psi}^{*}(y, t, z)=(t, z)=(\phi^{*}(y, t), z)$$

and the result is immediate from Lemma 3.9.

Proof of Corollary 3.5. Let $\eta\colon X\to\Omega SX$ be the inclusion $\eta(x)(t)=(x, t)$. It is sufficient to prove $\eta_{*}[y, z]=0$. But $\eta_{*}[y, z]=[\eta_{*}\, y, \eta_{*}\, z]=0$ by Theorems 3.3 and 3.4.

Proof of Theorem 3.6. (i) By Lemma 3.9

$$\bar{\psi}^{*}(y_1+y_2, t, z)=(\phi^{*}(y_1+y_2, t)+[y_1+y_2, z], z)$$

and since $\bar{\psi}$ is a coaction and by Lemmas 3.9 and 6.3

$$\begin{aligned}\bar{\psi}^{*}(y_1+y_2, t, z)&=\bar{\psi}^{*}(y_1, \bar{\psi}^{*}(y_2, t, z))\\ &=\bar{\psi}^{*}(y_1, \phi^{*}(y_2, t)+[y_2, z], z)\\ &=(\phi^{*}(y_1, \phi^{*}(y_2, t)+[y_2, z])+[y_1, z], z).\end{aligned}$$

(ii) $\quad \psi^{*}(y, t, z_1+z_2)=(\phi^{*}(y, t)+[y, z_1+z_2], z_1+z_2).$

Also by Lemmas 6.3, 3.8 and 3.9

$$\begin{aligned}\bar{\psi}^*(y, t, z_1+z_2) &= \psi^*\big(y, (t, z_1)+(0, z_2)\big)\\ &= \big(\phi^*(y, t)+[y, z_1], z_1\big)+([y, z_2], z_2)\\ &= \big(\phi^*(y, t)+[y, z_1]+[y, z_2]^{z_1}, z_1+z_2\big).\end{aligned}$$

(iii) The product is linear in its first variable when A is an M'-space by Lemma 6.2(i). If B is homotopy abelian then so is $A^+ \barwedge B$ and therefore $(\cdot)^z$ is the identity and the product is linear in its second variable.

Proof of Theorem 3.7. Let $S^0=\{0, 1\}$ and $S^{0+}=\{0, 1, 2\}$ where the base points are 0 and 2 respectively. Then taking $A=S^0$, and with the obvious notation, the sequence of Lemma 3.1 becomes

$$B_1 \underset{c}{\overset{j_1}{\rightleftarrows}} B_0 \vee B_1 \underset{r}{\overset{\pi_2}{\rightleftarrows}} B_1$$

where j_1 and j_2 are the inclusions into the first and second factors, c is the folding map, π_2 is the projection onto the second factor and $r=j_2/j_1$. Also

$$\psi=(1_{B_0} \vee \phi)\colon\ B_0 \vee B_1 \to S^1_0 \vee S^1_1 \vee B_0 \vee B_1 \to S^1_1 \vee B_0 \vee B_1.$$

Thus

$$\omega=(1 \vee c)(1 \vee \phi)(j_2/j_1)=\phi/\iota_2.$$

4. The Commutator Definition

In this Section I consider the case where B is an h-cogroup. The cooperator product was defined by means of the coaction $A^+ \barwedge B \to SA \vee (A^+ \barwedge B)$ and the splitting function r in the short exact h-cogroup sequence $B \to A^+ \barwedge B \to A \barwedge B$. The latter sequence is now "embedded" in a short exact h-cogroup sequence $SA \vee B \to SA \vee A^+ \barwedge B \to A \barwedge B$ and the cooperator product is shown to correspond to a commutator in the group $[SA \vee A^+ \barwedge B, SA \vee B]$. When $B=SK$ is a suspension the short exact h-cogroup sequence $SA \vee SK \to SA \vee A^+ \barwedge SK \to A \barwedge SK$ is shown to be equivalent to the sequence $SA \vee SK \to S(A \times K) \to S(A \barwedge K)$ and in this case the cooperator product coincides with the Whitehead product. The spaces A, B and K are assumed to have nondegenerate basepoints.

Define a semi-direct sum h-cogroup structure on $SA \vee A^+ \barwedge B$ by means of the coaction $\big((-1) \vee 1\big)\psi$. Thus the comultiplication

$$m\colon\ SA \vee A^+ \barwedge B \to (SA \vee A^+ \barwedge B)_1 \vee (SA \vee A^+ \barwedge B)_2$$

is given by

$m|SA\colon\ SA \to SA_1 \vee SA_2$ is the suspension comultiplication

$$m|A^+ \barwedge B=\big((-1_{SA_2}) \vee 1_{A^+ \barwedge B_1}\big)\psi+1_{A^+ \barwedge B_2}.$$

By Lemma 6.3 m is a cogroup structure: the inverse for m is $(-1_{SA}) \vee (-\psi)$.

Proposition 4.1. *Let A and B have nondegenerate base points and let B be an h-cogroup, then the sequence*

$$SA \vee B \xrightarrow{1 \vee i} SA \vee A^+ \barwedge B \xrightarrow{(*, q)} A \barwedge B$$

is a short exact sequence of h-cogroups with respect to the sum comultiplication on $SA \vee B$ and the comultiplication m on $SA \vee A^+ \barwedge B$.

Let ρ_1 and ρ_2 be the composite maps

$$\rho_1 \colon SA \vee A^+ \barwedge B \xrightarrow{(1, *)} SA \subset SA \vee B$$

and

$$\rho_2 \colon SA \vee A^+ \barwedge B \xrightarrow{(*, p)} B \subset SA \vee B$$

and let $\langle \rho_1, \rho_2 \rangle = (\rho_1 + \rho_2) - (\rho_2 + \rho_1)$ be their commutator obtained by using the comultiplication m.

Theorem 4.2. *Let A and B have nondegenerate basepoints and let B be an h-cogroup, then the following diagram is homotopy commutative*

$$\begin{array}{ccc} SA \vee A^+ \barwedge B & \xrightarrow{\langle \rho_1, \rho_2 \rangle} & SA \vee B \\ {\scriptstyle (*,1)} \downarrow & \nearrow {\scriptstyle (1 \vee p)\psi - p} & \\ A^+ \barwedge B & & \end{array}$$

and thus the cooperator product class $\{\omega\}$ is the unique class satisfying $(*, q)^* \{\omega\} = \langle \rho_1, \rho_2 \rangle$.

Proof. From the definition of the cogroup structure on $SA \vee A^+ \barwedge B$,

$$\begin{aligned} \psi(*, 1) &\sim * \vee \big((-(-1_{SA})) \vee 1_{A^+ \barwedge B}\big) \psi \\ &\sim (1_{SA} \vee 1_{A^+ \barwedge B}) + \big((-1_{SA}) \vee *\big) \end{aligned}$$

and therefore

$$\begin{aligned} \big((1 \vee p)\psi - p\big)(*, 1) &\sim (1 \vee p) + \big((-1) \vee *\big) - (* \vee p) \\ &\sim (\rho_1 + \rho_2) - (\rho_2 + \rho_1). \end{aligned}$$

As well as giving an alternative definition of the cooperator product in the general case, Theorem 4.2 is also the first step in showing that the cooperator product coincides with the Whitehead product in case $B = SK$ is a suspension. In this latter case then define $\tau \colon S(A \times K) \to SA \vee A^+ \barwedge SK$ by $\tau = \tau_1 + \tau_2$ where τ_1 is the projection $S(A \times K) \to SA \subset SA \vee A^+ \barwedge SK$ and $\tau_2 \colon S(A \times K) \to A^+ \barwedge SK \subset SA \vee A^+ \barwedge SK$ is given by $\tau_2(a, k, t) = (a, k, t)$.

Proposition 4.3. *Let $B = SK$ be a suspension and let A and K have nondegenerate base points, then τ is an isomorphism and the two h-cogroup*

extensions of $S(A \ltimes\!\!\times K)$ *by* $SA \vee SK$ *are equivalent*[7,8].

$$\begin{array}{ccccc} SA \vee SK & \xrightarrow{S\iota} & S(A \times K) & \xrightarrow{S\pi} & S(A \ltimes\!\!\times K) \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle 1} \\ SA \vee SK & \xrightarrow{1 \vee i} & SA \vee A^{+} \ltimes\!\!\times SK & \xrightarrow{(*,\, q)} & A \ltimes\!\!\times SK. \end{array}$$

Proof. The diagram above is homotopy commutative since

$$(\tau_1 + \tau_2)(S\iota) = \tau_1(S\iota) + \tau_2(S\iota) \sim (1 \vee *) + (* \vee i) \sim 1 \vee i, \quad \text{and} \quad (*, q)\,\tau \sim S\pi.$$

Also, as maps $S(A \times K) \to (SA \vee A^{+} \ltimes\!\!\times SK)_1 \vee (SA \vee A^{+} \ltimes\!\!\times SK)_2$,

$$\begin{aligned} m\tau &= m(\tau_1 + \tau_2) \\ &= m\tau_1 + m\tau_2 \\ &\sim (1_{SA_1} + 1_{SA_2})\,\tau_1 + \big(((-1_{SA_2}) \vee 1_{(A^{+} \ltimes\!\!\times SK)_1})\,\psi + 1_{(A^{+} \ltimes\!\!\times SK)_2}\big)\,\tau_2 \\ &\sim (\tau_1)_1 + (\tau_1)_2 - (\tau_1)_2 + (\tau_2)_1 + (\tau_1)_2 + (\tau_2)_2 \\ &\sim \big((\tau_1)_1 + (\tau_2)_1\big) + \big((\tau_1)_2 + (\tau_2)_2\big) \\ &= (\tau)_1 + (\tau)_2 \end{aligned}$$

where m is the comultiplication in $SA \vee A^{+} \ltimes\!\!\times SK$. This proves that τ is linear. Let σ_1, σ_2 and σ_3 be the projections from $S(A \times K)$ to SA, $A \ltimes\!\!\times SK$ and SK. Then

$$\big(1 \vee (q + p)\big)\,\tau \sim \sigma_1 + \sigma_2 + \sigma_3 \colon\ S(A \times K) \to SA \vee A \ltimes\!\!\times SK \vee SK.$$

Since $\big(1 \vee (q + p)\big)$ and $\sigma_1 + \sigma_2 + \sigma_3$ are homotopy equivalences, τ is also[9] a homotopy equivalence.

Let

$$\pi_1 \colon\ S(A \times K) \to SA \to SA \vee SK \quad \text{and} \quad \pi_2 \colon\ S(A \times K) \to SK \to SA \vee SK$$

be the projections onto the factors. Then there is a unique class represented by a map $W \colon\ S(A \ltimes\!\!\times K) \to SA \vee SK$ satisfying $(S\pi)^{*}\{W\} = \{(\pi_1 + \pi_2) - (\pi_2 + \pi_1)\}$: W is the universal example for the Whitehead product. Since $\rho_1\tau \sim \pi_1$ and $\rho_2\tau \sim \pi_2$, the following Theorem is an immediate consequence of Theorem 4.2 and Proposition 4.3.

---

[7] The first sequence is exact for A and K having nondegenerate basepoints by Satz 16 of [3].

[8] Since the composite $A \times K \times I \to (A \ltimes\!\!\times K) \times I \to S(A \ltimes\!\!\times K)$ is an identification, the identity function $1 \colon S(A \ltimes\!\!\times K) \to A \ltimes\!\!\times SK$ is a continuous bijection. It is a homotopy equivalence from the proof of Satz 18 of [3].

[9] This follows alternatively from the five lemma.

Theorem 4.4. *Let A and K have nondegenerate base points, then the following diagram is homotopy commutative*

$$\begin{array}{ccc} S(A \divideontimes K) & \xrightarrow{W} & SA \vee SK \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} \\ (A \divideontimes SK) & \xrightarrow{\omega} & SA \vee SK \end{array}$$

and hence, in this case, the cooperator product

$$[SA, X] \times [SK, X] \to [A \divideontimes SK, X] \cong [S(A \divideontimes K), X]$$

coincides with the Whitehead product.

Remark 4.5. The homotopy equivalence

$$\sigma_1 + \sigma_2 + \sigma_3 \colon S(A \times K) \to SA \vee S(A \divideontimes K) \vee SK$$

induces an h-cogroup structure on $SA \vee S(A \divideontimes K) \vee SK$ from the suspension structure on $S(A \times K)$: the corresponding group structure on $[SA, X] \times [S(A \divideontimes K), X] \times [SK, X]$ is given by

$$(y_1, t_1, z_1) + (y_2, t_2, z_2) = (y_1 + y_2, t_1^{-y_2} + [-y_2, z_1] + t_2^{z_1}, z_1 + z_2)$$

where for example $(\cdot)^z$ is the automorphism induced by the inner automorphism $t \to z + t - z$ of $[S(A \times K), X]$ (cf. Lemma 3.8).

5. An Identity for Iterated Cooperator Products

Let A, B and D have nondegenerate base points and let

$$A \divideontimes B \divideontimes D = A \times B \times D / A \triangle B \triangle D$$

where

$$A \triangle B \triangle D = A \times B \times * \cup * \times B \times D \cup A \times * \times D.$$

From the proof of Satz 18 of [3] the identity maps $A \divideontimes B \divideontimes D \to (A \divideontimes B) \divideontimes D$, $A \divideontimes B \divideontimes D \to A \divideontimes (B \divideontimes D)$ and $A \divideontimes B \divideontimes D \to B \divideontimes A \divideontimes D \to B \divideontimes (A \divideontimes D)$ are homotopy equivalences. Let D be an h-cogroup and consider the following commutative diagram of three short exact sequences of h-cogroups.

$$\begin{array}{ccccc} & & & & (A \vee B) \divideontimes D \\ & & & & \downarrow \\ D & \longrightarrow & (A \times B)^+ \divideontimes D & \longrightarrow & (A \times B) \divideontimes D \\ \downarrow{\scriptstyle 1} & & \downarrow & & \downarrow \\ D & \longrightarrow & (A \divideontimes B)^+ \divideontimes D & \longrightarrow & (A \divideontimes B) \divideontimes D. \end{array}$$

By Lemma 6.3 the h-cogroup coactions

$$\psi\colon (A\times B)^+ \between D \to S(A\times B)\vee (A\times B)^+ \between D$$

and

$$\phi\colon (A\times B)\between D \to S(A\times B)\vee (A\times B)\between D$$

are by automorphisms. Furthermore, since the composite function

$$(A\vee B)\between D \to (A\times B)\between D \xrightarrow{\phi} S(A\times B)\vee (A\times B)\between D \to SA\vee (A\between B)\between D$$

is nulhomotopic, it is clear from the vertical exact sequence that ϕ induces the coaction $\phi_A\colon (A\between B)\between D \to SA\vee (A\between B)\between D$, and this coaction via automorphisms is, because of the functorial nature of the construction, easily seen to be the same as the one obtained from the cofibre sequence $A\to A\between (B\between D)^+ \to A\between (B\between D)\to\cdots$.

Similarly ϕ induces a coaction

$$\phi_B\colon (A\between B)\between D \to SB\vee (A\between B)\between D.$$

Theorem 5.1. *Let A, B and D have nondegenerate base points and let D be an h-cogroup then the iterated cooperator products $[SA, X]\times[SB, X]\times[D, X]\to[A\between B\between D, X]$ satisfy the identity*[10,11]

$$\phi_B^*(b, [[-b, a], d]) - [a, -[b, -d]]^d - \phi_A^*(a, [b, [-a, d]]) = 0.$$

Proof. Since the projection $u\colon (A\times B)^+\between D \to A\between B\between D$ induces a monomorphism $u^*\colon [A\between B\between D, X]\to[(A\times B)^+\between D, X]$ it is sufficient to prove the corresponding identity in $[(A\times B)^+\between D, X]$. Now by Theorem 4.4 and Lemma 6.3 $u^*\phi_B^*(b, [[-b, a], d])$ is represented by[12]

$$(b, (\langle -b, a\rangle, d)\psi - d)\psi \sim (a+b-a, d)\psi - (b, d)\psi.$$

Similarly $u^*(-[a, -[b, -d]]^d)$ is represented by

$$\begin{aligned} d - ((a, -((b, -d)\psi - (-d)))\psi - (-((b, -d)\psi - (-d)))) - d \\ \sim d - (-(a, d)\psi + (a+b, d)\psi - (b, d)\psi + d) - d \\ \sim (b, d)\psi - (a+b, d)\psi + (a, d)\psi - d. \end{aligned}$$

---

[10] The twisting map $A\between B\between D\to B\between A\between D$ is implicit in the first and third terms of this formula.

[11] The iterated products are composite functions

$$A\between B\between D \to S(A\between B)\vee D \to SA\vee SB\vee D,$$

$$A\between B\between D \to SA\vee (B\between D)\to SA\vee SB\vee D$$

and

$$A\between B\between D \to SB\vee (A\between D)\to SA\vee SB\vee D.$$

[12] To avoid complication of notation the symbol a, for example, is used for $S(A\times B)\to SA\to X$ as well as for $SA\to X$.

Finally $u^*(-\phi_A^*(a,[b,[-a,d]]))$ is represented by

$$\begin{aligned}&-(a,(b,(-a,d)\psi-d)\psi-((-a,d)\psi-d))\psi\\&\sim-(a,(b-a,d)\psi-(b,d)\psi+d-(-a,d)\psi)\psi\\&\sim d-(a,d)\psi+(a+b,d)\psi-(a+b-a,d)\psi.\end{aligned}$$

The result is now immediate.

Corollary 5.2. *The following identity is satisfied*

$$[[a,b],d]-[a,[b,d]]-[b,-[a,d]]=0$$

if either A and B are M′-spaces and D is homotopy abelian or SA and SB are homotopy abelian and D is a homotopy abelian suspension[13].

Proof. By Lemma 6.2 $\phi_B^*(b,y)=y$ if either B is an M'-space or SB is homotopy abelian and the comultiplication on $A\ltimes D$ is a suspension structure (e.g. if either A or D is a suspension). Also, since SB is homotopy abelian, $(\cdot)^b=1$ on $A\ltimes SB$. Thus $[-b,a]=[-b,a]^b=[a,b]$ by Theorem 4.4, and it follows that $\phi_B^*(b,[[-b,a],d])=[[a,b],d]$. Similarly if either A is an M'-space or SA is homotopy abelian and the comultiplication on $B\ltimes D$ is a suspension structure then

$$\phi_A^*(a,[b,[-a,d]])=[b,[-a,d]]=[b,-[a,d]]$$

since

$$\begin{aligned}q^*[-a,d]&=q^*\phi^*(a,[-a,d])\\&=\psi^*(a,(-a,d)\psi-d)\\&=d-\psi^*(a,d)\\&=q^*(-[a,d]).\end{aligned}$$

Finally if D is homotopy abelian then $(\cdot)^d=1$ in $[A\ltimes B\ltimes D,X]$, and in $[B\ltimes D,X]$, and thus

$$\begin{aligned}q^*[b,-d]&=\psi^*(b,-d)-(-d)\\&=-(-d)-\psi^*(b,d)\\&=-(\psi^*(b,d)-d)\\&=-q^*[b,d].\end{aligned}$$

The result is immediate.

Remarks 5.3. An alternative formulation of Theorem 5.1 is obtained by means of the monomorphism

$$[A\ltimes B\ltimes D,X]\to[S(A\times B)\vee(A\times B)^+\ltimes D,X].$$

[13] Several other special cases of Theorem 5.1 follow easily from the proof of Corollary 5.2.

In this case $\phi_A^*(a,\cdot)$ and $\phi_B^*(b,\cdot)$ correspond to the appropriate inner automorphisms in the second group as in the proof of Lemma 6.2(ii) and, by Theorems 4.2 and 4.4, the cooperator products correspond to commutators. The commutator identity obtained in this way is the Witt identity (see [4]). By defining $[y,x]=-[x,y]$ whenever the cooperator product $[x,y]$ is defined, the formula of Corollary 5.2 becomes the Jacobi identity

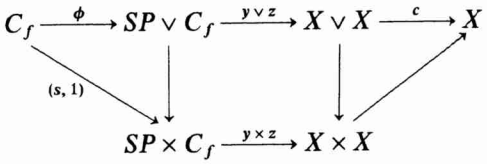
$$[[a,b],d]+[[b,d],a]+[[d,a],b]=0$$

and therefore Corollary 5.2 extends Theorem 1 of [4].

6. Three Lemmas on Coactions

Lemma 6.1. *Let $P \xrightarrow{f} Q \xrightarrow{r} C_f \xrightarrow{s} SP \to \cdots$ be an induced cofibre sequence with coaction $\phi\colon C_f \to SP \vee C_f$ and let X be an M-space, then ϕ^*: $[SP,X]\times[C_f,X]\to[C_f,X]$ satisfies $\phi^*(y,z)=s^*y+z$.*

Proof. The result is immediate from the homotopy commutative diagram



where $c\colon X \vee X \to X$ is the folding map.

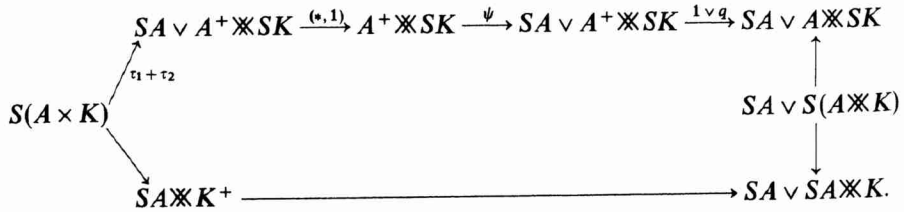
Lemma 6.2. *Let B have non-degenerate base point then the coaction $\phi\colon A \barwedge B \to SA \vee A \barwedge B$ is homotopic to the inclusion if either*

(i) *A is an M'-space, or*

(ii) *SA is homotopy abelian and B is a suspension.*

Proof. (i) Since ϕ is a homomorphism with respect to the comultiplications induced by that on A, and since $A \barwedge B \to SA$ is nulhomotopic, the result is immediate.

(ii) Consider the following diagram



The right hand vertical maps are bijective isomorphisms from the proof of Satz 18, [3]. Since $\tau_1+\tau_2$ is a homomorphism, then, as in the proof of Theorem 4.2,

$$(1\vee q)\psi(*,1)(\tau_1+\tau_2)\sim(1\vee q)(\tau_1+\tau_2)+(-1\vee *)(\tau_1+\tau_2)$$
$$\sim 1_{SA}+S\pi-1_{SA}.$$

Now 1_{SA} and $S\pi$ both factor through $SA\ltimes K^+$ which is abelian and thus

$$1_{SA}+S\pi-1_{SA}\sim S\pi.$$

Therefore $\iota_2 q\sim(1\vee q)\psi\sim\phi q$. The result follows.

Lemma 6.3. *Let B be an M'-space having nondegenerate base point then for each fixed a in $[SA, X]$ the functions $x\to\psi^*(a,x)$ and $x\to\phi^*(a,x)$ are automorphisms of $[A^+\ltimes B, X]$ and $[A\ltimes B, X]$ respectively.*

Proof. Let m: $B\to B\vee B$ be the comultiplication then the commutativity of the diagram

$$\begin{array}{ccccc} S^0 & \longrightarrow & B^+ & \longrightarrow & B \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle m^+} & & \downarrow{\scriptstyle m} \\ S^0 & \longrightarrow & (B\vee B)^+ & \longrightarrow & B\vee B \end{array}$$

implies the homotopy commutativity of the diagram of coactions

$$\begin{array}{ccc} B & \xrightarrow{\phi} & S^1\vee B \\ \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle 1\vee m} \\ B\vee B & \longrightarrow & S^1\vee B\vee B. \end{array}$$

Replacing m successively by the inclusions onto the first and second factors it follows that the coaction $B\vee B\to S^1\vee B\vee B$ can be taken to be $\phi\vee\phi$. Hence $\phi^*(a,\cdot)$ is a homomorphism: it is an automorphism since $\phi^*\big(-a,\phi^*(a,x)\big)=x=\phi^*\big(a,\phi^*(-a,x)\big)$.

References

1. Berstein, I.: A note on spaces with non-associative comultiplication. Proc. Cambridge Philos. Soc. **60**, 353–354 (1964).
2. Ganea, T.: Cogroups and suspensions. Inventiones Math. **9**, 271–281 (1969/70).
3. Puppe, D.: Homotopiemengen und ihre induzierten Abbildungen I. Math. Z. **69**, 299–344 (1958).

4. Rutter, J. W.: Two theorems on Whitehead products. J. London math. Soc. **43**, 509–512 (1968), and II. Ser. **1**, 20 (1969).
5. Rutter, J. W.: A coclassifying map for the inclusion of the wedge in the product. Math. Z. **129**, 173–183 (1972).

Dr. J. W. Rutter
Department of Mathematics
University of Liverpool
P.O. Box 147, Liverpool, L 69 3 BX
England

(Received April 24, 1973)