Math. Z. 132, 361 – 366 (1973)

Remarks on Set-Contractions and Condensing Maps

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Introduction

Let X be a complex Banach space and let $T: X \to X$ be a linear condensing map. The main purpose of this paper is to prove that there exists an equivalent norm on X such that, with respect to this new norm, T is already a strict set-contraction. By means of a counterexample we also show that, in general, this result does not hold for nonlinear maps. Moreover, we give a general necessary condition for a (nonlinear) set-contraction to be a strict set-contraction with respect to some equivalent norm, and we prove that for linear maps, this condition is also sufficient. Along the way we observe some consequences of our results.

1.

Let X be a metric space and let A be a bounded subset of X. Following Kuratowski [3], we define $\gamma(A)$, the measure of noncompactness of A, to be $\inf\{d>0 \mid$ there exists a finite number of sets $S_1, \ldots, S_n$ such that diameter $(S_i) \leqq d$ and $A = \bigcup_{i=1}^{n} S_i\}$. Let X_1 and X_2 be metric spaces with measures of noncompactness γ_1 and γ_2, respectively, and let $f: X_1 \to X_2$ be a continuous map. We say that f is a k-set-contraction if there exists $k \in \mathbb{R}_+$ such that, given any bounded set A in X_1, $\gamma_2(f(A)) \leqq k\,\gamma_1(A)$. A continuous map $f: X_1 \to X_2$ is said to be a set-contraction if it is a k-set-contraction for some $k \in \mathbb{R}_+$. If f is a set-contraction, we define $\gamma(f)$, the measure of noncompactness of f, to be $\inf\{k \geqq 0 \mid f$ is a k-set-contraction$\}$, and if $\gamma(f) < 1$, we call f a strict set-contraction. Finally, we say that f is a condensing map if for every bounded set A in X_1 with $\gamma(A) \neq 0$, we have $\gamma_2(f(A)) < \gamma_1(A)$. Obviously every strict set-contraction is a condensing map, but Nussbaum [5] has shown that there exist condensing maps which are not strict set-contractions.

We shall use the following convention. Let $(X, \|\cdot\|)$ be a normed linear space and let $\|\cdot\|_1$ be a norm equivalent to $\|\cdot\|$. Then $\gamma(A)$, $\gamma(f)$, and diam(A) will refer to the original norm, whereas $\gamma_{\|\cdot\|_1}(A)$, $\gamma_{\|\cdot\|_1}(f)$, and diam$_{\|\cdot\|_1}(A)$ will be taken with respect to $\|\cdot\|_1$.

Let $(X, \|\cdot\|)$ be a normed vector space and let $f\colon D\to X$, $D\subset X$, be a set-contraction. Then f is obviously a set-contraction with respect to any equivalent norm on X. In the following we shall say that f is a topological strict set-contraction if there exists an equivalent norm $\|\cdot\|_1$ such that f is a strict set-contraction with respect to $\|\cdot\|_1$.

Denote by $L(X)$ the space of bounded linear operators on the Banach space $(X, \|\cdot\|)$. Every $T\in L(X)$ is obviously a set-contraction with $\gamma(T)\leqq \|T\|$. It was observed in [6] that the sequence $\{(\gamma(T^n))^{1/n}\}_{n=1}^{\infty}$ is convergent, with $\lim_{n\to\infty}(\gamma(T^n))^{1/n}=\inf_n\{(\gamma(T^n))^{1/n}\}$. For the proof one needs only that for S and T in $L(X)$,

$$\text{(a) } \gamma(ST)\leqq\gamma(S)\cdot\gamma(T) \quad \text{and} \quad \text{(b) } \gamma(T)\geqq 0,$$

two properties which are also valid for nonlinear set-contractions S and T, provided the composition ST is defined. Thus we conclude that if f is a set-contraction mapping the subset Domain$(f)\subset(X, \|\cdot\|)$ into itself, then $\lim_{n\to\infty}(\gamma(f^n))^{1/n}$ exists and equals $\inf_n\{(\gamma(f^n))^{1/n}\}$. The following proposition shows that this number is a lower bound for the measures of noncompactness of f taken over all norms equivalent to $\|\cdot\|$.

Proposition 1. *Let G be a subset of the Banach space $(X, \|\cdot\|)$, and let $f\colon G\to G$ be a set-contraction. If $\|\cdot\|_1$ is a norm on X equivalent to $\|\cdot\|$, then* $\lim_{n\to\infty}(\gamma(f^n))^{1/n}\leqq\gamma_{\|\cdot\|_1}(f)$.

Proof. By assumption there exists a constant $c>1$ such that for every $x\in X$, $c^{-1}\|x\|_1\leqq\|x\|\leqq c\|x\|_1$. Let A be a bounded subset of G, and let $\varepsilon>0$ be given. By the definition of $\gamma_{\|\cdot\|_1}(A)$ there exist finitely many sets $S_1,\dots,S_n$ with diam$_{\|\cdot\|_1}(S_i)<\gamma_{\|\cdot\|_1}(A)+\frac{\varepsilon}{c}$ such that $A=\bigcup_{i=1}^n S_i$. Then for each i, $1\leqq i\leqq n$,

$$\begin{aligned}\operatorname{diam}(S_i)&=\sup_{x,y\in S_i}\|x-y\|\leqq\sup_{x,y\in S_i}c\cdot\|x-y\|_1\\&=c\cdot\sup_{x,y\in S_i}\|x-y\|_1=c\cdot\operatorname{diam}_{\|\cdot\|_1}(S_i)\end{aligned}$$

so that

$$\begin{aligned}\gamma(A)&\leqq\max_{1\leqq i\leqq n}\operatorname{diam}(S_i)\leqq\max_{1\leqq i\leqq n}c\cdot\operatorname{diam}_{\|\cdot\|_1}(S_i)\\&<c\,\gamma_{\|\cdot\|_1}(A)+\varepsilon.\end{aligned}$$

This shows that $\gamma(A)\leqq c\,\gamma_{\|\cdot\|_1}(A)$, and by symmetry, $\gamma_{\|\cdot\|_1}(A)\leqq c\,\gamma(A)$. Thus if $g\colon G\to G$ is an arbitrary set-contraction, we have for each bounded set $B\subset G$

$$\gamma_{\|\cdot\|_1}(g(B))\leqq c\,\gamma(g(B))\leqq c\,\gamma(g)\,\gamma(B)\leqq c^2\,\gamma(g)\,\gamma_{\|\cdot\|_1}(B).$$

It follows that $\gamma_{\|\cdot\|_1}(g) \leqq c^2 \gamma(g)$, and again by symmetry, that $c^{-2}\gamma(g) \leqq \gamma_{\|\cdot\|_1}(g)$. Thus for each $n = 1, 2, 3, \ldots$,

$$c^{-2}\gamma(f^n) \leqq \gamma_{\|\cdot\|_1}(f^n) \leqq c^2 \gamma(f^n).$$

The preceding inequality implies that

$$\lim_{n\to\infty} (\gamma(f^n))^{1/n} = \lim_{n\to\infty} (\gamma_{\|\cdot\|_1}(f^n))^{1/n},$$

from which the assertion follows since for each $n = 1, 2, 3, \ldots$ we have

$$(\gamma_{\|\cdot\|_1}(f^n))^{1/n} \leqq ((\gamma_{\|\cdot\|_1}(f))^n)^{1/n} = \gamma_{\|\cdot\|_1}(f).$$

2.

We will need several facts about Fredholm and semi-Fredholm operators. We list these here briefly, and refer the reader to the articles of Lebow and Schechter [4] and of Nussbaum [6] for more detailed discussions and complete lists of references.

Let X be a complex Banach space and suppose $T \in L(X)$. Denote by Φ_T the set of $\lambda \in \mathbb{C}$ such that $T - \lambda$ is Fredholm, and by $\tilde{\Phi}_T$ the set of $\lambda \in \mathbb{C}$ such that $T - \lambda$ is semi-Fredholm. Then $\tilde{\Phi}_T$ is a union of open components on each of which the index $i(T-\lambda)$ is constant. The union of those components where $|i(T-\lambda)| < \infty$ is Φ_T. Following Browder [1], we define $\sigma_e(T)$, the essential spectrum of T, to be the set of λ in $\sigma(T)$, the spectrum of T, for which at least one of the following conditions holds:

(1) The range of $\lambda - T$ is not closed;

(2) λ is a limit point of $\sigma(T)$;

(3) $\bigcup_{r \geqq 1} N(\lambda - T)^r$ is infinite dimensional, where $N(S)$ denotes the nullspace of a linear operator S. Gohberg and Krein [2] showed that this set coincides with the complement of the union of all components of Φ_T containing points of the resolvent set of T. Nussbaum [6] proved that the radius of the essential spectrum, $r_e(T) \equiv \max_{\lambda \in \sigma_e(T)} |\lambda|$, satisfies

$$r_e(T) = \lim_{n\to\infty} (\gamma(T^n))^{1/n}, \tag{2.1}$$

an equality which will be of much use to us.

We have shown that a necessary condition for the function f of Proposition 1 to be a topological strict set-contraction is that $\lim_{n\to\infty} (\gamma(f^n))^{1/n} < 1$. It is a consequence of the next theorem that if $f \in L(X)$, that condition in fact insures that f is a topological strict set-contraction.

Theorem 1. *Let $(X, \|\cdot\|)$ be a complex Banach space and suppose $T \in L(X)$. Given an arbitrary $\varepsilon > 0$, there exists an equivalent norm $\|\cdot\|_\varepsilon$ on X such that $\gamma_{\|\cdot\|_\varepsilon}(T) < r_e(T) + \varepsilon$.*

Proof. By Lemma 6 in [6] there exists a compact operator $K \in L(X)$ such that the spectrum of $S \equiv T+K$ is contained in $\{\lambda \in C \,|\, |\lambda| < r_e(T)+\varepsilon/2\}$. Denote by $r(S)$ the spectral radius of S, and choose N such that $n \geqq N$ implies $\|S^n\|^{1/n} < r(S)+\varepsilon/2$. Define the norm $\|\cdot\|_\varepsilon$ on X by

$$\|x\|_\varepsilon = \|x\| + \frac{\|S x\|}{r(S)+\varepsilon/2} + \frac{\|S^2 x\|}{(r(S)+\varepsilon/2)^2} + \cdots + \frac{\|S^{N-1} x\|}{(r(S)+\varepsilon/2)^{N-1}}, \qquad x \in X.$$

Then $\|\cdot\|_\varepsilon$ is equivalent to $\|\cdot\|$ and

$$\|S\|_\varepsilon \equiv \sup_{\|x\|_\varepsilon = 1} \|S x\|_\varepsilon \leqq r(S)+\varepsilon/2.$$

Since K is compact, we have $\gamma_{\|\cdot\|_\varepsilon}(T) = \gamma_{\|\cdot\|_\varepsilon}(T+K)$ (see, e.g., [5]), so that

$$\gamma_{\|\cdot\|_\varepsilon}(T) = \gamma_{\|\cdot\|_\varepsilon}(S) \leqq \|S\|_\varepsilon \leqq r(S)+\varepsilon/2 < r_e(T)+\varepsilon. \qquad \text{Q.E.D.}$$

We note that by (2.1) and Proposition 1 the inequality $r_e(T) \leqq \gamma_{\|\cdot\|_\varepsilon}(T)$ always holds.

Remark 1. It is now easy to show that the measure of noncompactness of a function can vary with a change to a different, equivalent norm. For let T be any quasinilpotent, noncompact operator on an infinite dimensional complex Banach space $(X, \|\cdot\|)$. Then $\gamma(T) > 0$, and for a suitable constant α, $\gamma(\alpha T) = \alpha \gamma(T) > 1$. But since $r_e(\alpha T) = 0$, there exists a norm $\|\cdot\|_1$ on X, equivalent to $\|\cdot\|$, such that $\gamma_{\|\cdot\|_1}(\alpha T) < 1$.

As a corollary to Theorem 1 we obtain a new proof of a fixed point theorem of Nussbaum [5].

Corollary. *Suppose $T \in L(X)$ with $r_e(T) < 1$. Let C be a closed, bounded, convex set in X and $f\colon C \to X$ a compact (not necessarily linear) map. Assume that $T+f\colon C \to C$. Then $T+f$ has a fixed point.*

Proof. Choose an equivalent norm $\|\cdot\|_1$ on X such that $\gamma_{\|\cdot\|_1}(T) < 1$. Since $\gamma_{\|\cdot\|_1}(T+f) = \gamma_{\|\cdot\|_1}(T)$, the assertion follows from Darbo's fixed point theorem on strict set-contractions (see, e.g., [5]).

Theorem 2. *Let $(X, \|\cdot\|)$ be a complex Banach space and suppose that $T \in L(X)$ is condensing. Then T is a topological strict set-contraction.*

Proof. It was proved in [6] that if $|\lambda| > \lim_{n\to\infty} (\gamma(T^n))^{1/n}$ then $\lambda - T$ is a Fredholm operator of index zero. Denote by $(\Phi_T)_0$ the component of Φ_T containing the set $\{\lambda \in \mathbb{C} \,|\, |\lambda| > \lim_{n\to\infty} (\gamma(T^n))^{1/n}\}$. Since every condensing map is a 1-set-contraction, it follows that $1 \geqq \gamma(T) \geqq \lim_{n\to\infty} (\gamma(T^n))^{1/n}$. Thus $(\Phi_T)_0$ contains the set $\{\lambda \in \mathbb{C} \,|\, |\lambda| > 1\}$. Using the fact that λT is condensing for a fixed $\lambda \in \mathbb{C}$ of modulus one, one can show by standard arguments (see, e.g., [7, 8]) that $I - \lambda T$ has finite dimensional nullspace and closed range. We conclude that the connected set $\{\lambda\in\mathbb{C}\,|\,|\lambda|=1\}$ is contained in some component $(\tilde{\Phi}_T)_\omega$ of $\tilde{\Phi}_T$. But since $(\tilde{\Phi}_T)_\omega$ is open it must contain points of $\{\lambda\in\mathbb{C}\,|\,|\lambda|>1\}$ and hence points of $(\Phi_T)_0$. Therefore $(\tilde{\Phi}_T)_\omega=(\Phi_T)_0$. Thus $(\Phi_T)_0$ contains the set $\{\lambda\in\mathbb{C}\,|\,|\lambda|\geqq 1\}$, and since $(\Phi_T)_0$ contains points of the resolvent set of T, the essential spectrum of T is contained in the complement of $(\Phi_T)_0$ and hence in the set $\{\lambda\in\mathbb{C}\,|\,|\lambda|<1\}$. This implies $r_e(T)<1$, so that by Theorem 1, T is a topological strict set-contraction. Q.E.D.

The following proposition improves the "Fredholm Alternative" of Petryshyn (Theorem 10, [7]).

Proposition 2. *Let $T\in L(X)$ be condensing. Then there exists $r>1$ such that $I-\lambda T$ is Fredholm of index zero whenever $|\lambda|<r$.*

Proof. In proving Theorem 2 we showed that $r_e(T)<1$. If

$$\lim_{n\to\infty}(\gamma(T^n))^{1/n}=r_e(T)=0$$

then $\lambda^{-1}-T=\lambda^{-1}(I-\lambda T)$ is Fredholm of index zero for each $\lambda\neq 0$ (see [6]). If $r_e(T)>0$, then for each $\lambda\neq 0$ with $|\lambda|<(r_e(T))^{-1}$, $r_e(\lambda T)=|\lambda|\,r_e(T)<1$, so that $\lambda^{-1}-T=\lambda^{-1}(I-\lambda T)$ is Fredholm of index zero. The assertion follows immediately.

3.

Let ρ: $[0,1]\to\mathbb{R}$ be a strictly decreasing nonnegative continuous function such that $\rho(0)=1$. Let B denote the unit ball in an infinite dimensional Banach space $(X,\|\cdot\|)$ and define a map f: $B\to B$ by $f(x)=\rho(\|x\|)\,x$. It was shown by Nussbaum in [5] that f is a condensing map but is not a strict set-contraction. We show here that in fact, f is not a topological strict set-contraction.

Denote by B_r the closed ball about 0 of radius r. It is easy to see that $f(B_r)\supset B_{\rho(r)r}$, $0<r<1$. Now let N be a fixed positive integer, and let $\varepsilon>0$ be given. Choose r_1, $0<r_1<1$, small enough so that $\rho(r_1)>(1-\varepsilon)^{1/N}$. Setting $r_n=\rho(r_{n-1})\,r_{n-1}$, $2\leqq n\leqq N+1$, we obtain the inclusions

$$\begin{aligned} &f(B_{r_1})\supset B_{r_2},\\ &f^2(B_{r_1})\supset f(B_{r_2})\supset B_{r_3},\\ &f^N(B_{r_1})\supset f^{N-1}(B_{r_2})\supset\cdots\supset B_{r_{N+1}}.\end{aligned}\tag{3.1}$$

Furthermore, we have

$$\begin{aligned} r_{N+1}&=\rho(r_N)\,r_N=\rho(r_N)\,\rho(r_{N-1})\,r_{N-1}\\ &>(\rho(r_{N-1}))^2\,r_{N-1}=(\rho(r_{N-1}))^2\,\rho(r_{N-2})\,r_{N-2}\\ &>(\rho(r_{N-2}))^3\,r_{N-2}=\cdots>(\rho(r_1))^N\,r_1>(1-\varepsilon)\,r_1.\end{aligned}\tag{3.2}$$

According to Proposition 5 in [5], for each $r \geqq 0$, $\gamma(B_r)=2r$. Thus from (3.1) and (3.2) we obtain the inequality

$$\gamma(f^N(B_{r_1})) \geqq \gamma(B_{r_{N+1}}) = 2r_{N+1} > 2(1-\varepsilon)\, r_1$$
$$= (1-\varepsilon)\,\gamma(B_{r_1}).$$

Since ε was arbitrary, we conclude that $1 \leqq \gamma(f^N) \leqq (\gamma(f))^N = 1$, that is, $\gamma(f^N)=1$. Therefore $\lim_{n\to\infty} (\gamma(f^n))^{1/n} = 1$, so that by Proposition 1, f cannot be a topological strict set-contraction.

Remark 2. It is easily seen that results corresponding to Proposition 1 and Theorem 1 hold for the widely used ball measure of noncompactness, $\tilde{\gamma}$ (see [6] for definitions). However, it follows immediately from Lemma 1 in [6] that every linear ball-condensing map is already a strict ball-set-contraction, so that the result for $\tilde{\gamma}$ corresponding to Theorem 2 is trivial.

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(Received March 5, 1973)