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Demi-Semi-Primal Algebras and Mal'cev-Type Conditions

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1. Introduction

In this paper we study two generalizations of primal algebras: demi-primal and demi-semi-primal algebras. Intuitively speaking, a demi-primal algebra has as many polynomials as possible consistent with its automorphisms while a demi-semi-primal algebra has as many polynomials as possible consistent with its automorphisms and subalgebras (the meanings of "polynomial" and other undefined terms in this paper will be that of [8]). Structure and characterization theorems are given for both classes of algebras and demi-semi-primals are compared to the more general quasi-primal algebras of Pixley [14, 15].

Typical examples of Mal'cev-type conditions are that of Mal'cev [12] for the permutability of congruences in equational classes and that of Jónsson [11] for the distributivity of congruence lattices in equational classes. We show that all presently known Mal'cev-type conditions hold in quasi-primal algebras. It is conjectured that every Mal'cev-type condition concerning congruences holds in every quasi-primal algebra. One of these Mal'cev-type conditions is regularity (see Grätzer [9]). We give a simple example of a demi-semi-primal algebra which does not satisfy the strongest set of polynomial identities for regularity but does satisfy the next strongest. Finally, we consider injective and projective algebras in equational classes generated by quasi-primal algebras.

2. Mal'cev-Type Conditions

A Mal'cev-type condition (P) is one for which there is for each positive integer n a set of functional identites Σ_n such that Σ_{n+1} is derivable from Σ_n and (P) holds in an equational class $\mathscr{K}$ iff there is an n such that Σ_n is satisfied by some polynomials of $\mathscr{K}$.

Note that the Σ_n need not be distinct. The first Mal'cev-type condition (due, of course, to Mal'cev) is an example of this. Let $\mathscr{K}$ be an equational class of algebras. By $\Pi(\mathscr{K})$ is meant the condition that every algebra in $\mathscr{K}$ has permutable congruences (if $\mathscr{K}$ is generated by a single algebra $\mathfrak{A}$ we will write $\Pi(\mathfrak{A})$ in place of $\Pi(\mathscr{K})$). Mal'cev proved [12]:

Theorem 2.1. *$\Pi(\mathscr{K})$ iff there is a polynomial $p(x, y, z)$ of $\mathscr{K}$ such that*

$$p(x, x, z)=p(z, x, x)=z. \tag{1}$$

By $\Delta(\mathscr{K})$ is meant that every algebra in $\mathscr{K}$ has a distributive congruence lattice. Jónsson [11] has found a Mal'cev-type condition for $\Delta(\mathscr{K})$. However, we will only need the following result of Pixley [13]:

Theorem 2.2. a) *If there is a polynomial $m(x, y, z)$ of $\mathscr{K}$ such that*

$$m(x, x, y)=m(x, y, x)=m(y, x, x)=x \quad \text{then } \Delta(\mathscr{K}). \tag{2}$$

b) *$\Delta(\mathscr{K})$ and $\Pi(\mathscr{K})$ iff there are polynomials $p(x, y, z)$ and $m(x, y, z)$ of $\mathscr{K}$ such that* (1) *and* (2) *hold.*

Corollary 2.3. *$\Delta(\mathscr{K})$ and $\Pi(\mathscr{K})$ iff there is a polynomial $q(x, y, z)$ of $\mathscr{K}$ such that*

$$q(x, x, z)=q(z, x, x)=q(z, x, z)=z. \tag{3}$$

Proof. We need only show (3) is equivalent to (1) and (2). If (3) holds let $p(x, y, z)=q(x, y, z)$ and $m(x, y, z)=q(x, q(x, y, z), z)$; then (1) and (2) hold. If (1) and (2) hold let $q(x, y, z)=p(x, m(x, y, z), z)$; (3) then holds.

We will call an algebra $\mathfrak{A}$ regular if whenever two congruences θ, φ of $\mathfrak{A}$ have a common congruence class then $\theta=\varphi$. By $R(\mathscr{K})$ is meant that every algebra in $\mathscr{K}$ is regular. Grätzer [9] has shown that $R(\mathscr{K})$ is a Mal'cev-type condition. The statement of this condition will be defered until Section 4. There are other Mal'cev-type conditions which we will only mention. There is Day's condition [1] for modularity of congruence lattices and the condition of Fraser and Horn [7] for the congruence lattice of a direct product to be the direct product of the congruence lattices of the factors. Both of these conditions are implied by $\Delta(\mathfrak{A})$. Grätzer [9] also considers weak regularity which, naturally, is weaker than regularity.

3. Demi-Semi-Primal Algebras

Recall that a primal algebra is a finite non-trivial algebra $\mathfrak{A}$ such that every function $f\colon A^n \to A$ is a polynomial of $\mathfrak{A}$. The following theorem of Foster and Pixley [6] gives a very nice characterization of primal algebras:

Theorem 3.1. *A finite non-trivial algebra $\mathfrak{A}$ is primal iff*

a) *$\mathfrak{A}$ is simple,*

b) *$\mathfrak{A}$ has no proper automorphisms,*

c) *$\mathfrak{A}$ has no proper subalgebras,*

d) *$\Delta(\mathfrak{A})$ and $\Pi(\mathfrak{A})$.*

An obvious generalization due to Foster and Pixley [5] is to permit $\mathfrak{A}$ to have subalgebras. We say $f\colon A^n \to A$ preserves $\mathscr{S}(\mathfrak{A})$ (the subalgebra lattice

of $\mathfrak{A}$) if for any $a_1, \ldots, a_n \in A$ we have $f(a_1, \ldots, a_n) \in [a_1, \ldots, a_n]$ (the subalgebra of $\mathfrak{A}$ generated by $a_1, \ldots, a_n$). Thus a finite non-trivial algebra $\mathfrak{A}$ is semi-primal if every function $f\colon A^n \to A$ which preserves $\mathscr{S}(\mathfrak{A})$ is a polynomial of $\mathfrak{A}$. The following characterization theorem is also due to Foster and Pixley [6]:

Theorem 3.2. *A finite non-trivial algebra $\mathfrak{A}$ is semi-primal iff*

a) *every subalgebra of $\mathfrak{A}$ is simple,*

b) *every subalgebra of $\mathfrak{A}$ has no proper automorphisms,*

c) *distinct non-trivial subalgebras are non-isomorphic,*

d) *$\Delta(\mathfrak{A})$ and $\Pi(\mathfrak{A})$.*

We shall also need the following lemma of Foster [3]:

Lemma 3.3. *If $\mathfrak{A}$ is semi-primal and $\mathfrak{A}_1, \mathfrak{A}_2$ are distinct non-trivial subalgebras then $\mathfrak{A}_1$ is semi-primal and $\mathfrak{A}_1 \not\cong \mathfrak{A}_2$.*

An obvious generalization of semi-primal algebras is to permit $\mathfrak{A}$ to have automorphisms also. Let $\mathscr{G}(\mathfrak{A})$ be the group of automorphisms of $\mathfrak{A}$; we say $f\colon A^n \to A$ preserves $\mathscr{G}(\mathfrak{A})$ if for every $a_1, \ldots, a_n \in A$ and every $\rho \in \mathscr{G}(\mathfrak{A})$ we have $\rho(f(a_1, \ldots, a_n)) = f(\rho(a_1), \ldots, \rho(a_n))$. Thus we define a finite non-trivial algebra to be demi-semi-primal if every function on A which preserves both $\mathscr{G}(\mathfrak{A})$ and $\mathscr{S}(\mathfrak{A})$ is a polynomial of $\mathfrak{A}$. Demi-semi-primal algebras are a generalization of what Foster calls infraprimal algebras [4]; as we shall see, the quasi-primal algebras of Pixley ([14, 15]) are more general than demi-semi-primal algebras. (In [14] quasi-primal algebras are called simple algebraic algebras.)

Given a demi-semi-primal algebra $\mathfrak{A}$ how many n-ary polynomials does it have? This is equivalent to asking what is the size of $\mathscr{F}_{\mathfrak{A}}(n)$, the free algebra on n generators over $\mathfrak{A}$. To answer this question let $\mathfrak{A}_1, \ldots, \mathfrak{A}_m$ be all non-trivial pairwise nonisomorphic subalgebras of $\mathfrak{A}$, and define the relation E_n on A^n by: $(a_1, \ldots, a_n) E_n (b_1, \ldots, b_n)$ iff there is a $\rho \in \mathscr{G}(\mathfrak{A})$ such that $\rho(a_i) = b_i$ for $i = 1, \ldots, n$. Clearly E_n is an equivalence relation. If $(a_1, \ldots, a_n) E_n (b_1, \ldots, b_n)$ then $[a_1, \ldots, a_n] \simeq [b_1, \ldots, b_n]$ and once an n-ary polynomial p is determined on $(a_1, \ldots, a_n)$ its value on $(b_1, \ldots, b_n)$ is determined; clearly $p(a_1, \ldots, a_n) \in [a_1, \ldots, a_n]$. But these are the only restrictions on p. Let $\alpha_{n,i}$ be the number of E_n equivalence classes generating subalgebras isomorphic to $\mathfrak{A}_i$, $i = 1, \ldots, m$.

Lemma 3.4. *If $\mathfrak{A}$ is demi-semi-primal, then with notation as above, $|\mathscr{F}_{\mathfrak{A}}(n)| = |\mathfrak{A}_1|^{\alpha_{n,1}} \times \cdots \times |\mathfrak{A}_m|^{\alpha_{n,m}}$.*

We will use this lemma to establish a characterization theorem for demi-semi-primal algebras. Let $\mathscr{I}(\mathfrak{A})$ be all isomorphisms between (not necessarily distinct) non-trivial subalgebras. Define the relation F_n on A^n by: $(a_1, \ldots, a_n) F_n (b_1, \ldots, b_n)$ iff there is a $\rho \in \mathscr{I}(\mathfrak{A})$ such that $\rho(a_i) = b_i$ for $i = 1, \ldots, n$; F_n is an equivalence relation. Clearly $E_n \subseteq F_n$ for $n \geqq 1$; the condition that $E_n = F_n$ for $n \geqq 1$ means that each $\rho \in \mathscr{I}(\mathfrak{A})$ can be extended to a $\rho' \in \mathscr{G}(\mathfrak{A})$.

Lemma 3.5. *If $\mathfrak{A}$ is demi-semi-primal then*

a) *every subalgebra of $\mathfrak{A}$ is simple,*

b) *$\Delta(\mathfrak{A})$ and $\Pi(\mathfrak{A})$,*

c) *$E_n = F_n$ for $n \geqq 1$.*

Proof. Define $q(x, y, z)$ by

$$q(x, y, z) = \begin{cases} z & \text{if } x = y, \\ x & \text{if } x \neq y. \end{cases} \tag{4}$$

This function is the ternary discriminator function of [15]. Clearly $q(x, y, z)$ preserves both $\mathscr{G}(\mathfrak{A})$ and $\mathscr{S}(\mathfrak{A})$ and so is a polynomial of $\mathfrak{A}$. Since $q(x, y, z)$ satisfies (3), b) holds. To see that $\mathfrak{A}$ (or any subalgebra) is simple, let $a \equiv b(\theta)$ with $a \neq b$. Then $c = q(a, a, c) \equiv q(a, b, c) = a$; hence a) holds.

To prove c) suppose $E_n \neq F_n$ for some n. Then some $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ are equivalent mod F_n but not mod E_n. Since $\mathscr{I}(\mathfrak{A})$ consists of isomorphisms between non-trivial subalgebras of $\mathfrak{A}$, we may assume that $n > 1$ and $a_i \neq a_j$ for some i, j. Then $b_i \neq b_j$ since there is an isomorphism $\varphi \in \mathscr{I}(\mathfrak{A})$ with $\varphi(a_k) = b_k$ for $k = 1, \ldots, n$. Define $f\colon A^n \to A$ to be equal its i-th argument except on the E_n class containing $(b_1, \ldots, b_n)$ where it is equal to its j-th argument. Thus f is a polynomial of $\mathfrak{A}$. But the isomorphism φ does not preserve f: $\varphi(f(a_1, \ldots, a_n)) = \varphi(a_i) = b_i \neq b_j = f(b_1, \ldots, b_n) = f(\varphi(a_1), \ldots, \varphi(a_n))$. Thus $E_n = F_n$ for $n \geqq 1$ so that c) holds.

Next we use a theorem of Jónsson [11] to obtain a structure theorem for equational classes generated by demi-semi-primal algebras. Given an algebra $\mathfrak{A}$. $\mathscr{K}(\mathfrak{A})$ denotes the equational class it generates.

Theorem 3.6. *Let $\mathfrak{A}$ be a finite algebra such that $\Delta(\mathfrak{A})$ holds. Then $\mathscr{K}(\mathfrak{A}) = IP_S HS(\mathfrak{A})$.*

Theorem 3.7. *If $\mathfrak{A}$ is demi-semi-primal then $\mathscr{K}(\mathfrak{A}) = IP_S S(\mathfrak{A}) = ISP(\mathfrak{A})$; if $\mathfrak{B} \in \mathscr{K}(\mathfrak{A})$ is finite then $\mathfrak{B} \in IPS(\mathfrak{A})$.*

Proof. Since $HS(\mathfrak{A}) = IS(\mathfrak{A})$ the first part follows from Theorem 3.6. It is easily shown that a finite subdirect product of simple algebras is actually a direct product when $\Pi(\mathfrak{A})$ holds (see [6]).

Notice that we needed only a) and b) of Lemma 3.5 to prove Theorem 3.7. We will now show that a)–c) of Lemma 3.5 characterize demi-semi-primal algebras. Let $\mathfrak{A}$ be a finite algebra satisfying a) and b) of Lemma 3.5. Then $\mathscr{F}_{\mathfrak{A}}(n)$ is finite and hence a direct product of subalgebras of $\mathfrak{A}$.

Let $\mathfrak{A}_1, \ldots, \mathfrak{A}_m$ be all non-trivial pairwise nonisomorphic subalgebras of $\mathfrak{A}$. Thus $\mathscr{F}_{\mathfrak{A}}(n) \simeq \mathfrak{A}_1^{\beta_{n,1}} \times \cdots \times \mathfrak{A}_n^{\beta_{n,m}}$. Let $\mathscr{C}(\mathscr{F}_{\mathfrak{A}}(n))$ be the congruence lattice of $\mathscr{F}_{\mathfrak{A}}(n)$. Since $\Delta(\mathfrak{A})$ holds, $\mathscr{C}(\mathscr{F}_{\mathfrak{A}}(n))$ is Boolean with $\beta_{n,1} + \cdots + \beta_{n,m}$ dual atoms (see [5]). Each dual atom is the kernel of the projection of $\mathscr{F}_{\mathfrak{A}}(n)$ onto one of its direct factors. Let $x_1, \ldots, x_n$ be the free generators of $\mathscr{F}_{\mathfrak{A}}(n)$; then each dual atom is induced by some map $x_i \to a_i$ for $a_i \in A$, $i = 1, \ldots, n$. Let $\theta(a_1, \ldots, a_n)$ be the dual atom induced by the map $x_i \to a_i$, $i = 1, \ldots, n$. Let $|[a_1, \ldots, a_n]| > 1$.

Lemma 3.8. $\theta(a_1, \ldots, a_n)=\theta(b_1, \ldots, b_n)$ *iff* $(a_1, \ldots, a_n) F_n(b_1, \ldots, b_n)$.

Proof. Clearly $p \equiv q(\theta(a_1, \ldots, a_n))$ iff $p(a_1, \ldots, a_n)=q(a_1, \ldots, a_n)$. Thus $\theta(a_1, \ldots, a_n)=\theta(b_1, \ldots, b_n)$ iff $[p(a_1, \ldots, a_n)=q(a_1, \ldots, a_n)$ iff $p(b_1, \ldots, b_n)=q(b_1, \ldots, b_n)]$. Define φ: $[a_1, \ldots, a_n] \to [b_1, \ldots, b_n]$ by $\varphi(a)=b$ iff $a=p(a_1, \ldots, a_n)$ and $b=p(b_1, \ldots, b_n)$ for some $p \in \mathscr{F}_{\mathfrak{A}}(n)$. φ is an isomorphism iff it is well defined and 1-1, i.e. $\varphi \in \mathscr{I}(\mathfrak{A})$ iff $[p(a_1, \ldots, a_n)=q(a_1, \ldots, a_n)$ iff $p(b_1, \ldots, b_n)=q(b_1, \ldots, b_n)]$. (We may ignore the case when $|[a_1, \ldots, a_n]|=1$.) Since $\varphi(a_i)=b_i$ for $i=1, \ldots, m$ the lemma proved.

Thus $\beta_{n,i}$ is the number of F_n equivalence classes generating subalgebras isomorphic to $\mathfrak{A}_i$.

Theorem 3.9. *A finite non-trivial algebra* $\mathfrak{A}$ *is demi-semi-primal iff* a)–c) *of Lemma* 3.5 *hold.*

Proof. Lemma 3.5 says a)–c) are necessary. Suppose a)–c) hold. Then by Lemma 3.7. $\mathscr{F}_{\mathfrak{A}}(n) \simeq \mathfrak{A}_1^{\beta_{n,1}} \times \cdots \times \mathfrak{A}_m^{\beta_{n,m}}$. By Lemma 3.4 $\mathfrak{A}$ will be demi-semi-primal if $|\mathscr{F}_{\mathfrak{A}}(n)|=|\mathfrak{A}_1^{\alpha_{n,1}}| \times \cdots \times |\mathfrak{A}_m|^{\alpha_{n,m}}$. Hence we are done if $\beta_{n,i}=\alpha_{n,i}$. But this is what condition c) guarantees.

Define $\mathfrak{A}$ to be demi-primal if $\mathfrak{A}$ is demi-semi-primal and has no proper subalgebras. A necessary and sufficient condition for a group of permutations $\mathscr{G}$ on a finite non-trivial set A to be the automorphism group of a demi-primal algebra based on A is that if $\varphi(a)=a$ for some $\varphi \in \mathscr{G}$ and some $a \in A$ then φ is the identity map on A. Characterization and structure theorems for demi-primal algebras are obvious and are omitted.

We say f: $A^n \to A$ preserves $\mathscr{I}(\mathfrak{A})$ if $\varphi(f(a_1, \ldots, a_n))=f(\varphi(a_1), \ldots, \varphi(a_n))$ for all $\varphi \in \mathscr{I}(\mathfrak{A})$ and all $a_1, \ldots, a_n \in A$. A finite non-trivial algebra $\mathfrak{A}$ is quasi-primal if every function which preserves $\mathscr{S}(\mathfrak{A})$ and $\mathscr{I}(\mathfrak{A})$ is a polynomial of $\mathfrak{A}$. This definition is due to Pixley [14] who then proves the following theorem:

Theorem 3.10. *A finite non-trivial algebra* $\mathfrak{A}$ *is quasi-primal iff* a) *and* b) *of Lemma* 3.5 *hold.*

Thus a quasi-primal algebra is demi-semi-primal iff $E_n=F_n$ for $n \geqq 1$. This condition is a surprisingly strong one. If we expand the set of operations of quasi-primal $\mathfrak{A}$ to get a new algebra $\mathfrak{A}'$ then $\mathfrak{A}'$ is quasi-primal; however, if $\mathfrak{A}$ is demi-semi-primal then $\mathfrak{A}'$ need not be.

Theorem 3.11. *Let* $\mathfrak{A}$ *be a demi-semi-primal algebra which has a proper automorphism* φ *and a proper non-trivial subalgebra* $\mathfrak{B}$ *such that* $\varphi|_B$ *is not the identity map on B. Then by adding a finite number of new operations to* $\mathfrak{A}$ *we can obtain an algebra* $\mathfrak{A}'$ *with no proper automorphisms but which is not semi-primal and hence not demi-semi-primal.*

Proof. Let $|A|=n$. For each proper automorphism $\rho \in \mathscr{G}(\mathfrak{A})$ there is an $a_\rho \in A$ such that $\rho(a_\rho)=b_\rho \neq a_\rho$. Define

$$f_\rho(a_1, \ldots, a_n)=\begin{cases} a_1 & \text{if } a_i=a_j \quad \text{for some } i \neq j, \text{ else} \\ a_\rho & \text{if } a_1=b_\rho, \\ a_1 & \text{otherwise.} \end{cases}$$

Note that f_ρ preserves $\mathscr{S}(\mathfrak{A})$ but does not preserve ρ. If $\mathfrak{A}=\langle A; F\rangle$ let $\mathfrak{A}'=\langle A; F\cup\{f_\rho|\rho\in\mathscr{G}(\mathfrak{A}) \text{ is proper}\}\rangle$. Thus $\mathfrak{A}'$ has no proper automorphisms, but $\mathscr{S}(\mathfrak{A}')=\mathscr{S}(\mathfrak{A})$. Look at the subalgebra $\mathfrak{B}$ of $\mathfrak{A}'$. The f_ρ are all first projections on $\mathfrak{B}$ so $\varphi|_B$ is either a proper automorphism of $\mathfrak{B}$ or an isomorphism from $\mathfrak{B}$ to another subalgebra of $\mathfrak{A}'$. In either case Lemma 3.3 shows that $\mathfrak{A}'$ is not semi-primal.

Example: $\mathfrak{B}$ is a topological Boolean algebra if it is a Boolean algebra with an added unary operation I such that $I(a)\leqq a$, $I(a)\wedge I(b)=I(a\wedge b)$, $I(I(a))=I(a)$, and $I(1)=1$. It is easy to show that a topological Boolean algebra is simple iff $I(a)=0$ for $a\neq 1$. This remark is essentially due to Halmos [10] (Halmos' simple monadic algebras are dual to simple topological Boolean algebras). Let $\mathfrak{B}$ be a finite non-trivial simple topological Boolean algebra. Since $\mathfrak{B}$ is a Boolean algebra, $\Delta(\mathfrak{B})$ and $\Pi(\mathfrak{B})$; clearly every subalgebra $\mathfrak{B}$ is simple. Thus $\mathfrak{B}$ is quasi-primal. Note that the automorphisms of $\mathfrak{B}$ correspond to permutations of the atoms of $\mathfrak{B}$. Suppose $\mathfrak{B}$ has $n\geqq 3$ atoms $a_1,\ldots,a_n$. Then $\{0, a_1, a_2\vee\cdots\vee a_n, 1\}$ is a proper subalgebra of $\mathfrak{B}$ with an automorphism φ which interchanges a_1 and $a_2\vee\cdots\vee a_n$. But φ is induced by no automorphism of $\mathfrak{B}$ so that $\mathfrak{B}$ is not demi-semi-primal. On the other hand, if $\mathfrak{B}$ has exactly two atoms then $\mathfrak{B}$ is demi-semi-primal. Hence the only simple topological Boolean algebras which are demi-semi-primal are the 2 and 4 element ones.

4. Mal'cev-Type Conditions Continued

Theorem 3.10 shows that every quasi-primal algebra has $\Delta(\mathfrak{A})$ and $\Pi(\mathfrak{A})$. We will show that every quasi-primal algebra has the regularity property $R(\mathfrak{A})$.

Let p_i^j for $0\leqq i<n$, $0\leqq j\leqq n$, $r_k^{i,j}$ for $0\leqq i, j<n$, $1\leqq k<n$, q_i for $0\leqq i\leqq n$, t_j^i for $0\leqq i<n$, $1\leqq j<n$ be ternary polynomial symbols such that $p_i^0(x_0,x_1,x_2)=x_2$ for $0\leqq i<n$, $q_0(x_0,x_1,x_2)=x_0$, and $q_n(x_0,x_1,x_2)=x_1$. Let r_i^j for $0\leqq i, j<n$ and t_i for $0\leqq i<n$ be n-ary polynomial symbols. Let Σ_n be the following set of identites:

$$r_i^j(x_0, r_1^{i,j},\ldots,r_{n-1}^{i,j})=\begin{cases}p_i^j & j \text{ even}\\ p_i^{j+1} & j \text{ odd}\end{cases}\qquad 0\leqq i, j<n, \tag{5.1}$$

$$r_i^j(x_1, r_1^{i,j},\ldots,r_{n-1}^{i,j})=\begin{cases}p_i^{j+1} & j \text{ even}\\ p_i^j & j \text{ odd}\end{cases}\qquad 0\leqq i, j<n, \tag{5.2}$$

$$t_i(x_2, t_1^i,\ldots,t_{n-1}^i)=\begin{cases}q_i & i \text{ even}\\ q_{i+1} & i \text{ odd}\end{cases}\qquad 0\leqq i<n, \tag{5.3}$$

$$t_i(p_i^n, t_1^i,\ldots,t_{n-1}^i)=\begin{cases}q_{i+1} & i \text{ even}\\ q_i & i \text{ odd}\end{cases}\qquad 0\leqq i<n. \tag{5.4}$$

Theorem 4.1 (Grätzer [9]). *$R(\mathscr{K})$ iff there is a natural number n and polynomials of $\mathscr{K}$ such that Σ_n holds.*

We shall prove $R(\mathfrak{A})$ for quasi-primal algebras in two ways, first algebraically and then by exhibiting the appropriate polynomials. We will need two lemmas for this; the first is from Grätzer [9].

Lemma 4.2. *$R(\mathscr{K})$ iff $\mathscr{F}_{\mathscr{K}}(3)$ is regular.*

Lemma 4.3. *If there is a ternary polynomial $s(x, y, z)$ of $\mathscr{K}$ such that*

$$s(x,y,z)=\begin{cases} z & \text{if } x=y,\\ y & \text{if } x=z,\\ x & \text{otherwise,}\end{cases} \qquad \text{then } \Sigma_3 \text{ holds and hence } R(\mathscr{K}). \tag{6}$$

Proof. Take $r_i^j(x, y, z)=t_i(x, y, z)=s(x, y, z)$, $r_1^{i,j}(x, y, z)=t_1^i(x, y, z)=x$, and $r_2^{i,j}(x, y, z)=t_2^i(x, y, z)=z$.

Theorem 4.4. *If $\mathfrak{A}$ is quasi-primal then $R(\mathfrak{A})$.*

Proof 1. $\mathscr{F}_{\mathfrak{A}}(3)$ is the direct product of, say, n copies of non-trivial subalgebras of $\mathfrak{A}$ so that the congruence lattice of $\mathscr{F}_{\mathfrak{A}}(3)$ is Boolean with n atoms. Thus $\mathscr{F}_{\mathfrak{A}}(3)$ is easily seen to be regular; hence by Lemma 4.2, $R(\mathfrak{A})$.

Proof 2. Clearly $s(x, y, z)$ as defined by (6) preserves both $\mathscr{S}(\mathfrak{A})$ and $\mathscr{I}(\mathfrak{A})$ and so is a polynomial of $\mathfrak{A}$. Then by Lemma 4.3, $R(\mathfrak{A})$.

An important but difficult problem about Mal'cev-type conditions is whether or not Σ_{n+1} is strictly weaker than Σ_n. An understanding of the way in which Σ_n was found for $\Delta(\mathscr{K})$ and $R(\mathscr{K})$ indicates that this should be the case, but actual models where Σ_{n+1} holds but Σ_n doesn't are all but unknown. We will now give an example which shows that for regularity Σ_3 is strictly weaker than Σ_2.

Let $A=\{0, 1\}$ and let $q(x, y, z)$ be defined by (4). Then $\mathfrak{A}=\langle A; q\rangle$ is quasi-primal and in fact is demi-semi-primal (any two element quasi-primal algebra is demi-semi-primal). By Theorem 4.4 Σ_3 is satisfied. Note that the only binary polynomials of $\mathfrak{A}$ are $p_1(x, y)=x$ and $p_2(x, y)=y$. In part Σ_2 requires:

a) $t_0(x_2, t_1^0(x_0, x_1, x_2))=x_0$,

b) $t_0(p_0^2(x_0, x_1, x_2), t_1^0(x_0, x_1, x_2))=t_1(p_1^2(x_0, x_1, x_2), t_1^1(x_0, x_1, x_2))$,

c) $t(x_2, t_1^1(x_0, x_1, x_2))=x_1$.

Thus a) requires $t_0(x_0, x_1)=x_1$ and $t_1^0(x_0, x_1, x_2)=x_0$ while c) requires $t_1(x_0, x_1)=x_1$ and $t_1^1(x_0, x_1, x_2)=x_1$. But then b) yields $x_0=x_1$. Hence Σ_2 cannot hold in $\mathfrak{A}$.

Theorem 4.5. *For $R(\mathscr{K})$ Σ_3 does not imply Σ_2.*

We have seen that six different Mal'cev-type conditions hold in quasi-primal algebras. These six conditions are of two distinct kinds. The first kind is of the form: $p\leqq q$ holds in the congruence lattice of every algebra in the equational class $\mathscr{K}$. Here p and q are polynomials in $\wedge$, $\vee$ and $\circ$ (congruence meet, join, and product). The conditions for permutability, distributivity, and modularity are of this form. Wille [17] has an algorithm for reducing $p\leqq q$

to a doubly indexed family of identities (see also Pixley [16]). In the case that $\Pi(\mathscr{K})$ holds, this doubly indexed family reduces to a genuine Mal'cev condition.

Theorem 4.6. *If $\Pi(\mathscr{K})$ and $\Delta(\mathscr{K})$ hold and $p \leqq q$ holds in the two element lattice then the corresponding Mal'cev condition holds in $\mathscr{K}$.*

Proof. Note that $\circ = \vee$ in any lattice of equivalence relations which is a chain. Since $\Pi(\mathscr{K})$ holds we have a genuine Mal'cev condition. Thus $p \leqq q$ reduces to a lattice inequality. Since $p \leqq q$ holds in the two element lattice, it holds in all distributive lattices. Hence by $\Delta(\mathscr{K})$ the Mal'cev condition holds in $\mathscr{K}$.

The second kind of Mal'cev condition makes use of the following lemma [9]. Let $\mathfrak{A}$ be an algebra and $\theta(a, b)$ the smallest congruence on $\mathfrak{A}$ collapsing a to b.

Lemma 4.7. *$x \equiv y\ (\theta(a, b))$ iff there is an integer n, a sequence $x = z_0, z_1, \ldots, z_n = y$ of elements of A, and a sequence $p_0, \ldots, p_{n-1}$ of unary algebraic functions such that for $i = 0, 1, \ldots, p-1$,*

$$p_i(a) = \begin{cases} z_i & \text{for } i \text{ even}, \\ z_{i+1} & \text{for } i \text{ odd}; \end{cases} \qquad p_i(b) = \begin{cases} z_{i+1} & \text{for } i \text{ even}, \\ z_i & \text{for } i \text{ odd}. \end{cases}$$

Let $\mathscr{K}$ be an equational class of algebras, $\mathscr{F}_{\mathscr{K}}(2)$ ($\mathscr{F}_{\mathscr{K}}(3)$) the free algebra in $\mathscr{K}$ with free generating set $\{x_0, x_1\}$ (respectively $\{y_0, y_1, y_2\}$). Let $\mathscr{C}(\mathfrak{A})$ be the congruence lattice of $\mathfrak{A}$. The following lemma is the basis for [7].

Lemma 4.8. *The natural embedding of $\mathscr{C}(\mathfrak{A}_1) \times \mathscr{C}(\mathfrak{A}_2)$ into $\mathscr{C}(\mathfrak{A}_1 \times \mathfrak{A}_2)$ is an isomorphism for all $\mathfrak{A}_1, \mathfrak{A}_2 \in \mathscr{K}$ iff $(x_0, y_2) \equiv (x_1, y_2)(\theta((x_0, y_0), (x_1, y_1)))$ holds in $\mathscr{C}(\mathscr{F}_{\mathscr{K}}(2) \times \mathscr{F}_{\mathscr{K}}(3))$.*

It is straightforward to calculate the identities of the Mal'cev condition by using Lemma 4.7. It is by no means clear that all Mal'cev conditions of this kind hold in all quasi-primal algebras. However, it seems likely that they do.

5. Injectives and Projectives

Let $\mathscr{K}$ be a class of algebras, $\mathfrak{A} \in \mathscr{K}$. $\mathfrak{A}$ is *projective in* $\mathscr{K}$ if for any $\mathfrak{A}_1, \mathfrak{A}_2 \in \mathscr{K}$, any epimorphism $f\colon \mathfrak{A}_1 \to \mathfrak{A}_2$, and any homomorphism $h\colon \mathfrak{A} \to \mathfrak{A}_2$, there is a homomorphism $g\colon \mathfrak{A} \to \mathfrak{A}_1$ such that $h = f \circ g$. If $\mathscr{K}$ is an equational class, it is well known that the projective algebras in $\mathscr{K}$ are precisely the retracts of free algebras in $\mathscr{K}$ ($\mathfrak{A}$ is a *retract* of $\mathfrak{A}'$ if there are homomorphisms $f\colon \mathfrak{A} \to \mathfrak{A}'$, $g\colon \mathfrak{A}' \to \mathfrak{A}$ such that $g \circ f$ is the identity mapping on A).

Let $\mathfrak{A}$ be quasi-primal and $\mathscr{K}(\mathfrak{A})$ the equational class it generates. Let $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ be all minimal non-trivial subalgebras of $\mathfrak{A}$; there may be none. Clearly each $\mathfrak{A}_i$ is demi-primal.

Theorem 5.1. *If $\mathfrak{B}$ is projective in $\mathscr{K}(\mathfrak{A})$ then $\mathfrak{B}$ has $\mathfrak{A}_i$ as a homomorphic image for $i = 1, \ldots, n$.*

Proof. Let $\mathfrak{B}'$ be a simple subdirect factor of $\mathfrak{B}$ with $g\colon \mathfrak{B} \to \mathfrak{B}'$ the projection onto $\mathfrak{B}'$. Consider $\mathfrak{B}' \times \mathfrak{A}_i$; let Π_1 be the projection of $\mathfrak{B}' \times \mathfrak{A}_i$ onto $\mathfrak{B}'$

and Π_2 the projection onto $\mathfrak{A}_i$. Since $\mathfrak{B}$ is projective there is a homomorphism $f\colon \mathfrak{B} \to \mathfrak{B}' \times \mathfrak{A}_i$ such that $\Pi_1 \circ f = g$. Since $\mathfrak{A}_i$ has no proper subalgebras, $f(B) = B' \times A_i$ by Theorem 3.7 (we need only assume $\mathfrak{A}$ is quasi-primal to prove 3.7). Thus $\Pi_2 \circ f$ is a homomorphism from $\mathfrak{B}$ onto $\mathfrak{A}_i$.

Theorem 5.2. *If $\mathfrak{B} \in \mathscr{K}(\mathfrak{A})$ is finite then $\mathfrak{B}$ is projective iff $\mathfrak{B}$ has $\mathfrak{A}_i$ as a direct factor for $i = 1, \ldots, n$. If there is no minimal non-trivial $\mathfrak{A}_i$ then every finite algebra in $\mathscr{K}(\mathfrak{A})$ is projective.*

Proof. Since $\mathfrak{B}$ is a finite direct product of subalgebras of $\mathfrak{A}$ by Theorem 3.7, Theorem 5.1 shows that the condition is necessary. Let $\mathfrak{B}$ have each $\mathfrak{A}_i$ as a direct factor. Let $\mathscr{F}$ be a finite free algebra in $\mathscr{K}(\mathfrak{A})$ such that $\mathfrak{B}$ is a homomorphic image of $\mathscr{F}$ (since $\mathfrak{B}$ is finite such an $\mathscr{F}$ exists). Thus $\mathscr{F} \simeq \mathfrak{B} \times \mathfrak{B}'$. Let $\Pi\colon \mathscr{F} \to \mathfrak{B}$ be the projection onto $\mathfrak{B}$. Since each direct factor of $\mathfrak{B}'$ contains some $\mathfrak{A}_i$ or has a one element subalgebra, it is easy to construct an embedding $f\colon \mathfrak{B}' \to \mathscr{F}$ so that $\Pi \circ f$ is the identity on $\mathfrak{B}'$. Hence $\mathfrak{B}'$ is a retract of $\mathscr{F}$ and so is projective.

Question. If $\mathfrak{A}$ is primal then every countable algebra in $\mathscr{K}(\mathfrak{A})$ is projective. Is it true that if $\mathfrak{B}$ satisfies the conditions of Theorem 5.1 and is countable then $\mathfrak{B}$ is projective?

Let $\mathscr{K}$ be a class of algebras, $\mathfrak{A} \in \mathscr{K}$. $\mathfrak{A}$ is *injective in* $\mathscr{K}$ if for any $\mathfrak{A}_1, \mathfrak{A}_2 \in \mathscr{K}$, any homomorphism $h\colon \mathfrak{A}_2 \to \mathfrak{A}$, and any monomorphism $g\colon \mathfrak{A}_2 \to \mathfrak{A}_1$, there is a homomorphism $f\colon \mathfrak{A}_1 \to \mathfrak{A}$ such that $f \circ g = h$. It is well known that a product of injectives is injective and a retract of an injective is injective. We will use the concept of a Boolean extension (see § 22 of [8]) and some results of Day [2] to calculate the injectives in $\mathscr{K}(\mathfrak{A})$ when $\mathfrak{A}$ is quasi-primal.

An algebra $\mathfrak{A}$ is *self-injective* if every homomorphism from a subalgebra of $\mathfrak{A}$ into $\mathfrak{A}$ can be extended to an endomorphism of $\mathfrak{A}$. If $\mathfrak{A}$ is quasi-primal then $\mathfrak{A}$ is self-injective iff $\mathfrak{A}$ is demi-semi-primal. The following is an immediate consequence of Theorem 2.3 of [2].

Lemma 5.3. *If $\mathfrak{A}$ is quasi-primal then $\mathfrak{A}$ is injective in $\mathscr{K}(\mathfrak{A})$ iff $\mathfrak{A}$ is self-injective (i.e. $\mathfrak{A}$ is demi-semi-primal).*

Thus let $\mathfrak{A}$ be demi-semi-primal. Hence every direct power of $\mathfrak{A}$ is injective in $\mathscr{K}(\mathfrak{A})$. Since every algebra in $\mathscr{K}(\mathfrak{A})$ can be embedded in a direct power of $\mathfrak{A}$, the injectives in $\mathscr{K}(\mathfrak{A})$ are precisely the retracts of direct powers of $\mathfrak{A}$. To find a better description of the injectives in $\mathscr{K}(\mathfrak{A})$ we use the following corollary to Theorem 2.25 of [2]:

Lemma 5.4. *If $\mathfrak{A}$ is primal then the injectives in $\mathscr{K}(\mathfrak{A})$ are precisely those algebras isomorphic to the extensions of $\mathfrak{A}$ by complete Boolean algebras.*

Let $\mathfrak{A}$ be demi-semi-primal and $\mathfrak{A}^*$ the algebra obtained from $\mathfrak{A}$ by making each element of $\mathfrak{A}$ the value of a nullary operation. Clearly $\mathfrak{A}^*$ is primal. By Lemma 5.4 the injectives in $\mathscr{K}(\mathfrak{A}^*)$ are the extensions of $\mathfrak{A}^*$ by complete Boolean algebras. Each is a retract of a direct power of $\mathfrak{A}^*$. Ignoring the added

nullaries, each injective in $\mathscr{K}(\mathfrak{A}^*)$ is an algebra in $\mathscr{K}(\mathfrak{A})$ and is a retract of a direct power of $\mathfrak{A}$. Hence in $\mathscr{K}(\mathfrak{A})$ every extension of $\mathfrak{A}$ by a complete Boolean algebra is injective. In fact, these are all the injectives in $\mathscr{K}(\mathfrak{A})$. To show this we will use the following lemma.

Lemma 5.5. *If $\mathfrak{A}$ is quasi-primal and $\mathfrak{B}$ is injective in $\mathscr{K}(\mathfrak{A})$ then $\mathfrak{B}$ is isomorphic to an extension of $\mathfrak{A}$ by a complete Boolean algebra.*

Proof (due to Grätzer). By Lemma 5.4 we need only show that by properly interpreting the nullary operations of $\mathfrak{A}^*$ in $\mathfrak{B}$, $\mathfrak{B}^*$ belongs to $\mathscr{K}(\mathfrak{A}^*)$. But since $\mathfrak{B}$ can be embedded in $\mathfrak{A}^I$ for some I and $\mathfrak{B}$ is injective, there is a homomorphism φ from $\mathfrak{A}^I$ onto $\mathfrak{B}$. Then the restriction of φ to the diagonal subalgebra of $\mathfrak{A}^I$, $\{p \in A^I | p(i)=p(j) \text{ for } i, j \in I\}$, properly interprets the nullary operations. Thus $\mathfrak{B}^*$ ($\mathfrak{B}$ with the nullaries interpreted by φ) is a homomorphic image of $(\mathfrak{A}^*)^I$ and so $\mathfrak{B}^* \in \mathscr{K}(\mathfrak{A}^*)$.

Theorem 5.6. *Let $\mathfrak{A}$ be quasi-primal and $\mathscr{K}(\mathfrak{A})$ the equational class generated by $\mathfrak{A}$.*

(a) *If $\mathfrak{A}$ is demi-semi-primal then the injectives in $\mathscr{K}(\mathfrak{A})$ are precisely those algebras isomorphic to the extensions of $\mathfrak{A}$ by complete Boolean algebras.*

(b) *If $\mathfrak{A}$ is not demi-semi-primal then there are no non-trivial injectives in $\mathscr{K}(\mathfrak{A})$.*

Proof. Only (b) need be proved. Let $\mathfrak{B}$ be a non-trivial injective in $\mathscr{K}(\mathfrak{A})$. By Lemma 5.5 $\mathfrak{B}$ is a diagonal subdirect power of $\mathfrak{A}$. Since $\mathfrak{A}$ is a homomorphic image of $\mathfrak{B}$, $\mathfrak{A}$ is a retract of $\mathfrak{B}$ and hence injective. This contradicts the assumption that $\mathfrak{A}$ is not demi-semi-primal.

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