Werk

Titel: Maps and Equivalences into Equalizing Fibrations and from Coequalizing Cofibratio...

Autor: Rutter, John W.

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Kontakt/Contact

Digizeitschriften e.V.
SUB Göttingen
Platz der Göttinger Sieben 1
37073 Göttingen

info@digizeitschriften.de

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Maps and Equivalences into Equalizing Fibrations and from Coequalizing Cofibrations

JOHN W. RUTTER

Introduction

Let $h, k: B \to K$ be maps of topological spaces and let $P_k^h \to B$ be their equalizer in the homotopy category: the standard construction for this is the following pullback in the topological category

$$\begin{array}{ccc} P_k^h & \longrightarrow & K^I \\ \downarrow & & \downarrow \\ B & \longrightarrow & K \times K \end{array}$$

and $\Omega K \to P_k^h \to B$ is then a fibration. An *equalizing fibration* is a fibration obtained in this way: an important special case is the fibration $P_k = P_k^* \to B$ used in the construction of simply connected Postnikov systems. Since the homotopy category has products the concepts of equalizing fibration and of fibration obtained from the mapping track of a triad introduced in [8], that is the following pullback in the homotopy category

$$\begin{array}{ccc} T & \longrightarrow & B_2 \\ \downarrow & & \downarrow \\ B_1 & \longrightarrow & K \end{array}$$

are obviously equivalent.

Let $F \to E \to B$ be a fibration[1], and $e: F \to \Omega K$ a map, then in Theorems 2.1 and 2.5 I obtain necessary and sufficient conditions in the non-based and based categories for the existence of an equalizing fibration $\Omega K \to P_k^h \to B$ and a map $E \to P_k^h$ over B giving the class e between the fibres: in particular conditions are given that the given fibration is equivalent to an equalizing fibration. With restrictions on dimension, the obstruction to the existence of such a fibration with $h = *$ is (Theorem 3.1) the Hopf construction $H^*(e)$ on e; and the obstruction to a given P_k^h having the fibre homotopy type of some P_g with the identity class on the fibre is (Theorem 3.2) the Whitehead product $\varepsilon_*[1, S\Omega k]$ in $[\Omega K * \Omega B, K]^*$. As a further application conditions are given (Theorem 4.3) that a map between two fibre sequences $\Omega K \to P_h \to B$ and $\Omega \bar{K} \to P_{\bar{k}} \to \bar{B}$ can

[1] In the sense of Hurewicz or Dold, or with some restrictions in the sense of Serre.

be extended to the right to give a homotopy commutative diagram

$$\begin{array}{ccccccc} \Omega K & \longrightarrow & P_h & \longrightarrow & B & \longrightarrow & K \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Omega\bar{K} & \longrightarrow & P_k & \longrightarrow & \bar{B} & \longrightarrow & \bar{K}. \end{array}$$

The duals of all these results are discussed concurrently. In this dual case the following problem arises: given a map between two cofibrations $A \to X \to B$ and $A \to X' \to B'$ which is the identity on A and a homotopy equivalence on the cofibres, what conditions ensure that $X \to X'$ is a cofibre homotopy equivalence? In § 5 I show that in a suitable category it is sufficient that $\pi_1(A) \to \pi_1(X)$ is zero on each path component of A.

I am indebted to Professor C.T.C. Wall for discussions on the content of § 5.

1. Preliminaries

I consider categories of based and non-based topological spaces: in particular $\mathscr{W}$ is the category of spaces having the base point preserving homotopy type of CW complexes. Base points are generally denoted by $*$. Base point preserving maps and homotopies are represented f and $f \sim g$: $(X, *) \to (Y, *)$, and non-based ones f and $f \sim g$: $X \to Y$: the corresponding homotopy classes are $[X, Y]^*$ and $[X, Y]$. The functors of unreduced join, suspension and cone are denoted $\tilde{*}$, $\tilde{S}$ and $\tilde{C}$; the reduced join, suspension and cone are denoted $*$, S and C. The reduced torus is $TX = X \times S^1/X \times * \cup * \times S^1$ and the reduced cylinder is $X \barwedge I = X \times I/* \times I$.

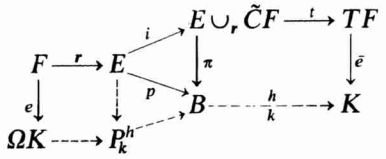
The equalizing fibration of h, k: $X \to Y$ is given by $P_k^h = \{(x, l) \in X \times Y^I : h(x) = l(0), k(x) = l(1)\}$ with $P_k^h \to X$ by $(x, l) \to x$: the reduced and unreduced coequalizing cofibrations by $C_k^h = Y \cup (X \barwedge I)$ and $\tilde{C}_k^h = Y \cup (X \times I)$ where $X \times \{0\}$ and $X \times \{1\}$ are attached in each case by h and k respectively and the inclusions $Y \subset C_k^h$ and $Y \subset \tilde{C}_k^h$ are the obvious ones: the reduced and unreduced mapping cones of k are $C_k = C_k^* = Y \cup_k CX$ and $\tilde{C}_k = Y \cup_k \tilde{C}X$; of course $\tilde{C}_k^* \not\equiv \tilde{C}_k$, and $\tilde{C}_k^h \not\equiv C_k^h$.

An m-dimensional space is a space X having non zero homology groups only in the range $r \leqq m$ with $H_m(X)$ free.

The space of unbased loops on X is denoted $\Omega_* X$: the end point map $\Omega_* X \to X$ is then a fibration.

2. The Mapping Theorems

Let $F \xrightarrow{r} E \xrightarrow{p} B$ be a fibration with B path connected where r is the inclusion $F = p^{-1}(*) \subset E$, and let e: $F \to \Omega K$. Consider the following diagram



where the upper sequence is the unbased cofibre sequence induced by r with t collapsing E to a point and identifying the vertices of the suspension $\tilde{S}F$ thus obtained without changing orientation, and the lower sequence is the equalizing fibre sequence induced by some h and k. The function π is given by $\pi|E=p$ and $\pi|\tilde{C}F=*$, and $\bar{e}\colon (TF,*)\to(K,*)$ is the adjoint of $e\colon F\to\Omega K$, thus $\bar{e}=\varepsilon(Te)$ where $\varepsilon\colon T\Omega K\to K$ is the evaluation given by $\varepsilon(l,s)=l(s)$; adjunction defined in this way determines a group isomorphism[2] $[F,\Omega K]\to[TF,K]^*$. The coaction for the non-based cofibre sequence is $\mu\colon E\cup\tilde{C}F\to TF\vee E\cup\tilde{C}F$ defined in the obvious way.

Fibre Space Mapping Theorem 2.1. *Let $F\xrightarrow{r}E\xrightarrow{p}B$ be a fibration[3] with B path connected[4], and $e\colon F\to\Omega K$ a map, then there is an equalizing fibration $p'\colon P_k^h\to B$ and a map $\omega\colon E\to P_k^h$ over B with $\omega|F\sim e\colon F\to\Omega K$ if, and only if, there are h, $k\colon (B,*)\to(K,*)$ satisfying[5] $h\pi\sim(\bar{e},k\pi)\mu\colon (E\cup\tilde{C}F,*)\to(K,*)$. Furthermore if B has a numerable weakly contractible cover[6] and $e\colon F\to\Omega K$ is a homotopy equivalence, then $\omega\colon E\to P_k^h$ is a fibre homotopy equivalence.*

Before the proof of this theorem is given, three important special cases are noted. First recall that $P_k\to B$ is induced from the path space $PK\to K$ by $k\colon B\to K$.

Corollary 2.2. *There is an induced fibre space $p'\colon P_k\to B$ and a map $\omega\colon E\to P_k$ over B with $\omega|F\sim e\colon F\to\Omega K$ if and only if there is a $k\colon (B,*)\to(K,*)$ satisfying*[7] $(-\bar{e})\,t\sim k\pi\colon (E\cup\tilde{C}F,*)\to(K,*)$.

For the second special case let $Q_k\to B$ be the fibration induced by $k\colon B\to K$ from the fibration $\Omega_* K\to K$. Now under the group action $\mu^*\colon \pi_1^F(K;*)\times[E\cup_r\tilde{C}F,K]^*\to[E\cup_r\tilde{C}F,K]^*$, the isotropy group of $\{k\pi\}$ is the image of $r^*\colon \pi_1^E(K;kp)\to\pi_1^F(K;*)$ (see [12]). The next corollary is now immediate.

Corollary 2.3. *There is an induced fibre space $p'\colon Q_k\to B$ and a map $\omega\colon E\to Q_k$ over B with $\omega|F\sim e\colon F\to\Omega K$ if and only if there is $k\colon (B,*)\to(K,*)$ such that*[8] $\{\bar{e}\}\in r^*\,\pi_1^E(K;kp)$.

As a further application of the theorem replace $F\to E\to B$ by $\Omega K\to P_k^h\to B$ and replace e and ω by the identities, then the following corollary is immediate.

Corollary 2.4. *There is a homotopy $h\pi\sim(\varepsilon,k\pi)\mu\colon (P_k^h\cup\tilde{C}\Omega K,*)\to(K,*)$. In particular, if $h=*$, the following diagram is homotopy commutative by a base point preserving homotopy*

$$\begin{array}{ccc} P_k\cup\tilde{C}\Omega K & \xrightarrow{t} & T\Omega K \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle -\varepsilon} \\ B & \xrightarrow[k]{} & K. \end{array}$$

---

[2] TF is clearly an H'-space in the based category.

[3] The first part of this theorem is valid for any sequence $F\xrightarrow{r}E\xrightarrow{p}B$ for which $pr=*$.

[4] This is, if course, no real restriction.

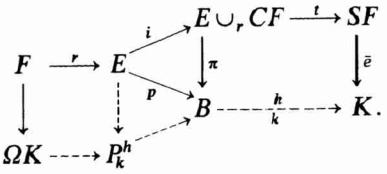
[5] The base point of $E\cup\tilde{C}F$ is the vertex of the cone.

[6] For example if B is a classifying space or a locally contractible paracompact space, e.g. a CW complex (Theorem 6.3 of [3]) or if B has the homotopy type of a CW complex.

[7] This is equivalent to $*\sim(\bar{e},k\pi)\mu$.

[8] In case F and E are H'-spaces then $r^*\,\pi_1^E(K;kp)=\Gamma(kp,r)\,\pi_1^E(K;*)$ as in § 3.2 of [9]. In the base point case Γ is calculated in Theorem 3.4.3 of [9].

Consider now the category of maps and homotopies preserving basepoint. Let $(F, *) \to (E, *) \to (B, *)$ and $e\colon (F, *) \to (\Omega B, *)$ be base point preserving, the base point of ΩB being the constant path[9]: the adjoint of e is $\bar{e}\colon (SF, *) \to (B, *)$. The diagram relevant to the fibre space mapping theorem in this case is the following one:



The coaction for the based cofibre sequence is $\mu\colon E \cup CF \to SF \vee E \cup CF$. The base point version of the fibre space mapping theorem is now stated; its proof and corollaries are similar to those of Theorem 2.1.

Fibre Space Mapping Theorem 2.5. *Let $(F, *) \xrightarrow{r} (E, *) \xrightarrow{p} (B, *)$ be a fibration with B path connected, and $e\colon (F, *) \to (\Omega K, *)$ a map, then there is an equalizing fibration $p'\colon (P_k^h, *) \to (B, *)$ and a map $\omega\colon (E, *) \to (P_k^h, *)$ over B with $\omega | F \sim e\colon (F, *) \to (\Omega K, *)$ if, and only if, there are $h, k\colon (B, *) \to (K, *)$ satisfying $h\pi \sim (\bar{e}, k\pi)\mu\colon (E \cup CF, *) \to (K, *)$. Furthermore if $(F, *) \to (E, *) \to (B, *)$ is a fibration in $\mathscr{W}$ and $(F, *) \to (\Omega K, *)$ a homotopy equivalence with K in $\mathscr{W}$[10], then $\omega\colon (E, *) \to (P_k^h, *)$ is a (based) fibre homotopy equivalence.*

Corollary 3.5 of [7] is the latter part of this theorem applied to the case $h = *$.

Suppose now that $(F, *) \to (E, *) \to (B, *)$ is freely fibre homotopy equivalent to $\Omega K \to P_k \to B$ by $e\colon F \to \Omega B$ on the fibre and let $e(*) = \lambda$ and λ_l^{-1} denote in P_k left multiplication by λ^{-1}, then, in $\mathscr{W}$, $\lambda_l^{-1} \circ e\colon (F, *) \to (\Omega B, *)$ and $\lambda_l^{-1} \circ \omega\colon (E, *) \to (P_k, *)$ are, up to homotopy, base point preserving equivalences. The proof of the next proposition is now straightforward.

Proposition 2.6. *Let $(F, *) \to (E, *) \to (B, *)$ be a Hurewicz fibration freely fibre homotopically equivalent to $\Omega K \to P_k \to B$ by an equivalence $e\colon F \to \Omega B$ on the fibre, then, if all spaces are in $\mathscr{W}$, the fibrations are fibre homotopically equivalent by a base point preserving equivalence homotopic to $\lambda_l^{-1} \circ e\colon (F, *) \to (\Omega K, *)$ on the fibre.*

In case the space K has an H-structure[11] the fibrations $P_k^h \to B$ take on a simple form and the fibrations $Q_k \to B$ are trivial. The proof of the following result is similar to that of Proposition 5.1 of [10].

Proposition 2.7. *Let K have an H-structure, then $P_k^h \to B$ has the same fibre homotopy type as $P_{k-h} \to B$ by an equivalence restricting to the identity class on the fibre. In case the H-structure preserves base point, so does the equivalence.*

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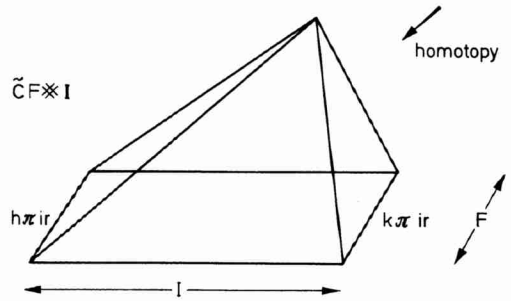
[9] With any other base point l_0, there is a similar theorem to the one below; note that in this case the standard H-structure on ΩK does not in general preserve base point even up to homotopy.

[10] Or more generally the numerable weakly contractible cover of B respects base point and certain homotopy extension properties are satisfied.

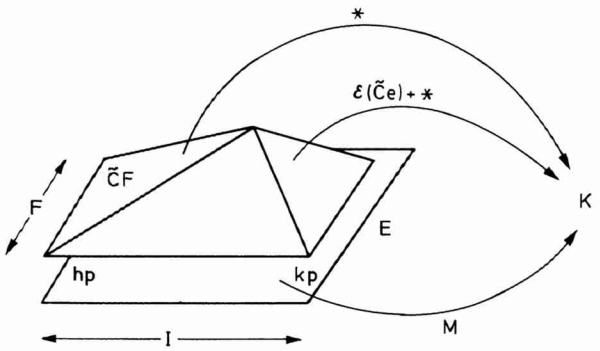
[11] A homotopy associative multiplication with homotopy identity and homotopy inverse.

Proof of Theorem 2.1. (i) Suppose there are h, k: $B \to K$ and a homotopy $J: h\pi \sim (\bar{e}, k\pi)\mu: (E \cup \tilde{C}F) \ast I \to K$, then $Ji: hp \sim kp$ may, using the exponential theorem, be regarded as a function $E \to K^I$ and thus determines a function $\omega: E \to P_k^h$ over B given by $\omega(x) = (p(x), Ji(x))$. It is now sufficient to prove that $\omega|F \sim e$: $F \to \Omega K$. Let L: $\tilde{C}F \to E \cup_r \tilde{C}F$ be the projection. The following diagram represents the restriction of the homotopy J to $\tilde{C}F$.



The base of the cylinder is the class of Jir in the track group[12] $\pi_1^F(K; h\pi ir, k\pi ir) = \pi_1^F(K; *)$ and the ends are respectively the homotopies (going upwards) $h\pi(-L) = *$ and (going downwards) $(\bar{e}, k\pi)\mu L = \bar{e} + * \sim \bar{e}$ in $\pi_1^F(K; *)$. Thus since the diagram clearly determines a homotopy[13] between Jir and $\bar{e} + *$, it follows that $\omega|F \sim e$.

(ii) Conversely if $\omega: E \to P_k^h$ satisfies $\omega|F \sim e$, replace e by $\omega|F$. The composite function $E \xrightarrow{\omega} P_k^h \to K^I$ determines a homotopy M: $E \times I \to K$ between $h\pi i = hp$ and $k\pi i = kp$ with $M|F \times I = \varepsilon(e \times 1)$: $F \times I \to \Omega K \times I \to K$. The homotopy M can be extended in an obvious way (essentially the constant homotopy) to $h\pi \sim (\bar{e}, k\pi)$: $(E \cup_r \tilde{C}F) \ast I \to K$ as in the following diagram:

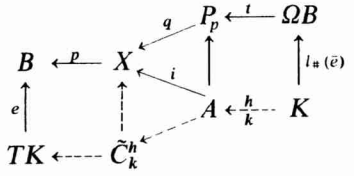


(iii) If e: $F \to \Omega K$ is a homotopy equivalence and B has a numerable weakly contractible cover, then ω: $E \to P_k^h$ is necessarily a fibre homotopy equivalence by Theorem 6.3 of [3].

[12] All track groups here are of unbased homotopies.

[13] There is a homeomorphism $(E \cup_r \tilde{C}F) \ast I \doteq (E \times I) \cup_{r \times 1} (\tilde{C}F \ast I)$.

Dually let $A \xrightarrow{i} X \xrightarrow{p} B$ be a cofibration with X path connected where $B = X/A$ has the standard base point. Any given $e\colon TK \to B$ is homotopic to a base point preserving map; assume therefore, that e is base point preserving. The group $\pi_1(B, *)$ operates on the group $[TK, B]^* \cong [K, \Omega B]$ by $(l, \xi) \to l_\# \xi = -l + \xi + l$ and the orbit set is bijective with $[TK, B]$. Consider the following diagram



where $\Omega B \to P_p$ is given by a choice of base point for A, the upper sequence is the fibre sequence induced by p and the lower sequence is the cofibre sequence induced by some h and k. The map ι is given by $\iota(a) = (i(a), *)$. The action for the fibre sequence is $m\colon \Omega B \times P_p \to P_p$.

Cofibre Space Mapping Theorem 2.8. *Let $A \to X \to B$ be a cofibration and let $e\colon (TK, *) \to (B, *)$ be a map, then there is a coequalizing cofibration $i'\colon A \to \tilde{C}_k^h$ and a map $\omega\colon \tilde{C}_k^h \to X$ extending the identity on A with $\omega/A \sim e\colon TK \to B$ if and only if there are a loop $l\colon * \to *$ in B and maps $h, k\colon K \to A$ satisfying $\iota h \sim m(l_\# \bar{e}, \iota k)\colon K \to P_p$. Furthermore if A, X and K have the homotopy type of CW complexes, each path component of A is simply connected, and $e\colon TK \to B$ is a homotopy equivalence, then $\omega\colon \tilde{C}_k^h \to X$ is a cofibre homotopy equivalence.*

The special cases of this theorem dual to Corollaries 2.2, 2.3 and 2.4 are clear. There is however a related unbased induced cofibre sequence $A \to \tilde{C}_h \to \tilde{S}K$, which is not a special case of the above one since $\tilde{C}_h$ and $\tilde{C}_h^*$ are distinct.

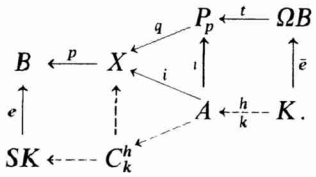
Let $e\colon \tilde{S}K \to B$, then up to homotopy e can be chosen to satisfy $e(0) = e(1) = *$: suppose then that e maps the end of the suspension to $*$. Denote by $(\lambda, l)_\#$ the self equivalence of $[TK, B]^* \cong [K, \Omega B]$ given by $\xi \to (\lambda, l)_\# \xi = -\lambda + \xi + l$ for elements λ and l of $\pi_1(B, *)$: this gives a group operation of $\pi_1(B, *) \times \pi_1(B, *)$ on $[TK, B]^*$ the orbit set of which is bijective with $[\tilde{S}K, B]$.

Using the same diagram as before the proof of the following theorem is similar to that of 2.8.

Cofibre Mapping Space Theorem 2.9. *Let $A \to X \to B$ be a cofibration and $e\colon \tilde{S}K \to B$ a map satisfying $e(0) = e(1) = *$, then there is an induced cofibre space $i'\colon A \to \tilde{C}_k$ and a map $\omega\colon \tilde{C}_k \to X$ extending the identity on A with $\omega/A \sim e\colon \tilde{S}K \to B$ if and only if there are loops $l, \lambda\colon * \to *$ in B and a map $k\colon K \to A$ satisfying $t(-((\lambda, l)_\# \bar{e})) \sim \iota k\colon K \to P_p$. Furthermore if A, X and K have the homotopy type of CW complexes and each path component of A is simply connected and $e\colon SK \to B$ is a homotopy equivalence, then $\omega\colon \tilde{C}_k \to X$ is a cofibre homotopy equivalence.*

Consider now the category of maps and homotopies preserving base point. Let $(A, *) \to (X, *) \to (B, *)$ preserve base point where $B = X/A$ and $* = A/A \in B$, and let $e\colon (SK, *) \to (B, *)$ be a map. The diagram relevant to the cofibre space

mapping theorem in this case is the following one:



The proof of the base point cofibre space mapping theorem is similar to that of Theorem 2.8 with $K \times I$ replacing $K \times I$.

Cofibre Space Mapping Theorem 2.10. *Let $(A,*)\to(X,*)\to(B,*)$ be a cofibration with X path connected and $e\colon (SK,*)\to(B,*)$ a map, then there is a coequalizing cofibration $i'\colon (A,*)\to(C_k^h,*)$ and a map $\omega\colon (C_k^h,*)\to(X,*)$ extending the identity of A with $\omega/A\sim e\colon (SK,*)\to(B,*)$ if and only if there are maps $h,k\colon (K,*)\to(A,*)$ satisfying $\iota h\sim m(\bar{e},\iota k)\colon (K,*)\to(P_p,*)$. Furthermore if A, X and K are in $\mathscr{W}$ and each path component of A is simply connected*[14] *and $e\colon (SK,*)\to(B,*)$ is a homotopy equivalence, then $\omega\colon (C_k^h,*)\to(X,*)$ is a cofibre homotopy equivalence.*

Let $\eta\colon K\to\Omega SK$ be given by $\eta(k)(t)=(k,t)$, then replacing $(A,*)\to(X,*)\to(B,*)$ by $(A,*)\to(C_k^h,*)\to(SK,*)$, the following corollary is dual to Corollary 2.4.

Corollary 2.11. *There is a homotopy $\iota h\sim m(\eta,\iota k)\colon (K,*)\to(P_p,*)$. In particular if $h=*$, the following diagram is homotopy commutative by a base point preserving homotopy*

$$\begin{array}{ccc} K & \xrightarrow{k} & A \\ {\scriptstyle -\eta}\downarrow & & \downarrow{\scriptstyle \iota} \\ \Omega SK & \xrightarrow{t} & P_p. \end{array}$$

The dual of Proposition 2.6 *is, of course, trivial.*

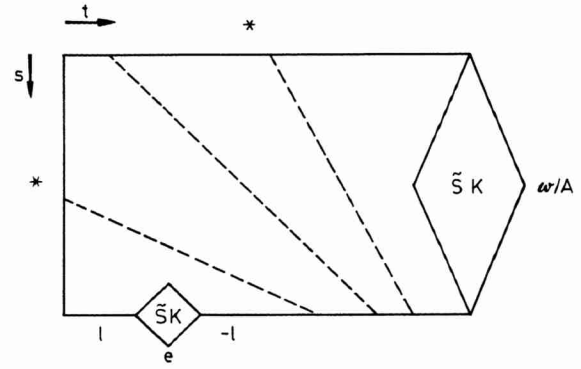
Proposition 2.12 (Proposition 5.1* of [10]). *Let $(K,*)$ have an H'-structure*[15]*, then $(A,*)\to(C_k^h,*)$, induced by $h,k\colon (K,*)\to(A,*)$, has the same cofibre homotopy type as $(A,*)\to(C_{k-h},*)$ by an equivalence determining the identity class on the cofibre.*

Proof of Theorem 2.8. (i) Suppose there are l, h and k and a homotopy $J\colon \iota h\sim m(-l+\bar{e}+l,\iota k)$, then $qJ\colon ih\sim ik\colon K\times I\to X$ determines $\omega=1\cup qJ\colon A\cup(K\times I)=\tilde{C}_k^h\to X$ which extends the identity on A. It is sufficient to prove that $\omega/A\sim e\colon TK\to B$. Let $\alpha\colon P_p\to PB$ be the projection and $\beta\colon PB\to B$ the end point map, then $\alpha J\colon K\times I\to PB$ determines by adjunction $H\colon K\times I\times I\to B$

[14] Without some restriction on fundamental groups, this part of the theorem is false: a counter-example follows from Remark 5.2.

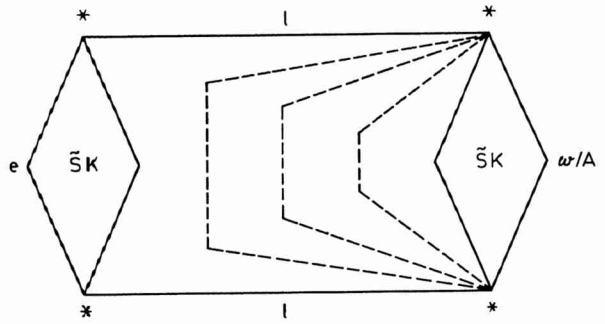
[15] A homotopy associative comultiplication with homotopy identity and homotopy inverse.

given by $H(k, s, t) = \alpha J(k, s)(t)$. The function H is represented in the following diagram

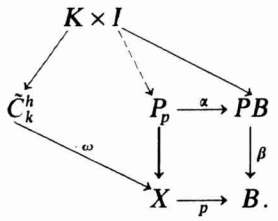


This can clearly be deformed, as indicated by the dotted lines, to give a homotopy $\omega/A \sim e$: $TK \to B$.

(ii) Conversely suppose there is ω: $\tilde{C}^h_k \to X$, then $\omega/A \sim e$ can be represented as in the following diagram for some path l



and this determines a function $K \times I \times I \to B$ as in the diagram of (i) and hence by adjunction $K \times I \to PB$ making the following diagram commutative



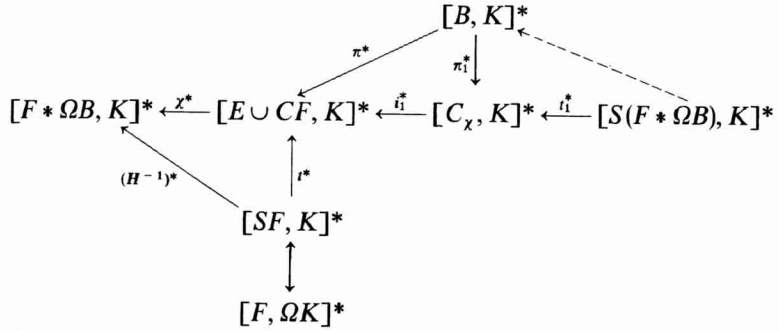
The pullback map $K \times I \to P_p$ is the required homotopy.

(iii) If e: $TK \to B$ is a homotopy equivalence and the other conditions are satisfied then ω is a cofibre homotopy equivalence by Theorems 5.1 and 5.3.

3. Some Obstructions

Let $(F, *) \xrightarrow{r} (E, *) \xrightarrow{p} (B, *)$ be a fibration in $\mathscr{W}$ and define $\pi: E \cup_r CF \to B$ by $\pi|E = p$ and $\pi(CF) = *$. By Theorem 1.1 of [4] the fibre of π is the reduced[16] join $F * \Omega B$ with map, say, $\chi: F * \Omega B \to E \cup CF$. Let $H: F * \Omega B \to S(F \times \Omega B) \xrightarrow{S\rho} SF$ be the Hopf construction with respect to the action $\rho: F \times \Omega B \to F$ of ΩB on the fibre and let H^{-1} denote the composite $(-1) \circ H$.

Consider now the following commutative diagram



where the triangle on the left is commutative by Theorem 1.4 of [4]. The horizontal sequence is exact. The fibre of $\pi_1: C_\chi \to B$ is $F * \Omega B * \Omega B$ and thus π_1 is $2m+n$ connected if F and B are respectively $(n-1)$ and $(m-1)$ connected $(m \geqq 1)$; hence π_1^* is bijective by Lemma 8.2 of [11] in case K has non vanishing homotopy groups only in the range $1 \leqq r \leqq 2m+n-1$. The following theorem is now immediate from the fibre space mapping Theorem 2.5.

Theorem 3.1. *Let $(F, *) \to (E, *) \to (B, *)$ be a fibration in $\mathscr{W}$ with F and B respectively $(n-1)$ and $(m-1)$ connected $(m \geqq 1)$ and let K have non vanishing homotopy groups only in the range $1 \leqq r \leqq 2m+n-1$. Given $e: (F, *) \to (\Omega K, *)$, there is an induced fibre space $p': (P_k, *) \to (B, *)$ and a map $\omega: (E, *) \to (P_k, *)$ over B with $\omega|F \sim e: (F, *) \to (\Omega K, *)$ if, and only if, $H^*(\bar{e}) = 0$. Further ω is a based fibre homotopy equivalence if e is a based homotopy equivalence.*

In case K is simply connected, the last part of this theorem is proved in a different way by Ganea (Theorem 1.2 of [5]).

Theorem 3.2. *Let $(F, *) \to (E, *) \to (B, *)$ be a fibration in $\mathscr{W}$ with F and B respectively $(n-1)$ and $(m-1)$ connected $(m \geqq 1)$ and let K be in $\mathscr{W}$ and have non vanishing homotopy groups only in the range $1 \leqq r \leqq 2m+n-1$. Given a fibration $(P_k^h, *) \to (B, *)$ induced by $h, k: (B, *) \to (K, *)$, with a map $\omega: (E, *) \to (P_k^h, *)$ over $1: B \to B$, then there is a fibration $(P_g, *) \to (B, *)$ induced by some $g: (B, *) \to (K, *)$ and a map $\omega': (E, *) \to (P_g, *)$ over $1: B \to B$ with $e = \omega|F \sim \omega'|F: F \to \Omega K$ if, and only if, the Whitehead product class $[\bar{e}, k\varepsilon] = 0$ in $[F * \Omega B, K]^*$. In particular the equalizing fibration $(P_k^h, *) \to (B, *)$ has the based fibre homotopy type of an induced fibration $(P_g, *) \to (B, *)$ by a map restricting to the identity class on the fibre if, and only if, $\varepsilon_*[1, S\Omega k] = 0$ in $[\Omega K * \Omega B, K]^*$.*

[16] Assuming given a base point preserving lifting function.

Proof. There is an H'-structure on $F * \Omega B$ with comultiplication $\phi' : F * \Omega B \to F * \Omega B \vee F * \Omega B$ given by

$$\phi'(x,s,\lambda)=\begin{cases}(x,4s,\lambda)\vee * & 0\leqq 4s\leqq 1\\ (*,2-4s,\lambda)\vee * & 1\leqq 4s\leqq 2\\ *\vee(x,3-4s,*) & 2\leqq 4s\leqq 3\\ *\vee(x,4s-3,\lambda) & 3\leqq 4s\leqq 4.\end{cases}$$

Define $\alpha : S(F\times\Omega B)\to F * \Omega B$ by

$$\alpha(x,\lambda,s)=\begin{cases}(x,1-3s,*) & s\leqq\frac{1}{3}\\ (x,3s-1,\lambda) & \frac{1}{3}\leqq s\leqq\frac{2}{3}\\ (*,3-3s,\lambda) & \frac{2}{3}\leqq s\end{cases}$$

then α is a homomorphism with respect to the H'-structure on $F * \Omega B$ given above and the projection $\beta : F * \Omega B \to S(F\times\Omega B)$ given by $\beta(x,s,\lambda)=(x,\lambda,s)$ satisfies $\alpha\beta\sim 1$ $\quad S(F\times\Omega B)\underset{\beta}{\overset{\alpha}{\leftrightarrows}} F * \Omega B.$

By 1.2 of [4], $\chi : F * \Omega B \to C_r$ is given by

$$\chi(x,s,\lambda)=\begin{cases}\sigma(x,\lambda)(2s) & s\leqq\frac{1}{2}\\ (\sigma(x,\lambda)(1),2-2s) & \frac{1}{2}\leqq s\end{cases}$$

where σ: $\{(e,l)\in B\times E^I : p(e)=l(0)\}\to E^I$ is the homotopy lifting function for $p : E\to B$. Let $\rho : F\times\Omega B\to F$ be the action given by σ, that is $\rho(x,\lambda)=\sigma(x,\lambda)(1)$; and let $\partial : \Omega B\to F$ be the boundary operator of the fibre sequence. Then as maps $S(F\times\Omega B)\to SF\vee B$,

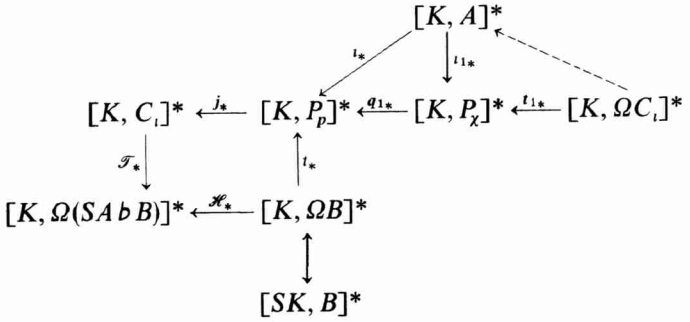
$$(1\vee\pi)\,\mu\,\chi\,\alpha\sim 1_{SF}-*+\varepsilon-S\rho+S\partial-\varepsilon$$

where ε: $S\Omega B\to B$ is the evaluation. By fibre space mapping Theorem 2.5 $h\pi\sim(\bar e, k\pi)\,\mu : (E\cup CF,*)\to(K,*)$, thus $*\sim(\bar e,k)(1\vee\pi)\,\mu\,\chi$ from which $\bar e+k\varepsilon-\bar e\circ S\rho+\bar e\circ S\partial-k\varepsilon\sim *$. On the other hand as maps $S(F\times\Omega B)\to SF$,

$$(S\rho)\,\beta\,\alpha\sim 1_{SF}-S\rho+S\partial$$

from which $\bar e H\alpha\sim\bar e-\bar e\circ S\rho+\bar e\circ S\partial$: thus $\bar e H\alpha\sim *$ if, and only if, $\bar e$ and $k\varepsilon$ commute in $[S(F\times\Omega B),K]^*$ and the result is immediate.

In the dual case let $(A,*)\xrightarrow{i}(X,*)\xrightarrow{p}(B,*)$ be a cofibration in $\mathscr{W}$, and consider the following commutative diagram



where the exact horizontal sequence is obtained from the fibre sequence induced by $\chi\colon P_p \to C_i$. The functions $\mathscr{T}\colon C_i \to \Omega(SA \flat B)$ and Hopf invariant $\mathscr{H}\colon \Omega B \to \Omega(SA \flat B)$ make the square commutative and are defined as follows (§4 of [4]). For the cofibration $(A,*) \xrightarrow{i} (X,*) \xrightarrow{p} (B,*)$ let $h\colon X \ltimes I \to SA \vee B$ be given by a homotopy extension function followed by projection onto $SA \vee B$ from the mapping cylinder $X \cup (A \ltimes I)$, thus $h(x,0)=(*,p(x))$, $h(x,1)=\rho p(x)$ where $\rho\colon B \to SA \vee B$ is the coaction, and $h(i(a),s)=((s,a),*)$. Let $\theta\colon B \ltimes I \to B$ be the homotopy from $B \xrightarrow{\rho} SA \vee B \to B$ to the identity and define $D\colon P_p \to \Omega(SA \vee B)$ by

$$D(x,l)=P\rho \circ l+(1 \vee -\theta)(-h(x))-\theta_0 \circ l.$$

As in § 4 of [4] a function $D_1\colon C_i \to \Omega(SA \vee B)$ is obtained extending D. Consider the split sequence

$$\Omega(SA \flat B) \underset{T}{\overset{\Omega L}{\rightleftarrows}} \Omega(SA \vee B) \underset{M}{\overset{\Omega J}{\rightleftarrows}} \Omega(SA \times B)$$

where $M=\Omega\alpha_1+\Omega\alpha_2$, for the projections $\alpha_1\colon SA \times B \to SA \to SA \vee B$ and $\alpha_2\colon SA \times B \to B \to SA \vee B$. Then $\mathscr{T}$ is defined by the unique class $C_i \to \Omega(SA \flat B)$ for which $\Omega L \circ \mathscr{T} \sim (1-M \circ \Omega J) \circ D_1$ or equivalently $\mathscr{T}=T \circ D_1$. The Hopf invariant $\mathscr{H}$ is defined by the unique class $\Omega B \to \Omega(SA \flat B)$ for which $\Omega L \circ \mathscr{H} \sim (1-M \circ \Omega J) \circ \Omega\rho$ or equivalently $\mathscr{H}=T \circ \Omega\rho$.

Let A be $(n-1)$-connected and $A \to X$ m-connected $(m \geqq 1, n \geqq 2)$ then, provided all spaces are in $\mathscr{W}$, ι is $(m+n-1)$-connected and ι_1 is $(m+2n-2)$-connected by Lemma 3.1 of [4], and $\mathscr{T}$ is $(m+n-1+\min(m,n))$-connected by Lemma 4.1 of [4]; thus by Lemma 8.2* of [11], $i_{1_*}\colon [K,A]^* \to [K,P_\chi]^*$ is bijective and $\mathscr{T}_*\colon [K,C_i]^* \to [K,\Omega(SA \flat B)]$ is injective provided K is simply connected and $(m+n-2+\min(m,n))$-dimensional. The following theorem is now proved.

Theorem 3.3. *Let $(A,*) \to (X,*) \to (B,*)$ be a cofibration in $\mathscr{W}$ with A and B respectively $(n-1)$ and m-connected $(m \geqq 1, n \geqq 2)$, and let K be simply connected $(m+n-2+\min(m,n))$-dimensional. Given $e\colon (SK,*) \to (B,*)$, there is an induced cofibre space $i'\colon (A,*) \to (C_k,*)$ and a map $\omega\colon (C_k,*) \to (X,*)$ extending the identity on A with $\omega/A \sim e\colon (SK,*) \to (B,*)$ if, and only if, $\mathscr{H}_*(\bar{e})=0$. Further ω is a cofibre homotopy equivalence if e is a based homotopy equivalence.*

The last part of the above theorem is proved in a different way by Ganea (Theorem 3.1 of [5]).

Let $\eta\colon B \to \Omega SB$ be given by $\eta(b)(t)=(b,t)$.

Theorem 3.4. *Let $(A,*) \to (X,*) \to (B,*)$ be a cofibration in $\mathscr{W}$ with A and B respectively $(n-1)$ and m-connected $(m \geqq 1, n \geqq 2)$, and let K be simply connected and $(m+n-2+\min(m,n))$-dimensional. Given an equalizing cofibration $(A,*) \to (C_k^h,*)$ induced by h, $k\colon (K,*) \to (A,*)$ with a map $\omega\colon (C_k^h,*) \to (X,*)$ extending $1\colon A \to A$, then there is a cofibration $(A,*) \to (C_g,*)$ induced by some $g\colon (K,*) \to (A,*)$ and a map $\omega'\colon (C_g,*) \to (X,*)$ extending $1\colon A \to A$ with $e=\omega/A \sim \omega'/A$:*

$SK \to B$ if and only if the flat product class $\{\eta k, \bar{e}\}=0$ in $[K, \Omega(SA \flat B)]$. In particular the equalizing cofibration $(A,*)\to(C_k^h,*)$ has the based cofibre homotopy type of an induced cofibration $(A,*)\to(C_g,*)$ by a map whose quotient is the identity class of SK if, and only if, $\eta^*\{\Omega S k, 1\}=0$ in $[K, \Omega(SA \flat SK)]$.

Proof. By the cofibre space mapping Theorem 2.10,

$$\imath h \sim m(\bar{e}, \imath k)\colon (K,*)\to(P_p,*); \quad \text{thus } \mathscr{T}_* j_* \{m(\bar{e}, \imath k)\}=0.$$

Consider the composite function

$$\Omega B \times A \xrightarrow{1\times\imath} \Omega B \times P_p \xrightarrow{m} P_p \xrightarrow{D} \Omega(SA \vee B) \xrightarrow{(1-M\circ\Omega J)} \Omega(SA \vee B)$$

$$P_p \xrightarrow{j} C_i \xrightarrow{D_1} \Omega(SA \vee B)$$

then, up to homotopy,

$$(l, a) \to (1 - M \circ \Omega J)(\Omega \rho\, l - \eta\, a - l)$$

and, since $\Omega J \circ \Omega \rho \sim \Omega(\partial, 1)\colon \Omega B \to \Omega S(A \times B)$ where $\partial\colon B \to SA$ is the boundary operator in the cofibre sequence, this map is homotopic to

$$(l, a) \to \Omega \rho\, l - \eta(a) - l + \eta(a) - \Omega \partial l.$$

This proves that

$$(1 - M \circ \Omega J)\, D_1\, j\, m(\bar{e}, \imath k) \sim \Omega \rho \circ \bar{e} - \eta\, k - \bar{e} + \eta\, k - \Omega \partial \circ \bar{e}$$

and thus the right hand side is nulhomotopic. Similarly

$$(1 - M \circ \Omega J)\, \Omega \rho \circ \bar{e} \sim \Omega \rho \circ \bar{e} - \bar{e} - \Omega \partial \circ \bar{e}.$$

It is clear now that $\mathscr{H}_*(\bar{e})=0$ if and only if ηk and $\bar{e}$ commute in $[K, \Omega(SA \vee B)]$ which is true (see 6.2 of [1]) if, and only if, their flat product is zero in $[K, \Omega(SA \flat SK)]$.

4. Extensions of Ladders

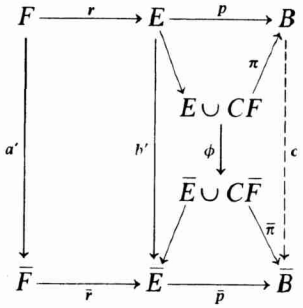
Two results are now given on maps between fibre sequences. Since obstruction theory is used extensively, attention is again restricted to the base point category $\mathscr{W}$.

Definition. A *map* between the fibre sequences $F \xrightarrow{i} E \xrightarrow{p} B$ and $\bar{F} \xrightarrow{\bar{\imath}} \bar{E} \xrightarrow{\bar{p}} \bar{B}$ is a triple (a, b, c) of maps making the diagram commutative:

$$\begin{array}{ccccc} F & \longrightarrow & E & \longrightarrow & B \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} \\ \bar{F} & \longrightarrow & \bar{E} & \longrightarrow & \bar{B}. \end{array}$$

Proposition 4.1. *In the category $\mathscr{W}$, let F and B be $(n-1)$ and $(m-1)$-connected respectively $(m \geqq 1)$, and $\bar{B}$ have non zero homotopy groups only in the range $1 \leqq r \leqq n+m-1$. Given $a' : F \to \bar{F}$ and $b' : E \to \bar{E}$ with $\bar{r}\, a' \sim b'\, r$, then there is a map (a, b, c) between the fibrations with $a \sim a'$ and $b \sim b'$.*

Proof. Consider the diagram



where ϕ is the union of b', the homotopy $\bar{r}\, a' \sim b'\, r : F \times I \to \bar{E}$ and $C\, a'$. Since $E \cup CF \to B$ is $(m+n)$-connected by Theorem 1.1 of [4], it follows by obstruction theory (e.g. Lemma 8.2 of [11]) that $[B, \bar{B}]^* \to [E \cup CF, \bar{B}]^*$ is bijective. Thus there is a c making the diagram homotopy commutative and the result follows by Lemma 1.1 of [5].

Of course the construction of ϕ and therefore of c is not unique: any two ϕ's differ essentially by a self homotopy of $b'\, r$ and thus the classes of two $\bar{\pi}\phi$'s differ by an element of $\pi_1^F(\bar{B}; \bar{p}\, b'\, r) \cong \pi_1^F(\bar{B}; *) = [F, \Omega\bar{B}]^*$. Also given any map $b' : E \to E'$ for which $\bar{p}\, b'\, r$ is nulhomotopic, there is, by the exactness of $\bar{F} \to \bar{E} \to \bar{B}$, a map a' with $\bar{r}\, a' \sim b'\, r$. The following corollary is now immediate.

Corollary 4.2. *Let F and B be respectively $(n-1)$ and 0-connected and let $\bar{B}$ have non vanishing homotopy groups only in the range $1 \leqq r \leqq n-1$. Given $b' : E \to \bar{E}$, there is a map (a, b, c) between the fibrations with $b \sim b'$, and the homotopy classes of a and c are unique.*

Consider now the case that the two fibre sequences are induced thus $\Omega K \to P_h \to B$ and $\Omega\bar{K} \to P_k \to \bar{B}$ are induced by $h : B \to K$ and $k : \bar{B} \to \bar{K}$, and let (a, b, c) be a map between these fibre sequences.

Theorem 4.3[17]. *In the category $\mathscr{W}$ let B be $(m-1)$-connected and K n-connected $(m \geqq 1)$ and let $\bar{K}$ have non zero homotopy groups only in a range $r \leqq n+m$. Suppose (a, b, c) is a map between the fibre sequences $\Omega K \to P_h \to B$ and $\Omega\bar{K} \to P_k \to \bar{B}$, and that a is homotopic to $\Omega\bar{a}$ for some $\bar{a} : K \to \bar{K}$, then $\bar{a}\, h$ is homotopic to $k\, c$*

$$\begin{array}{ccc} B & \xrightarrow{h} & K \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle \bar{a}} \\ \bar{B} & \xrightarrow{k} & \bar{K}. \end{array}$$

[17] The last part of Theorem 2.1 of [6] is a special case of this theorem.

Proof. Consider the following commutative diagram

$$\begin{array}{ccccc}
[S\Omega K, K]^* & \xrightarrow{i^*} & [P_h \cup C\Omega K, K]^* & \xleftarrow{\pi^*} & [B, K]^* \\
\downarrow{\scriptstyle \bar{a}_*} & & \downarrow{\scriptstyle \bar{a}_*} & & \downarrow{\scriptstyle \bar{a}_*} \\
[S\Omega K, \bar{K}]^* & \xrightarrow{i^*} & [P_h \cup C\Omega K, \bar{K}]^* & \xleftarrow{\pi^*} & [B, \bar{K}]^* \\
\uparrow{\scriptstyle (Sa)^*} & & \uparrow{\scriptstyle (b \cup Ca)^*} & & \uparrow{\scriptstyle c^*} \\
[S\Omega \bar{K}, \bar{K}]^* & \xrightarrow{\bar{i}^*} & [P_k \cup C\Omega \bar{K}, \bar{K}]^* & \xleftarrow{\pi^*} & [\bar{B}, \bar{K}]^*.
\end{array}$$

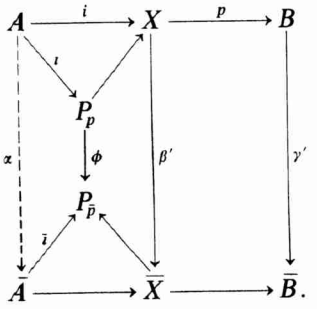
Now by the base point version of Corollary 2.4, $i^*\{-\varepsilon\} = \pi^*\{h\}$ and $\bar{i}^*\{-\bar{\varepsilon}\} = \pi^*\{k\}$; but $\bar{a}_*\{-\varepsilon\} = (Sa)^*\{-\bar{\varepsilon}\}$ and therefore $\pi^* \bar{a}_*\{h\} = \pi^* c^*\{k\}$. However, as above, by Theorem 1.1 of [4] $P_h \cup C\Omega K \to B$ is $(n+m)$-connected and thus by Lemma 8.2 of [11] π^* is injective: the result is immediate.

Definition. A *map* between the cofibre sequences $A \xrightarrow{i} X \xrightarrow{p} B$ and $\bar{A} \xrightarrow{\bar{i}} \bar{X} \xrightarrow{\bar{p}} \bar{B}$ is a triple (α, β, γ) of maps making the diagram commutative:

$$\begin{array}{ccccc}
A & \longrightarrow & X & \longrightarrow & B \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\
\bar{A} & \longrightarrow & \bar{X} & \longrightarrow & \bar{B}.
\end{array}$$

Proposition 4.4. *In the category $\mathcal{W}$, let $\bar{A}$ be $(n-1)$-connected and $\bar{A} \to \bar{X}$ be m-connected $(n \geqq 2, m \geqq 1)$, and let A be simply connected and $(m+n-1)$-dimensional. Given maps $\beta' : X \to \bar{X}$ and $\gamma' : Y \to \bar{Y}$ with $\bar{p}\,\beta' \sim \gamma' p$, then there is a map (α, β, γ) between the cofibrations with $\beta \sim \beta'$ and $\gamma \sim \gamma'$.*

Proof. Consider the diagram



Since $\bar{A} \to P_{\bar{p}}$ is $(m+n-1)$-connected by Lemma 3.1 of [4], it follows by Lemma 8.2* of [11] that $[A, \bar{A}]^* \to [A, P_{\bar{p}}]^*$ is surjective and the result follows by the dual of Lemma 1.1 of [5].

Corollary 4.5. *Let $\bar{A}$ be simply connected and $\bar{A} \to \bar{X}$ m-connected $(m \geqq 1)$, and let A be m-dimensional. Given any map $\beta' : X \to \bar{X}$, there is a map (α, β, γ)*

between the cofibrations with $\beta \sim \beta'$ and the homotopy classes of α and γ are unique if also A is $(m-1)$-dimensional.

Theorem 4.6. *In the category $\mathscr{W}$ let $\bar{A}$ be $(m-1)$-connected $(m \geqq 2)$, let $\bar{K}$ be $(n-1)$-connected $(n \geqq 1)$, and let K be simply connected and $(m+n-2)$-dimensional. Suppose (α, β, γ) is a map between the cofibre sequences $A \to C_h \to SK$ and $\bar{A} \to C_k \to S\bar{K}$ and that γ is homotopic to $S\bar{\gamma}$ for some $\bar{\gamma}: K \to \bar{K}$, then $\alpha h \sim k\bar{\gamma}$.*

$$\begin{array}{ccc} K & \xrightarrow{h} & A \\ {\scriptstyle \bar{\gamma}}\downarrow & & \downarrow{\scriptstyle \alpha} \\ \bar{K} & \xrightarrow[k]{} & \bar{A}. \end{array}$$

Proof. The following diagram is commutative

$$\begin{array}{ccccc} [K, \Omega SK]^* & \xrightarrow{\iota_*} & [K, P_p]^* & \xleftarrow{\iota_*} & [K, A]^* \\ \downarrow{\scriptstyle (\Omega\gamma)_*} & & \downarrow{\scriptstyle (\beta \times P\gamma)_*} & & \downarrow{\scriptstyle \alpha_*} \\ [K, \Omega S\bar{K}]^* & \xrightarrow{\bar{\iota}_*} & [K, P_{\bar{p}}]^* & \xleftarrow{\bar{\iota}_*} & [K, \bar{A}]^* \\ \uparrow{\scriptstyle \bar{\gamma}^*} & & \uparrow{\scriptstyle \bar{\gamma}^*} & & \uparrow{\scriptstyle \bar{\gamma}^*} \\ [\bar{K}, \Omega S\bar{K}]^* & \xrightarrow{\bar{\iota}_*} & [\bar{K}, P_{\bar{p}}]^* & \xleftarrow{\bar{\iota}_*} & [\bar{K}, \bar{A}]^*. \end{array}$$

Since $\iota: \bar{A} \to P_{\bar{p}}$ is $(m+n-1)$-connected by Lemma 3.1 of [4], $\bar{\iota}_*: [K, \bar{A}]^* \to [K, P_{\bar{p}}]^*$ is bijective by Lemma 8.2* of [11]. The remainder of the proof is dual to that of Theorem 4.3 using Corollary 2.11.

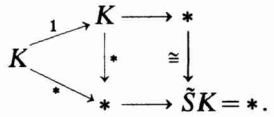
5. Conditions for Cofibre Equivalence

In Theorem 6.3 of [3], Dold shows that a map between two fibre spaces over the identity on their common path connected base B is, with topological restrictions on B, a fibre homotopy equivalence if, and only if, it gives a homotopy equivalence between the fibres. The dual of this is untrue without some restriction on fundamental groups.

Theorem 5.1. *Let $A \to X \to X \cup \tilde{C}A$ and $A \to X' \to X' \cup \tilde{C}A$ be induced cofibre sequences for which X and X' have universal covers, and let $f: X \to X'$ be a map extending the identity on A for which $f \cup \tilde{C}1: X \cup \tilde{C}A \to X' \cup \tilde{C}A$ is a weak homotopy equivalence. Then $f: X \to X'$ is also a weak homotopy equivalence provided $i: A \to X$ induces the zero homomorphism on the fundamental group of each path component of A*

$$\begin{array}{ccccc} & {\scriptstyle i} \nearrow & X & \longrightarrow & X \cup \tilde{C}A \\ A & & \downarrow{\scriptstyle f} & \cong & \downarrow{\scriptstyle f \cup \tilde{C}1} \\ & {\scriptstyle i'} \searrow & X' & \longrightarrow & X' \cup \tilde{C}A. \end{array}$$

Remark 5.2. Some restriction on fundamental groups is necessary. For example[18] let $K = K(\pi, 1)$ where π is a perfect group



Then $* \cup \tilde{C}1$ is a homotopy equivalence, though K and $*$ do not even have the same homotopy type.

Proof of Theorem 5.1. Assume without loss of generality that A meets each path component of X and let $P \subset A$ meet each path component of A in exactly one point. According to Brown's formulation of the van Kampen theorem (Theorem 4.2 of [2]) the following diagram of fundamental groupoids is a pushout in the category of groupoids

$$\begin{array}{ccc} \pi(A, P) & \longrightarrow & \pi(\tilde{C}A, P) \\ \downarrow & & \downarrow \\ \pi(X, \bar{P}) & \longrightarrow & \pi(X \cup \tilde{C}A, \bar{P}). \end{array}$$

In view of the hypotheses $\pi(A, P)$ may be replaced by the trivial groupoid on P, and further $\pi(\tilde{C}A, P)$ is the groupoid on P having exactly one road between any two objects. It follows that $\pi(X, \bar{P}) \to \pi(X \cup \tilde{C}A, \bar{P})$ is injective, and thus, since $\pi(X \cup \tilde{C}A, \bar{P}) \to \pi(X' \cup \tilde{C}A, \bar{P}')$ is an isomorphism on the vertex groups of $\pi(X \cup \tilde{C}A, \bar{P})$, $\pi(X, \bar{P}) \to \pi(X', \bar{P})$ is an isomorphism on the vertex groups of $\pi(X, \bar{P})$: in particular f induces isomorphisms of the fundamental groups of the components of X. Let the universal cover of a space Z be denoted $\tilde{Z}$. It is now sufficient to prove that $\tilde{f}\colon \tilde{X} \to \tilde{X}'$ is a homology equivalence on each path component. The spaces X and X' are assumed to be path connected: more generally $X \cup \tilde{C}A$ is the one point union of spaces of the form $X_1 \cup \tilde{C}A_1$ where X_1 is a path component of X and the extension to this case is immediate. Since $i\colon A \to X$ induces the zero homomorphism of fundamental groups, the universal cover of $X \cup \tilde{C}A$ is obtained by attaching a disjoint union $\bigcup \tilde{C}A$ of copies of $\tilde{C}A$ to a disjoint union $\bigcup \tilde{X}$ of copies of $\tilde{X}$; for a given $\tilde{C}A$, the path components of A are attached to different copies of $\tilde{X}$, and for a given $\tilde{X}$ and given path component A_1 of A, there are copies of CA attached to $\tilde{X}$ at A_1 for each element of $\pi_1(X)$. Thus there is a cofibration $\bigcup \tilde{X} \to \widetilde{X \cup \tilde{C}A} \to \vee \tilde{S}A$, and it is immediate from the homology sequences of this and the corresponding cofibration for $A \to X'$, using the five lemma, that $\tilde{f}\colon \tilde{X} \to \tilde{X}'$ is a homology equivalence. This completes the proof of Theorem 5.1.

[18] This example is due to C.A. Robinson. Another example with $A = S^1$, X the Poincaré 3-sphere, $X' = S^3$ and f a homology equivalence was pointed out to me by P.J. Hilton.

The proof of the following theorem is dual to that of 6.1 of [3].

Theorem 5.3. *Let* $A \xrightarrow{i} X$ *and* $A \xrightarrow{i'} X'$ *be cofibrations and let* $f: X \to X'$ *extend the identity on* A*, then* f *is a homotopy equivalence if and only if it is a cofibre homotopy equivalence.*

References

1. Arkowitz, M.: The generalized Whitehead product. Pacific J. Math. **12**, 7–23 (1962).
2. Brown, R.: Groupoids and Van Kampens theorem. Proc. London Math. Soc. (3) **17**, 385–401 (1967).
3. Dold, A.: Partitions of unity in the theory of fibrations. Ann. of Math. **78**, 223–255 (1963).
4. Ganea, T.: A generalization of the homology and homotopy suspension. Commentarii Math. Helvet. **39**, 295–322 (1965).
5. – Induced fibrations and cofibrations. Trans. Amer. Math. Soc. **127**, 442–459 (1967).
6. Kahn, D.W.: Induced maps for Postnikov systems. Trans. Amer. Math. Soc. **107**, 432–450 (1963).
7. Meyer, J. P.: Principal fibrations. Trans. Amer. Math. Soc. **107**, 177–185 (1963).
8. Nomura, Y.: An application of the path space technique to the theory of triads. Nagoya Math. J. **22**, 169–188 (1963).
9. Rutter, J.W.: A homotopy classification of maps into an induced fibre space. Topology **6**, 379–403 (1967).
10. – Secondary operations and the square construction. Quart. J. Math. Oxford (2) **19**, 337–350 (1968).
11. – Groups of self homotopy equivalences of induced spaces. Commentarii Math. Helvet. **45**, 236–255 (1970).
12. – Exact sequences of track sets. To be published.

Dr. J.W. Rutter
University of Liverpool
Department of Pure Mathematics
Liverpool L69 3BX
England

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